

A numerical procedure based on the Backward Euler method applied to an optimal control problem in flight mechanics

Fabio Silva Botelho
Department of Mathematics
Federal University of Santa Catarina, UFSC
Florianópolis, SC - Brazil
e-mail: fabio.botelho@ufsc.br

Abstract

This short communication develops a numerical procedure suitable for a large class of ordinary differential equation systems found in models in physics and engineering. More specifically, we have utilized the standard Backward Euler method combined with a MAT-LAB function for obtaining a method suitable for a class of two point boundary value ODE non-linear systems. In the last section, we apply the method to a control problem in flight mechanics.

Keywords: Two point boundary value ODE system, Backward Euler method, control problem in flight mechanics.

MSC: 65L10, 65L12

1 Introduction

In this article we develop an application of the Backward Euler method adapted for a two point boundary value ODE (ordinary differential equation) system to a control problem in flight mechanics.

The model in flight mechanics here addressed may be found in [1].

Details on the Sobolev spaces involved may be found in [2].

Related results to those concerning the generalized method of lines, may be found in the here referenced books of 2011 and 2020, [3, 4] respectively.

For a reference on standard finite differences schemes, please see [5].

Finally, related results concerning the fourth order Runge-Kutta and Backward Euler methods have been presented in [6, 7], respectively.

2 The main numerical method

Consider the first order system of ordinary differential equations given by

$$\frac{du_j}{dt} = f_j(\{u_l\}, a), \text{ on } [0, t_f] \forall j \in \{1, 2, 3\},$$

with the boundary conditions

$$u_1(0) = 0, \quad u_3(0) = 120, \quad u_1(t_f) = u_f = 11000.$$

Here $\mathbf{u} = \{u_l\} \in W^{1,2}([0, t_f], \mathbb{R}^3)$ and f_j are functions at least of C^1 class, $\forall j \in \{1, 2, 3\}$.

Moreover $a \in W^{1,2}([0, t_f])$ is a control variable.

For $K_8 = 50000000$, consider the optimal control problem of minimizing the functional J , where

$$\begin{aligned} J(\mathbf{u}, a, t_f) = & 30^2 t_f^2 + u_2(t_f)^2 + 50^2 (u_1(t_f) - 11000)^2 \\ & + K_8^2 \int_0^{t_f} (u_2''(t))^2 dt + 500^2 \int_0^{t_f} (u_1''(t))^2 dt, \end{aligned} \quad (1)$$

subject to

$$\frac{du_j}{dt} = f_j(\{u_l\}, a), \text{ on } [0, t_f] \forall j \in \{1, 2, 3\},$$

with the boundary conditions

$$u_1(0) = 0 \text{ and } u_3(0) = 120$$

and with the restriction

$$2 \pi/180 \leq a(t) \leq 18 \pi/180, \text{ on } [0, t_f].$$

Concerning a finite differences scheme, we denote the number of nodes on $[0, t_f]$ by m_8 and set $d = t_f/m_8$.

Based on the Backward Euler method, we propose the following algorithm.

In finite differences, (indeed we have made it through an appropriate MAT-LAB function), we define $t_f = t_0$ (here t_0 , the final time, it is also a unknown variable to be minimized through the minimization of the not relabeled functional J to be specified in the next lines),

$$u_1(0) = 0, \quad u_2(0) = G_0, \quad u_3(0) = 120;$$

Fixing $k_{10} = 15$, $m_8 = 10000$, $d = t_f/m_8 = t_0/m_8$ we calculate $(u_1)_1, (u_2)_1$ and $(u_3)_1$ through the following iterations,

$$\text{Set } (u_{01})_1 = u_1(0), (u_{02})_1 = u_2(0), (u_{03})_1 = u_3(0).$$

For k_3 from 1 to k_{10} do

1. $(u_j)_1 = u_j(0) + f_j((u_{01})_1, (u_{02})_1, (u_{03})_1, a_1) d, \forall j \in \{1, 2, 3\},$
2. $(u_{0j})_1 = (u_j)_1, \forall j = \{1, 2, 3\}.$

end;

for k from 2 to m_8 do

$$(u_{0j})_k = (u_j)_{k-1}, \forall j \in \{1, 2, 3\}.$$

for k_3 from 1 to k_{10} do

$$1. (u_j)_k = (u_j)_{k-1} + f_j((u_{01})_k, (u_{02})_k, (u_{03})_k, a_k) d, \forall j \in \{1, 2, 3\},$$

$$2. (u_{0j})_k = (u_j)_k, \forall j = \{1, 2, 3\}.$$

end;

end;

Finally, having obtained

$$\{(u_j)_n(G_0, t_0, \{a_k\})\},$$

we obtain $(G_0, t_0, \{a_k\})$ by minimizing the functional

$$J(G_0, t_0, \{a_k\}).$$

where

$$\begin{aligned} J(G_0, t_0, \{a_k\}) &= 25^2 t_0^2 + 30^2 ((u_1)_{m_8}(G_0, t_0, \{a_k\}) - 11000)^2 + [(u_2)_{m_8}(G_0, t_0, \{a_k\})]^2 \\ &\quad + 500^2 \sum_{n=1}^{m_8-2} \left(\frac{((u_1)_{n+2} - 2(u_1)_{n+1} + (u_1)_n)}{d^2} \right)^2 \\ &\quad + K_8 \sum_{n=1}^{m_8-2} \left(\frac{((u_2)_{n+2} - 2(u_2)_{n+1} + (u_2)_n)}{d^2} \right)^2. \end{aligned} \quad (2)$$

subject to

$$2 \pi/180 \leq a_k \leq 18 \pi/180, \forall k \in \{1, \dots, m_8\}.$$

Remark 2.1. Through the minimization of such a functional J , we intend to obtain

$$(G_0, t_0, \{a_k\}) \in \mathbb{R}^3$$

which minimizes the final time $t_f = t_0$ and approximately satisfy the boundary conditions $u_1(t_f = t_0) = 11000$, $u_2(t_f = t_0) = 0$ and, at the same time, minimizes the integral of the second derivative squares of u_1 and u_2 (intending to minimize the trajectory curvature as well) along the time interval $[0, t_f]$.

We emphasize from the expression of J , the boundary condition $u_2(t_f = t_0) = 0$ is just approximately satisfied.

The problem is then approximately solved.

3 Applications to a flight mechanics model

We present numerical results for the following system of equations, which models the in plane climbing motion of an airplane (please, see more details in [1]).

$$\begin{cases} \dot{h} = V \sin \gamma, \\ \dot{\gamma} = \frac{1}{m_f V} (F \sin[a(t) + a_F] + L) - \frac{g}{V} \cos \gamma, \\ \dot{V} = \frac{1}{m_f} (F \cos[a(t) + a_F] - D) - g \sin \gamma \\ \dot{x} = V \cos \gamma, \end{cases} \quad (3)$$

with the boundary conditions,

$$\begin{cases} h(0) = h_0 = 0 \text{ m}, \\ v(0) = v_0 = 120 \text{ m/s} \\ x(0) = x_0 = 0 \text{ m} \end{cases} \quad (4)$$

with a with the following constraint

$$2 \pi/180 \leq a = a(t) \leq 18 \pi/180, \text{ on } [0, t_f],$$

where the optimal final time $t_f = t_0$ will be also calculated, h is the airplane altitude, V is its speed, γ is the angle between its velocity and the horizontal axis, and finally x denotes the horizontal coordinate position.

For numerical purposes, we assume (data close to those of an Air bus 320)

$$m_f = 120,000 \text{ Kg}, S_f = 260 \text{ m}^2, g = 9.8 \text{ m/s}^2,$$

$$\rho(h) = 1.225(1 - 0.0065h/288.15)^{4.225} \text{ Kg/m}^3,$$

$$a_F = 0.0175,$$

$$(C_L)_a = 5$$

$$(C_D)_0 = 0.0175,$$

$$K_1 = 0.0,$$

$$K_2 = 0.06$$

$$(C_L)_0 = 0,$$

$$C_D = (C_D)_0 + K_1 C_L + K_2 C_L^2,$$

$$C_L = (C_L)_0 + (C_L)_a a(t),$$

$$L = \frac{1}{2} \rho(h) V^2 C_L S_f,$$

$$D = \frac{1}{2} \rho(h) V^2 C_D S_f,$$

$$F = 240000$$

and where units refer to the International System.

To simplify the analysis, we redefine the variables as below indicated:

$$\begin{cases} h = u_1, \\ \gamma = u_2 \\ V = u_3 \\ x = u_4. \end{cases} \quad (5)$$

Thus, denoting $\mathbf{u} = (u_1, u_2, u_3, u_4) \in U = W^{1,2}([0, t_f]; \mathbb{R}^4)$, the system above indicated may be expressed by

$$\begin{cases} \dot{u}_1 = f_1(\mathbf{u}) \\ \dot{u}_2 = f_2(\mathbf{u}, a) \\ \dot{u}_3 = f_3(\mathbf{u}, a) \\ \dot{u}_4 = f_4(\mathbf{u}), \end{cases} \quad (6)$$

where,

$$\begin{cases} f_1(\mathbf{u}) = u_3 \sin(u_2), \\ f_2(\mathbf{u}, a) = \frac{1}{m_f u_3} (F \sin[a(t) + a_F] + L(\mathbf{u}, a(t))) - \frac{g}{u_3} \cos(u_2), \\ f_3(\mathbf{u}, a) = \frac{1}{m_f} (F \cos[a(t) + a_F] - D(\mathbf{u}, a(t))) - g \sin(u_2) \\ f_4(\mathbf{u}) = u_3 \cos(u_2). \end{cases} \quad (7)$$

We approximately solve this last system for the following boundary conditions:

$$\begin{cases} h(0) = 0 \text{ m}, \\ v(0) = 120 \text{ m/s}, \\ x(0) = 0 \text{ m}. \end{cases} \quad (8)$$

In summary, our final control problem consists in minimizing the functional J , where

$$\begin{aligned} J(h, \gamma, v, a, t_f) = & 30^2 t_f^2 + \gamma(t_f)^2 + 50^2 (h(t_f) - 11000)^2 \\ & + K_8^2 \int_0^{t_f} (\gamma''(t))^2 dt + 500^2 \int_0^{t_f} (h''(t))^2 dt, \end{aligned} \quad (9)$$

subject to

$$\begin{cases} \dot{h} = V \sin \gamma, \\ \dot{\gamma} = \frac{1}{m_f V} (F \sin[a(t) + a_F] + L) - \frac{g}{V} \cos \gamma, \\ \dot{V} = \frac{1}{m_f} (F \cos[a(t) + a_F] - D) - g \sin \gamma, \end{cases} \quad (10)$$

with the boundary conditions

$$h(0) = 0, \quad v(0) = 120$$

and with the restriction

$$2 \pi/180 \leq a(t) \leq 18 \pi/180, \text{ on } [0, t_f].$$

Observe that having calculated a solution (h, γ, V, a) we may easily calculate x through the fourth equation, that is,

$$x(t) = x(0) + \int_0^t V(t) \cos(\gamma(t)) dt$$

Thus, for a solution, we may consider just the first three equations in the three variables (h, γ, V) . We have numerically obtained $t_f = t_0 = 569.4254 \text{ s}$ and for the solutions for h, γ, V, a , please see the figures 1, 2, 3 and 4, respectively. We have set $m_8 = 10000$ nodes.

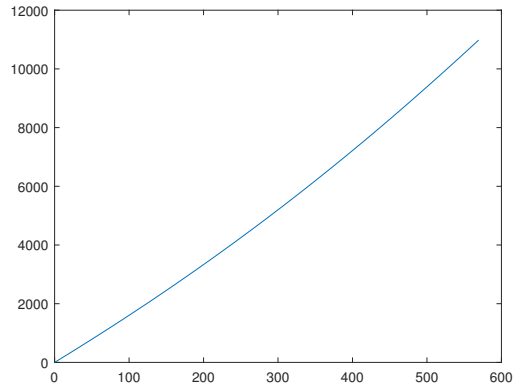


Figure 1: The solution h (in m) for $t_f = 569.4254s$.

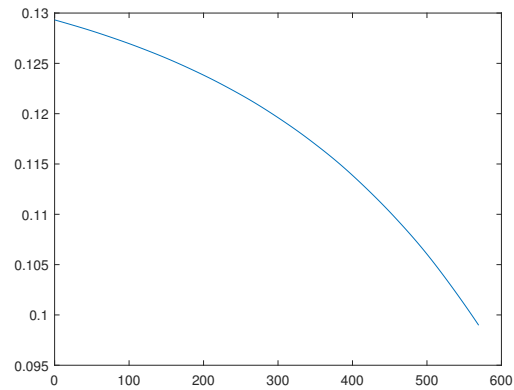


Figure 2: The solution γ (in rad) for $t_f = 569.4254s$.

Remark 3.1. For numerical calculation purposes we have approximated the control function $a(t)$ by a continuous piecewise linear one with just $M_1 + 1 = 6$ nodes ($M_1 = 5$ linear intervals).

Here we present the software in MAT-LAB entitled "FlightMech12Sept2026" through which such numerical results have been obtained.

```
1. clear all
   global m8 d tf CL CD CLo CDo CLa K1 K2 ho go vo Sf mf a aF F h v g W5 G K8 b22
   to k10 M1 a1
   m8=10000;
   M1=5;
   k10=10;
   K8=0;
   mf=120000;
```

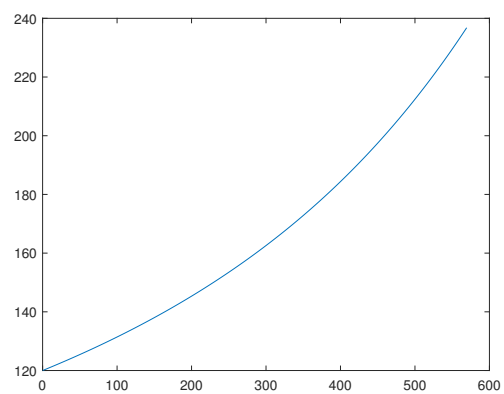


Figure 3: The solution V (in m/s) for $t_f = 569.4254s$.

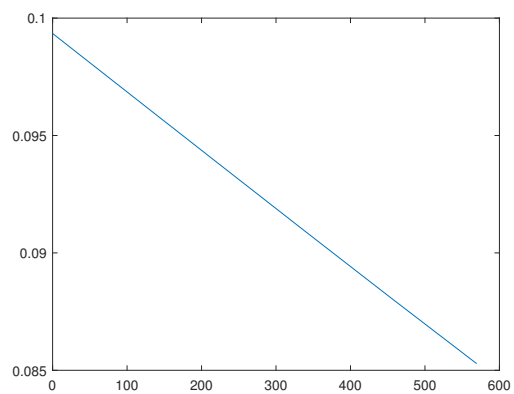


Figure 4: The solution a (in rad) for $t_f = 569.4254s$.

```

Sf=260;
G=9.81;
a=0.138;
aF=0.0175;
F=240000;
CLa=5;
CDo=0.0175;
K1=0.0;
K2=0.06;
CLo=0;
for i=1:M1+1
lb(i,1)=2.0*pi/180;
ub(i,1)=18*pi/180;
end;
lb(M1+1+1,1)=0.05;
ub(M1+1+1,1)=0.45;
lb(M1+1+2,1)=420;
ub(M1+1+2,1)=900;
for i=1:M1+1
xo(i,1)=a;
end;
xo(M1+1+1,1)=0.15;
xo(M1+1+2,1)=639;
b14=1;
k3=1;
while (b14 > 10-4) && (k3 < 5)
k3
k3=k3+1;
k10=k10+5;
b12=1;
k1=1;
while (b12 > 10-4) && (k1 < 5)
k1
k1=k1+1;
X=lsqnonlin('funFlightMech12Sept2026',xo,lb,ub);
b12=max(abs(X-xo));
xo=X;
h(m8/2,1);
end;

```

```

b14=max(abs(b22));
end;
for i=1:m8
x3(i,1)=i*d;
end;
plot(x3,h)

```

With the main function entitled "funFlightMech12Sept2026"

```

1. function W5=funFlightMech12Sept2026(x)
    global m8 d tf CL CD CLo CDo CLa K1 K2 ho go vo Sf mf a aF F h v g W5 G K8 M1
    b22 to k10 a1
    Ho(1,1)=0;
    Go(1,1)=x((M1+1)+1,1);
    Vo(1,1)=120;
    to=x((M1+1)+2,1);
    d=to/m8;
    for i=1:M1
    for j=1:m8/M1
    a1((i-1)*m8/M1+j,1)=x(i,1)*(1-j*d/(to/M1))+x(i+1,1)*j*d/(to/M1);
    end;
    end;
    k=1;
    CL=CLo+CLa*a1(k,1);
    CD = CDo + K1 * CL + K2 * CL^2;
    ho(k,1)=Ho(1,1);
    go(k,1)=Go(1,1);
    vo(k,1)=Vo(1,1);
    for k8=1:k10
    
$$r(k) = \text{abs}(1.225 * (1 - 0.0065 * ho(k,1)/288.15)^{4.225});$$

    
$$L = 1/2 * r(k) * vo(k,1)^2 * CL * Sf;$$

    
$$D = 1/2 * r(k) * vo(k,1)^2 * CD * Sf;$$

    h(k,1)=(Ho(1,1)+vo(k,1)*sin(go(k,1))*d);
    g(k,1)=(Go(1,1)+1/mf*(F*sin(a1(k,1)+aF)+L)/vo(k,1)*d-G/vo(k,1)*cos(go(k,1))*d);
    v(k,1)=(Vo(1,1)+1/mf*(F*cos(a1(k,1)+aF)-D)*d-G*sin(go(k,1))*d);
    b22(k,1)=abs(v(k,1)-vo(k,1));
    ho(k,1)=h(k,1);
    go(k,1)=g(k,1);
    vo(k,1)=v(k,1);
    end;
    end;
    W5=W5+1;
end;

```

```

end;
for k=2:m8
CL=CLo+CLa*a1(k,1);
CD = CDo + K1 * CL + K2 * CL2;
ho(k,1)=h(k-1,1);
go(k,1)=g(k-1,1);
vo(k,1)=v(k-1,1);
for k8=1:k10
r(k) = abs(1.225 * (1 - 0.0065 * ho(k,1)/288.15)4.225);
L = 1/2 * r(k) * vo(k,1)2 * CL * Sf;
D = 1/2 * r(k) * vo(k,1)2 * CD * Sf;
h(k,1)=(h(k-1,1)+vo(k,1)*sin(go(k,1))*d);
g(k,1)=(g(k-1,1)+1/mf*(F*sin(a1(k,1)+aF)+L)/vo(k,1)*d-G/vo(k,1)*cos(go(k,1))*d);
v(k,1)=(v(k-1,1)+1/mf*(F*cos(a1(k,1)+aF)-D)*d-G*sin(go(k,1))*d);
b22(k,1)=abs(v(k,1)-vo(k,1));
ho(k,1)=h(k,1);
go(k,1)=g(k,1);
vo(k,1)=v(k,1);
end;
end;
W5(1,1)=30*(h(m8,1)-11000);
W5(2,1)=25*to;
W5(3,1)=g(m8,1);
for i=1:m8-2
W5(i+3,1) = 500 * (h(i+2,1) - 2 * h(i+1,1) + h(i,1))/d2;
W5(m8-2+i+3,1) = 50000000 * (g(i+2,1) - 2 * g(i+1,1) + g(i,1))/d2;
end;
*****

```

4 Acknowledgements

The author is grateful to Professor Pedro Paglione of Technological Institute of Aeronautics, ITA, SP-Brazil, for his valuable suggestions and comments which help me a lot to improve some important parts of this text, in particular on the part concerning the model in flight mechanics addressed.

5 Conclusion

In this article we have presented a numerical procedure for solving a class of two point boundary value ODE systems with applications to a control problem in flight mechanics.

The numerical method is based on the Backward Euler approach adapted for a two point boundary value ODE system.

In a near future research we intend to address some other linear and non-linear cases.

1. **Conflict of interest declaration: The author declares no conflict of interest concerning this article.**

References

- [1] N.X. Vinh, Flight Mechanics of High Performance Aircraft, (Cambridge University Press, New York, 1993)
- [2] R.A. Adams and J.F. Fournier, Sobolev Spaces, 2nd edn. (Elsevier, New York, 2003).
- [3] F. Botelho, *Topics on Functional Analysis, Calculus of Variations and Duality*, Academic Publications, Sofia, (2011).
- [4] F.S. Botelho, Functional Analysis, Calculus of Variations and Numerical Methods for Models in Physics and Engineering, CRC Taylor and Francis, Florida, 2020.
- [5] J.C. Strikwerda, Finite Difference Schemes and Partial Differential Equations, SIAM, second edition (Philadelphia, 2004).
- [6] F. Botelho, A numerical procedure based on the fourth order Runge-Kutta method for two point boundary value ordinary differential equation (ODE) systems with applications to a model in flight mechanics. Engrxiv/Preprints, 2026. DOI: <https://doi.org/10.31224/8168>
- [7] F. Botelho, A numerical procedure based on the Backward Euler method for two point boundary value ordinary differential equation (ODE) systems with applications to a model in flight mechanics. Engrxiv/Preprints, 2026. DOI: <https://doi.org/10.31224/8190>