Theory and Model for Active Design of Spiral Bevel Gears

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Abstract Starting from the fundamental principles of differential geometry, this paper systematically establishes the mathematical foundation and model representation of spiral bevel gear tooth surfaces. Following a logical chain of "geometric foundations—parametrization principle—normal theory—integrability conditions—metric and curvature—closed-form equations—conjugate meshing", all core contents of spiral bevel gear tooth surface modeling are unified within an axiomatic framework. It is proved that the essence of a spiral bevel gear tooth surface is a smooth immersion from a two-dimensional parameter manifold into three-dimensional Euclidean space; the normal direction angle functions are the angular coordinate control functions of the Gauss map defined on this manifold; and the geometric essence of conjugate meshing is the constraint condition of relative velocity in the tangent space. This theoretical system not only provides a unified mathematical basis for precise modeling, active design, and modification optimization of spiral bevel gears, but also opens avenues for research on non-circular bevel gears, novel tooth profiles, and intelligent tooth surfaces.

Keywords Spiral bevel gears; Tooth surface modeling; Differential geometry; Differential manifold; Gauss map; Frobenius integrability; Closed-form equations; Conjugate meshing

0 Introduction

As a core element of intersecting-axis transmission, spiral bevel gears are widely used in aviation, automotive, marine, and robotics applications. Their tooth surfaces are complex surfaces in space, whose geometry arises from the coupling of three ingredients: cone geometry, spiral angle distribution, and involute tooth profile. For more than a century, the two major systems - Gleason and Klingelnberg - have dominated the design and manufacture of spiral bevel gears, with a core methodology based on the machining-simulation path of "cutter geometry + relative motion -> envelope surface". This methodology has been enormously successful in engineering, yet its mathematical essence - the existence, regularity, symmetry, and conjugacy conditions of the envelope surface - is often submerged in a maze of coordinate transformations and empirical formulas.

As high-end equipment imposes ever-increasing demands on gear transmission performance, the traditional methodology faces three major difficulties. First, its capacity for active tooth-surface design is insufficient: the designer can hardly control the contact characteristics and the modification law of the tooth surface directly. Second, its sign conventions are chaotic: the coupling of tooth side, hand of spiral, and tooth indexing leads to unclear concepts that are error-prone and hard to trace. Third, the design of novel tooth forms and non-conical bevel gears lacks a unified theoretical framework, so that every new problem must be derived from scratch.

The essence of a gear is a pair of conjugate surfaces, and the essence of conjugate surfaces is the interaction problem of constrained surfaces in differential geometry. Therefore, the theoretical foundation of tooth-surface modeling should, and indeed must, be built upon differential geometry.

Starting from the most elementary geometric concepts, this paper constructs the theoretical edifice of the spiral bevel gear layer by layer, and finally derives its closed-form equations and the whole-tooth-surface conjugacy theorem. The theoretical formulas established by this method are then programmed to obtain the three-dimensional model, and finally the gear product is realized by 5-axis CNC machining.

1 Cone Geometry and the Spherical Involute - The Geometric Foundation of Spiral Bevel Gears

1.1 Intrinsic Metric of the Cone Surface

Let the half-apex angle (base cone angle) of the cone surface be δ_b, with the cone apex at the coordinate origin and the axis coincident with the z axis. Introduce curvilinear coordinates (u,R) on the cone surface, where R is the distance from an arbitrary point on the generatrix to the cone apex (the cone distance) and u is the angular parameter of that point along the generatrix.$\delta_b z(u,R) R u$

The parametric equation of the cone surface is:

$$\mathbf{r}(u, R) = R \cdot \mathbf{q}(u) \tag{1.1}$$

where $\mathbf{q}(u)$ is the unit generatrix direction vector on the base cone surface. In Cartesian coordinates：

$$\mathbf{q}(u)=\begin{pmatrix}\sin\delta_b\cos u\\ -\sin u\\ \cos\delta_b\cos u\end{pmatrix} \tag{1.2}$$

From this we compute the coefficients of the first fundamental form:

$$E=\mathbf{r}_u\cdot\ \mathbf{r}_u=R^2\cos^2 u \tag{1.3}$$

$$F=\mathbf{r}_u\cdot\ \mathbf{r}_R=0 \tag{1.4}$$

$$G=\mathbf{r}_R\cdot\ \mathbf{r}_R=1 \tag{1.5}$$

It is seen that the parameter net of the cone surface is an orthogonal net ($F=0$), which is an important characteristic of the cone surface as a developable surface. The area element is:F=0

$$dA=\sqrt{EG-F^2}\ du\ dR=R\cos u\ du\ dR \tag{1.6}$$

Property 1.1 (Developability). The cone surface is a developable surface whose Gaussian curvature vanishes everywhere. Consequently, the cone surface can be unrolled into a plane sector without stretching. This property is the geometric origin of the fact that a spherical involute can be "flattened" into a planar involute.

1.2 Generation of the Spherical Involute

On the cone surface, the generating mechanism of the involute is entirely analogous to that of the planar involute on a cylinder surface, except that it takes place on a sphere.

Definition 1.1 (Spherical involute). Let a plane roll without slipping along the base cone. The locus swept out in space by a straight line on that plane passing through the cone apex is the spherical involute surface, and its intersection curve with an arbitrary sphere centred at the cone apex is called the spherical involute.

When the plane rolls around the base cone, let the azimuth of the contact generatrix between the plane and the base cone be χ ; then the deflection angle of the fixed line through the cone apex in the plane, relative to the base cone generatrix, is $u/\sin\delta_b$ (derived from the proportionality between arc length and angle). Hence the circumferential angle of the involute is:

$$\Phi(u)=\sigma\cdot\ \frac{u}{\sin\delta_b}+\phi_0 \tag{1.7}$$

where $\sigma=\pm 1$ is the involute unwinding direction and ϕ_0 is the initial phase.

The physical meaning of Eq. (1.7) is clear: u measures the "unwinding distance" along the base cone generatrix, and division by $\sin\delta_b$ converts arc length along the generatrix into a rotation angle within the tangent plane of the base cone.

When $\delta_b \to 90^\circ$, the base cone degenerates into a cylinder and $\sin\delta_b \to 1$; Eq. (1.7) then degenerates into the unwinding-angle formula of the planar involute, verifying that the cylindrical gear is the limiting case of the bevel gear.

1.3 The Base Cone Theorem

Theorem 1.1 (Base cone theorem). For an intersecting-axis gear transmission with constant transmission ratio, there exists a unique pair of base cones such that the two tooth surfaces are generated respectively by the pure rolling of one and the same plane along each base cone. This plane is the meshing plane.

The base cone theorem is the cornerstone of spiral bevel gear theory. It converts the dynamical concept of "conjugate meshing" into the static geometric problem of "a plane rolling along a cone surface". All subsequent tooth-surface equations, meshing conditions, and contact analyses take this theorem as their logical starting point.

The base cone angles δ_{b1},δ_{b2} and the pitch cone angles δ_1,δ_2 satisfy the relation:

$$\sin\delta_{bi}=\sin\delta_i\cdot\cos\alpha_t \quad (i=1,2) \tag{1.8}$$

where α_t is the transverse pressure angle. Equation (1.8) is the direct generalization to bevel gears of $r_b=r\cos\alpha$ for cylindrical gears, reflecting the intrinsic unity of the two gear theories.

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2 The Parametrization Principle - From Abstract Manifold to Spatial Surface

2.1 The Parameter Manifold

Definition 2.1 (Tooth-surface parameter manifold). Let the working region of a spiral bevel gear tooth surface be described by two independent parameters (u,R), where $u\in[u_f,u_a]$ is the tooth profile parameter (along the tooth height direction)

and $R\in[R_i,R_e]$ is the cone distance parameter (along the tooth width direction). The parameter domain

$$D=[u_f,u_a]\times[R_i,R_e]\subset\mathbb{R}^2 \tag{2.1}$$

itself constitutes a two-dimensional differentiable manifold, called the tooth-surface parameter manifold and denoted by M.

The parameter manifold M is a purely abstract space - it has no shape and no metric, only a topological structure (a two-dimensional connected region) and coordinate functions (u,R). The sole role of M is to assign an "identifier" to each point of the tooth surface. The actual geometric shape is conferred by the embedding map.

2.2 The Embedding Map and Regularity

Definition 2.2 (Tooth-surface embedding map). The tooth-surface embedding map is the smooth map

$$\mathbf{r}:M\rightarrow\mathbb{R}^3,\quad (u,R)\mapsto \mathbf{r}(u,R) \tag{2.2}$$

If at every point $p\in M$ the Jacobian matrix has rank 2, that is,

$$\mathbf{r}_u\times\mathbf{r}_R\neq\mathbf{0} \tag{2.3}$$

then $\mathbf{r}$ is called an immersion, and its image $S=\mathbf{r}(M)$ is a two-dimensional smooth submanifold of three-dimensional Euclidean space - the actual tooth surface.

Condition (2.3) is the basic regularity condition for the tooth surface to be a smooth surface. Physically, it guarantees that the tooth surface has a uniquely determined tangent plane and normal vector at every point. At locations such as the tooth tip edge and the boundary of the tooth root transition zone, the regularity condition may fail; these locations constitute the boundary or the singular set of the tooth surface.

2.3 Tangent Space and Pushforward Map

At every point p of the parameter manifold, the tangent space T_pM is the two-dimensional linear space spanned by the coordinate tangent vectors:

$$T_pM=\mathrm{span}\left\{\frac{\partial}{\partial u},\frac{\partial}{\partial R}\right\} \tag{2.4}$$

The differential of the embedding map (the pushforward map)

$$d\mathbf{r}_p: T_pM \to T_{\mathbf{r}(p)}S \subset \mathbb{R}^3 \tag{2.5}$$

maps an abstract tangent vector to a tooth-surface tangent vector in three-dimensional space. For an arbitrary tangent vector $X = X^u \frac{\partial}{\partial u} + X^R \frac{\partial}{\partial R} \in T_pM$, we have

$$d\mathbf{r}_p(X) = X^u \mathbf{r}_u + X^R \mathbf{r}_R \tag{2.6}$$

Therefore the tangent space of the tooth surface is:

$$T_{\mathbf{r}(p)}S = \mathrm{span}\{\mathbf{r}_u, \mathbf{r}_R\} \tag{2.7}$$

Conceptual clarification - strict distinction between the parametric level and the geometric level

Level	Object	Example elements
Parametric level	Abstract manifold M	u, R ; du, dR ; coordinate tangent vector $\partial/\partial u$
Geometric level	Embedded submanifold $S$$S$	$\mathbf{r}(u,R)$; $d\mathbf{r}$; spatial tangent vector $\mathbf{r}_u$

The two are related through the embedding map $\mathbf{r}$ and its differential $d\mathbf{r}$. The same surface admits many different parametrizations, but the intrinsic geometric properties of the surface (curvature, area, arc length) are independent of the choice of parameters.

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3 Normal Theory - The Gauss Map and the Direction Angle Functions

3.1 The Gauss Map

For a regular surface $S \subset \mathbb{R}^3$, at every point $p \in S$ there exists a unique unit normal vector $\mathbf{n}(p)$, which is orthogonal to the tangent plane T_pS. In parametric coordinates,

$$\mathbf{n}(u,R) = \frac{\mathbf{r}_u \times \mathbf{r}_R}{|\mathbf{r}_u \times \mathbf{r}_R|} \tag{3.1}$$

Definition 3.1 (Gauss map). The Gauss map is the map from the surface to the unit sphere

$$\mathrm{N}: S \to S^2, \quad p \mapsto \mathbf{n}(p) \tag{3.2}$$

where $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$ is the unit sphere.

3.2 The Mathematical Essence of the Normal Direction Angle Functions

In the normal-direction-angle-function model of spiral bevel gears, the normal vector is not given directly by three Cartesian components, but is parametrized by a set of angular functions. Specifically, in the local frame $\{\mathbf{e}_R(\Phi),\mathbf{e}_\phi(\Phi),\mathbf{e}_r\}$ of the base cone, the unit normal vector can be expressed as:

$$\mathbf{n}(u,R)=\cos\mu(u,R)\cdot\left[\sin u\,\mathbf{e}_R(\Phi)+\cos u\,\mathbf{e}_\phi(\Phi)\right]+\sin\mu(u,R)\cdot\mathbf{e}_r \tag{3.3}$$

where: - u is the tooth profile direction angle, controlling the rotation of the normal within the tangent plane of the base cone; - $\mu(u,R)$ is the spiral direction angle, describing the extent to which the normal deviates from the tangent plane of the base cone; - $\Phi(u,R)$ is the circumferential phase function, which determines the orientation of the local frame.

From the standpoint of differential geometry, Eq. (3.3) is not a tooth-surface equation but an angular-coordinate parametrization of the Gauss map:

$$(u,R)\mapsto(u,\mu,\Phi)\mapsto\mathbf{n}(u,R)\in S^2 \tag{3.4}$$

Therefore, the mathematical essence of the normal direction angle functions is:

$$\text{Normal direction angle functions = angular-coordinate description of the Gauss map on the parameter manifold} \tag{3.5}$$

3.3 The Three-Level Structure

Thus the three mathematical levels in spiral bevel gear tooth-surface modeling are now clearly presented:

Level	Object	Map	Core question
Parametric level	Parameter manifold M	—	How to parametrize? Are the parameters reasonable?
Normal level	Gauss map N	$M\xrightarrow{\mathrm{N}}S^2$	How are the normals distributed? What about curvature?
Geometric level	Embedding map **r**	$M\xrightarrow{\mathbf{r}}\mathbb{R}^3$	What is the shape of the surface?

The three are linked by the orthogonality condition:

$$\mathbf{n}\cdot\mathbf{r}_u=0,\quad \mathbf{n}\cdot\mathbf{r}_R=0 \tag{3.6}$$

Equation (3.6) is the fundamental bridge connecting the normal level and the geometric level.

4 Integrability Theory - Existence Conditions for Integral Surfaces of a Normal Vector Field

4.1 The Pfaff Equation

A displacement on the tooth surface must lie within the tangent plane, and therefore satisfies

$$\mathbf{n}\cdot d\mathbf{r}=0 \tag{4.1}$$

This is a first-order linear partial differential equation, called the Pfaff equation. If the normal vector field is regarded as a function $\mathbf{n}(\mathbf{x})$ of spatial position, one may define a 1-form$\mathbf{n}(\mathbf{x})$

$$\omega=n_x dx+n_y dy+n_z dz \tag{4.2}$$

The equation .$\omega=0$ requires that all tangent vectors of the integral surface lie in the kernel of ω , that is, the tangent plane of the surface is orthogonal to the normal vector.

4.2 The Frobenius Integrability Theorem

Theorem 4.1 (Frobenius integrability theorem). Let ω be a smooth 1-form on an open set U in $\mathbb{R}^3$ that is nowhere zero. Then $\omega=0$ defines a family of two-dimensional integral surfaces if and only if

$$\omega\wedge d\omega=0 \tag{4.3}$$

For $\omega=n_x dx+n_y dy+n_z dz$, condition (4.3) is equivalent to

$$\mathbf{n}\cdot(\nabla\times\mathbf{n})=0 \tag{4.4}$$

that is, the normal vector field must be orthogonal to its own curl.

The physical meaning of the Frobenius theorem: an arbitrarily prescribed normal vector field only specifies a plane direction (a tangent-plane distribution) at each point of space, but these planes cannot necessarily be "stitched together" into a family of smooth surfaces. Such surfaces exist only when the integrability condition is satisfied.

4.3 Verification of Integrability in Parametric Form

For a parametric surface $\mathbf{r}(u,R)$, the integrability condition reduces to the direct verification of

$$\mathbf{n}\cdot\mathbf{r}_u=0,\quad \mathbf{n}\cdot\mathbf{r}_R=0 \tag{4.5}$$

These two conditions guarantee that $\mathbf{n}$ is indeed the normal vector of the parametric surface.

For the "inverse problem" - given $\mathbf{n}(u,R)$, find $\mathbf{r}(u,R)$ - one must solve the system of partial differential equations (4.5). The compatibility condition for this system to be solvable is the parametric version of the Frobenius condition:

$$\frac{\partial}{\partial R}(\mathbf{n}\cdot\mathbf{r}_u)-\frac{\partial}{\partial u}(\mathbf{n}\cdot\mathbf{r}_R)=0 \tag{4.6}$$

After expansion, one obtains the constraint relations that must be satisfied among the derivatives of the direction angle functions up to all orders.

5 Metric and Curvature - The Geometric Measures of the Tooth Surface

5.1 The First Fundamental Form and the Intrinsic Metric

The metric structure of the tooth surface is induced by the inner product of three-dimensional space.

Definition 5.1 (First fundamental form). The first fundamental form is the symmetric bilinear form on the tangent space

$$\mathrm{I}_p(X,Y)=d\mathbf{r}_p(X)\cdot d\mathbf{r}_p(Y) \tag{5.1}$$

In coordinates(u,R) , its components are:

$$\begin{cases} E(u,R)=\mathbf{r}_u\cdot\mathbf{r}_u \\ F(u,R)=\mathbf{r}_u\cdot\mathbf{r}_R \\ G(u,R)=\mathbf{r}_R\cdot\mathbf{r}_R \end{cases} \tag{5.2}$$

Hence

$$\mathrm{I}=E\,du^2+2F\,du\,dR+G\,dR^2 \tag{5.3}$$

The first fundamental form completely describes the intrinsic geometry of the tooth surface: geometric quantities such as arc length, angle, area, and geodesics can all be computed from E,F,G without any knowledge of the concrete shape of the surface in space. The area element is:

$$dA=\sqrt{EG-F^2}\,du\,dR \tag{5.4}$$

5.2 The Second Fundamental Form and Extrinsic Bending

The first fundamental form tells us only the metric of the surface and is not sufficient to determine how the surface bends in space. It is the second fundamental form that describes the bending behaviour of the surface.

Definition 5.2 (Shape operator). The shape operator (Weingarten map) of the surface at a point p is the linear operator

$$\mathrm{S}_p:T_pS\to T_pS,\quad \mathrm{S}_p(v)=-d\mathrm{N}_p(v) \tag{5.5}$$

that is, the negative of the differential of the Gauss map. The shape operator is self-adjoint; its two real eigenvalues κ_1,κ_2 are the principal curvatures, and the corresponding orthogonal eigendirections are the principal directions.

Definition 5.3 (Second fundamental form). The second fundamental form is the symmetric bilinear form on the tangent space

$$\mathrm{II}_p(X,Y)=\mathrm{I}_p(\mathrm{S}_p(d\mathbf{r}(X)),d\mathbf{r}(Y)) \tag{5.6}$$

Its components are:

$$\begin{cases} L(u,R)=-\mathbf{n}_u\cdot\ \mathbf{r}_u=\mathbf{r}_{uu}\cdot\ \mathbf{n} \\ M(u,R)=-\mathbf{n}_u\cdot\ \mathbf{r}_R=\mathbf{r}_{uR}\cdot\ \mathbf{n} \\ N(u,R)=-\mathbf{n}_R\cdot\ \mathbf{r}_R=\mathbf{r}_{RR}\cdot\ \mathbf{n} \end{cases} \tag{5.7}$$

5.3 Curvature and Contact Characteristics

The principal curvatures are the solutions of the following generalized eigenvalue problem:

$$\det(\mathrm{II}-\kappa\mathrm{I})=0 \tag{5.8}$$

Expanding gives:

$$(EG-F^2)\kappa^2-(LG-2MF+NE)\kappa+(LN-M^2)=0 \tag{5.9}$$

From this we obtain:

Gaussian curvature (an intrinsic quantity, the Theorema Egregium):

$$K=\kappa_1\kappa_2=\frac{LN-M^2}{EG-F^2} \tag{5.10}$$

Mean curvature (an extrinsic quantity):

$$H=\frac{\kappa_1+\kappa_2}{2}=\frac{LG-2MF+NE}{2(EG-F^2)} \tag{5.11}$$

For gear engineering, curvature directly determines the contact characteristics of the tooth surface. According to Hertz contact theory, the size and shape of the contact patch are determined by the principal curvatures of the two contacting surfaces (the relative curvatures along the principal directions), while the contact stress is closely related to the curvature.

6 Closed-Form Representation of the Spiral Bevel Gear Tooth Surface

6.1 The General Master Equation

Within the theoretical framework set out above, the general master equation of the spiral bevel gear tooth surface can be expressed as:

$$\mathbf{r}_i^{(s,k)}(u,R)=R\cdot\ A_z\left(\Phi_i^{(s,k)}\right)\cdot\ \mathbf{q}_i(u) \tag{6.1}$$

where: -$i=1,2$ - index of the pinion and the gear; - $s=\pm1$ - tooth side index (involute unwinding direction; +1 for the concave/right flank, -1 for the convex/left flank); - $k=0,1,\ldots,z_i$ - tooth index; - $\mathbf{q}_i(u)$ - unit generatrix vector of the base cone of the i -th wheel; - $A_z(\Phi)$ - rotation matrix about the z axis; $\Phi_i^{(s,k)}(u,R)$ -circumferential phase function.

Equation (6.1) decomposes a spatial surface into three independent operations: 1. radial scaling along the base cone generatrix (multiplication by R); 2. parametrization along the tooth profile direction ($\mathbf{q}_i(u)$); 3. circumferential helical phase rotation ($A_z(\Phi)$).

6.2 The Three-Parameter Decoupled Structure of the Phase Function

The phase function $\Phi_i^{(s,k)}(u,R)$ is the core of the tooth-surface equation; it determines how the tooth profile, the tooth trace, and tooth indexing are coupled. Its key property is complete decoupling:

$$\Phi_i^{(s,k)}(u,R) = s\cdot\underbrace{\frac{u}{\sin\delta_{bi}}}_{1}+\underbrace{\varphi_{si}(R)}_{2}+\underbrace{\varphi_{0i}^{(s,k)}}_{3} \tag{6.2}$$

This decomposition reveals three independent degrees of freedom:

- **The tooth profile term $s\cdot u/\sin\delta_{bi}$: it depends only on the tooth side s and the profile parameter u, and determines the unwinding direction of the involute profile;The tooth trace term $\varphi_{si}(R)$: it depends only on the cone distance R , is determined by the spiral angle function, and controls the shape and hand of the tooth trace;**
- **The tooth indexing term $\varphi_{0i}^{(s,k)}$: it depends only on the tooth index k and the tooth side s , and determines the circumferential position and the thickness of each tooth.**

6.3 Component Expansion

Expanding Eq. (6.1) into Cartesian coordinate components:

$$\begin{cases} x=R[\sin\delta_{bi}\cos u\cos\Phi+s\sin u\sin\Phi] \\ y=R[\sin\delta_{bi}\cos u\sin\Phi-s\sin u\cos\Phi] \\ z=R\cos\delta_{bi}\cos u \end{cases} \tag{6.3}$$

The parameter domain is

$$u\in[u_{fi},u_{ai}],\quad R\in[R_i,R_e] \tag{6.4}$$

Equation (6.3) is the closed-form analytical expression of the spiral bevel gear tooth surface. It gives explicitly the spatial coordinates of an arbitrary point on the tooth surface, thereby avoiding the cumbersome process of the traditional methodology, in which tooth-surface points can be obtained only through envelope computation or numerical iteration.

6.4 Structural Symmetry of the Four Tooth Surfaces

The pinion and the gear together comprise two pairs of flanks, that is, four tooth surfaces, whose equations exhibit a clear symmetry:

No.	Tooth surface	i	s	Hand h_i	Pairing
F1	Pinion concave flank Σ_1^{cv}	1	+1	+1 (right-hand)	↔ F3
F2	Pinion convex flank Σ_1^{cx}	1	-1	+1	↔ F4
F3	Gear convex flank Σ_2^{cx}	2	+1	-1 (left-hand)	↔ F1
F4	Gear concave flank Σ_2^{cv}	2	-1	-1	↔ F2

The symmetry is manifested as follows: - the tooth side index s merely flips the sign of the y component, the sign of the $u/\sin\delta_b$ term, and the sign of the tooth-indexing phase correction term in the equation; - the hand index h_i changes only the sign of the tooth trace function $\varphi_{si}(R)$; - the pairing rule is $\mathrm{h}_1=-\mathrm{h}_2$, $s_1=s_2=s$, that is, "the concave flank of the pinion mates with the convex flank of the gear, and the convex flank of the pinion mates with the concave flank of the gear".

6.5 The Base Cone Tangent Plane Constraint

Property 6.1. At any point of the tooth surface, both the position vector and the normal vector lie in the tangent plane of the base cone to which that point belongs, that is,

$$\mathbf{n}_i^{(s)} \cdot \mathbf{e}_v(\Phi)\equiv 0, \quad \mathbf{r}_i^{(s,k)} \cdot \mathbf{e}_v(\Phi)\equiv 0 \tag{6.5}$$

where $\mathbf{e}_v(\Phi)$ is the unit normal vector of the base cone (pointing outward from the cone surface), and this plane passes through the cone apex.

This property is the geometric foundation of the whole-tooth-surface meshing theorem. It means that all normals of the tooth surface intersect the instantaneous axis - which is precisely the fundamental reason why spiral bevel gears can achieve continuous conjugate transmission.

7 Differential-Geometric Conditions for Conjugate Meshing

7.1 Tangent-Space Formulation of the Meshing Equation

The basic equation of gear meshing is

$$\mathbf{n}\cdot\ \mathbf{v}^{(12)}=0 \tag{7.1}$$

where $\mathbf{v}^{(12)}$ is the relative velocity of the two tooth surfaces at the contact point.

For bevel gear transmission with a constant transmission ratio, the relative velocity can be expressed as

$$\mathbf{v}^{(12)}=\boldsymbol{\omega}_{12}\times\mathbf{r}=\dot{\varphi}_1\ \mathbf{p}\times\mathbf{r} \tag{7.2}$$

where $\mathbf{p}$ is the unit vector along the instantaneous axis (the pitch line). Substituting into (7.1) gives

$$\mathbf{n}\cdot\ (\mathbf{p}\times\mathbf{r})=0 \tag{7.3}$$

Geometric meaning: the three vectors $\mathbf{n},\mathbf{p}$, and $\mathbf{r}$ are coplanar.

From the standpoint of differential geometry, the meshing condition can be stated in a more essential form:

Meshing condition <=> the relative motion velocity belongs to the tangent space of the tooth surface

that is,

$$\mathbf{v}^{(12)}\in T_pS \tag{7.4}$$

7.2 The Whole-Tooth-Surface Meshing Theorem

Theorem 7.1 (Whole-tooth-surface meshing theorem). Let the base cones of the two wheels be C_1 and C_2 respectively, with shaft angle Σ . If there exists a plane Π_s tangent to both base cones simultaneously, and the tooth surfaces of the two wheels are generated respectively by the pure rolling of this plane along their respective base cones, then the entire tooth surfaces achieve strictly conjugate meshing with a constant transmission ratio.

Proof: By Property 6.1, at any point of the tooth surface both the normal n and the position vector .r lie in the base cone tangent plane $\Pi(\Phi)$, and this plane passes through the cone apex

The meshing condition $\mathbf{n}\cdot(\mathbf{p}\times\mathbf{r})=0$ is equivalent to the coplanarity of the three vectors. Since $\mathbf{n}$ and $\mathbf{r}$ already lie in $\Pi(\Phi)$, the meshing condition reduces to: the instantaneous axis $\mathbf{p}$ must lie in the plane $\Pi(\Phi)$, that is,

$$\mathbf{p}\in\Pi(\Phi) \tag{7.5}$$

Let the unit normal vector of the plane Π be $\mathbf{m}$; then $\mathbf{p}\in\Pi$ is equivalent to $\mathbf{m}\cdot\mathbf{p}=$. Combining this with the base cone tangent plane condition $\mathbf{m}\cdot\mathbf{z}_i=\pm\sin\delta_{bi}$, one obtains the solution

$$\mathbf{m}_{\pm}=\cos\alpha_t\,\mathbf{q}\pm\sin\alpha_t\,\hat{\mathbf{y}} \tag{7.6}$$

where $\mathbf{q}=(-\cos\delta_1,0,\sin\delta_1)$. This shows that there exist two meshing planes .$\Pi_{\pm}$, corresponding respectively to the two sets of tooth surfaces $s=\pm1$, and that the two wheels share one and the same transverse pressure angle α_t

We have thus proved that, for all points satisfying $\mathbf{p}\in\Pi(\Phi)$, the meshing condition is satisfied automatically. These points constitute the entire tooth surface - because the tooth surface is precisely generated by the pure rolling of the meshing plane along the base cone. ∎

8 Engineering Example and Verification

Based on the above method, machining verification has been carried out for several sets of spiral bevel gear pairs on a five-axis machining centre. The typical machining parameters are: normal module $\boldsymbol{m_n}$=4~ 8, tooth number ratio z_1/z_2=11/39 , face width $\boldsymbol{b}$=30~ 50 mm; the material is 20CrMnTi, case-hardened by carburizing; and the machining accuracy reaches grade 5 of GB/T 11365.

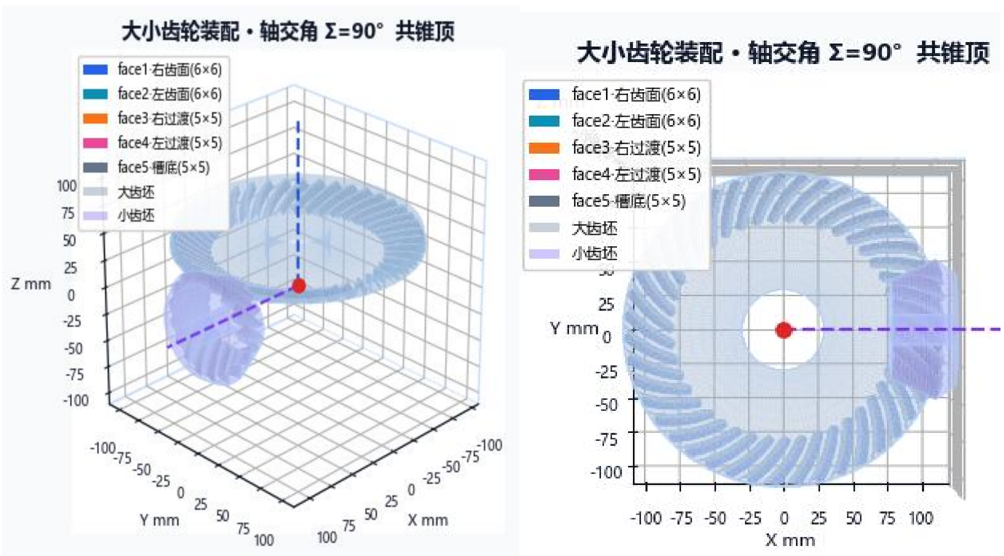
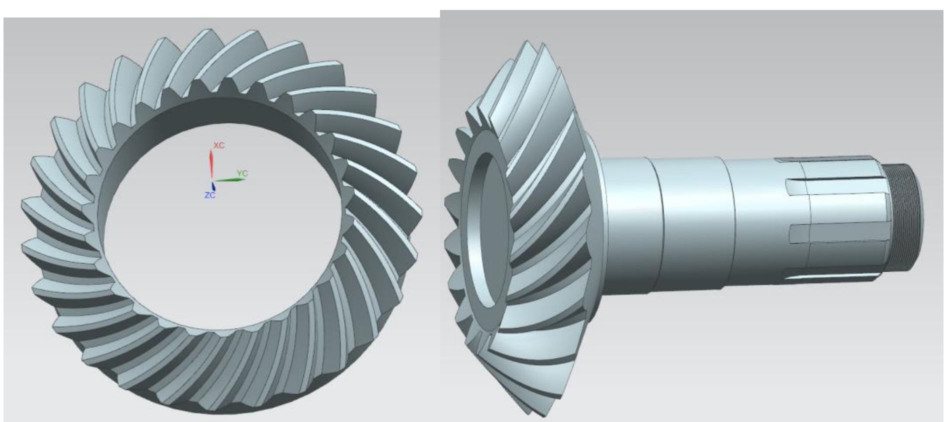


Fig. 8-1 Spiral bevel gear model generated according to the theory of this paper



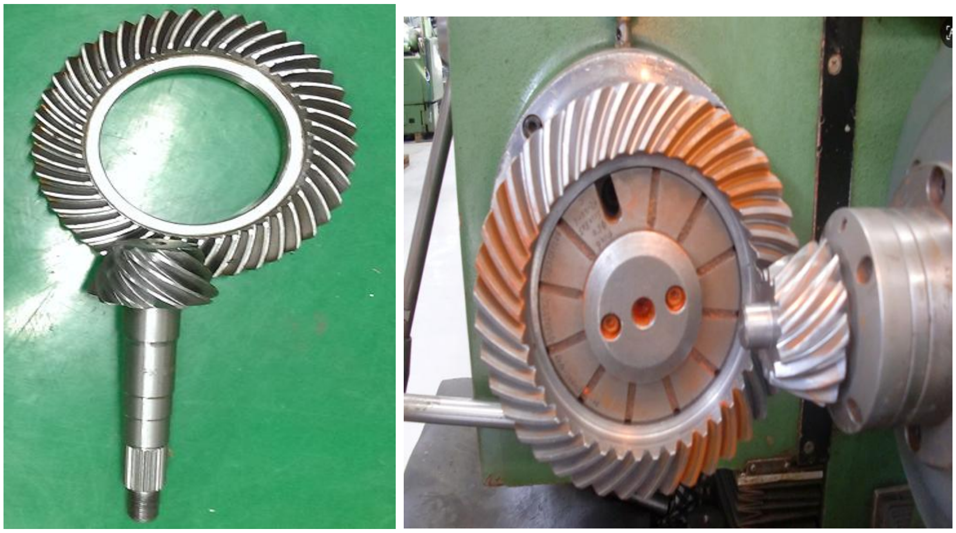


Fig. 8-6 Photograph of the machined parts (Caption: photograph of the finished spiral bevel gear pair, showing the tooth surface topography and the meshing state.)

The machining results show that the tooth-surface modeling and CNC machining method based on closed-form analytical equations can meet the accuracy requirements of precision-grade gears without any need for dedicated gear cutting machines or dedicated cutting tools, thus providing a feasible technical route for the manufacture of gears in small batches, in a wide variety of types, and with novel tooth forms.

9 Conclusions and Outlook

Taking differential geometry as its language and starting from the most elementary geometric concepts, this paper has systematically constructed the theoretical foundation and the model representation system of spiral bevel gears. The main conclusions can be summarized as follows:

1. **Geometric foundation: the geometric foundation of the spiral bevel gear is the cone surface and the spherical involute. The base cone theorem converts conjugate transmission into the pure rolling of a plane along a cone surface, providing an axiomatic starting point for the entire theory.**

2. **Parametrization principle: the mathematical essence of the tooth surface is a smooth immersion from a two-dimensional parameter manifold into three-**

dimensional Euclidean space. The parameters (u,R) are coordinates on the manifold, and the tooth-surface equation $\mathbf{r}(u,R)$ is the embedding map. Strictly distinguishing the parametric level from the geometric level is the key to avoiding conceptual confusion.

3. **Normal theory: the normal direction angle functions are not tooth-surface coordinates but an angular-coordinate parametrization of the Gauss map defined on the parameter manifold. Through the differential of the Gauss map they directly determine the second fundamental form and the principal curvatures, and hence directly determine the contact characteristics of the tooth surface. This is the fundamental reason why the direction-angle-function method becomes a natural tool for active tooth-surface design.**

4. **Integrability constraint: an arbitrarily prescribed set of normal direction angle functions does not necessarily generate a real tooth surface. The normal field must satisfy the Frobenius integrability condition ($\omega \wedge d\omega = 0$, equivalently $\mathbf{n} \cdot (\nabla \times \mathbf{n}) = 0$) in order to integrate to a smooth surface. The integrability condition delimits the boundary of the design freedom available for active tooth-surface design.**

5. **Machining realization: tooth-surface modeling based on closed-form equations can proceed directly into the engineering implementation stage; from parameter discretization, five-patch stitching, cutter location computation, tool path planning, and post-processing to on-machine inspection and error compensation, it forms a complete closed loop of theory - modeling - machining - inspection - compensation. The analytical normal vectors and curvature distributions provide a direct mathematical basis for optimizing machining accuracy and efficiency. Finally, this paper presents gear models and examples of CNC machining.**

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