

ARTICLE TYPE

An Inverse Predictor-Corrector Data Assimilation Method for One-Dimensional Discharge and Bathymetry Estimation from Surface Flow Measurements

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Abstract

This study presents the Predictor-Corrector Data Assimilation (IPCDA) method, a numerical algorithm for estimating discharge and bed topography in one-dimensional steady subcritical open-channel flows when boundary conditions and bathymetry are unavailable. The governing equation is derived from the one-dimensional Saint-Venant equation under steady-state conditions. The Working Grid (WG) is constructed from a subset of observational nodes selected according to a minimum water surface elevation drop criterion grounded in the diffusion-wave approximation and motivated by practical instrument resolution constraints. To address the inherent underdetermination arising from incomplete information, the algorithm proceeds in two sequential phases: in the Predictor-phase, a local inverse problem is solved independently within each cell of the WG, providing initial discharge estimates through closed-form algebraic expressions derived from the discretized energy balance equation; in the Corrector-phase, the global discharge estimate is refined through the iterative solution of a quadratic programming problem obtained via a derivative-free linearization of the nonlinear friction slope term. The methodology was assessed using two synthetic test cases generated with HEC-RAS. The first case corresponds to a 2.6 [km] trapezoidal channel with variable bed slope and uniform roughness coefficient, considering reference discharges of 1, 10, and 100 [m³/s]. The second case consists of a 7.5 [km] reach of the Biobío river in Chile under a flood discharge condition of 2000 [m³/s] and variable Manning's n . In both experiments, the estimated discharges generally exhibited relative errors below 5%, specifically when the tolerance for the creation of the WG uses the higher resolutions. In the trapezoidal channel, bed elevation profiles were successfully reconstructed with normalized errors below 2.3% for low-to-moderate flow conditions, increasing at high discharge where a relatively small-parameter hypothesis of the geometry assumption is less satisfied. For the Biobío river case, the flow area was recovered within $\pm 6\%$ at a representative cross section, despite the use of a rectangular approximation for a highly irregular natural geometry. A sensitivity analysis combining variations in the objective function weights and the Manning roughness coefficient n provides a practical framework for constructing a plausible discharge interval from IPCDA runs considering independent segments of the same river: by pooling the estimates across parameters configurations and assumptions, the range of estimated discharges serves as a data-driven uncertainty band. The width of this band is controlled primarily by the uncertainty in n . For the two cases tested, the median estimate remains within 4% of the reference discharge across all configurations, suggesting that the interval is approximately centered on the true discharge even under significant parametric uncertainty.

KEYWORDS

Open-channel flow, Surface measurements, Discharge estimation, Bathymetry reconstruction, Inverse problems, Data assimilation, Quadratic programming

1 | INTRODUCTION

Accurate discharge estimation in open streams is a critical challenge in hydraulic engineering. As a fundamental aspect of understanding the hydrological cycle, reliable discharge estimation plays a key role in water resource management, flood risk mitigation, the study of ecosystem dynamics, among other applications. While the implementation of streamflow gauging stations is commonly used to obtain a continuous record of discharge by a calibrated rating curve, its application in fluvial systems with high morphological dynamism is considerably hindered by the inherent technical complexity and the significant economic implications associated with their installation and operation. In-situ measurements technics, as the area-velocity method, which divides a cross-section into segments to measure velocity and area, although very popular, is time- consuming and potentially dangerous.

Indirect approaches based on data captured from remote observations have been widely studied as alternatives to address this issue. In this context, analyzing or estimating hydraulic variables by comparing hydraulic models outputs with observational data holds significant practical and academic value. Data assimilation techniques and the theory of inverse problems provide suitable frameworks to address discharge estimation. Using these frameworks, the formulation of a model to solve the problem may vary considerably, leading to different methods. Among the tools employed for addressing such problems are the Maximum Entropy Principle^{1,2,3,4,5}, power law-based models^{6,7,8,9} and direct-numerical methods^{10,11,12,13}.

Another line of research is the Optimal Control formulation, used to link the concept of Data Assimilation with the theory of Inverse Problems. This approach has been successfully used in understanding geophysical and hydraulic engineering phenomena, for example Variational Data Assimilation approach (VDA)^{14,15,16,17,18,19,20}, Data Assimilation with Ensemble Kalman-filter^{21,22,23,24,25}, Principal Component Geostatistical approach (PCGA)²⁶, with Deep Learning techniques^{27,28,29}, among many others. Despite the varying levels of complexity in the problems addressed in literature, a frequent requirement is the availability of prior information, a seed or a reference value to initialize simulations in direct or iterative processes. Keeping in mind the use of stepped models to address highly complex problems, a model with an apparently simple structure can provide valuable insights from different perspectives, yet presenting various challenges for standard numerical treatment.

Some of the aforementioned studies on steady state conditions and 1D problems adopt this approach, since the solutions provided by certain simplified models contain valuable information for unsteady or two-dimensional problems. Additionally, these models facilitate the characterization of steady solutions for evaluating the stability of space-time schemes and for examining the parametric dependence of system responses, among many other applications^{30,19,20}. Although the problems addressed in the literature exhibit different levels of complexity, a common requirement is the availability of prior information, such as an initial guess or a reference solution, to initialize simulations in direct or iterative procedures. This situation arises in previous studies involving 1D-2D computational simulations for the analysis of environmental and hydraulic problems, including inflow and outflow conditions in river networks, floodplain modeling, among others (see Ref. ^{31,32,33}). Furthermore, obtaining a reference solution is often necessary to initialize iterative solution strategies. Thus, obtaining results for an underdetermined one-dimensional steady problem, whose formulation appears simple but possesses a complex structure, may have implications from different perspectives and, therefore, attract the interest of a variety of approaches.

In this direction, the present work is primarily focused on the numerical treatment of the discharge estimation and bathymetry reconstruction problem in an underdetermined setting, complemented with information obtained from surface observations of velocity, water surface elevation, and channel width. To this end, we propose a new two-phase data assimilation algorithm, referred to as the Inverse Predictor-Corrector Data Assimilation (IPCDA) method. The main idea is to combine different models with varying levels of complexity in order to progressively complement and guide the approximations, that is, through an ensemble of low-complexity models.

Following ideas similar to those presented in previous studies^{30,34}, the Predictor-phase aims to generate local point-wise estimates that provide preliminary approximations of the discharge by solving auxiliary algebraic equations in terms of the discharge, while the remaining terms are approximated using observational data. The resulting discharge estimate obtained in this stage is subsequently employed in the Corrector-phase to complete the inverse problem through an iterative optimization procedure that updates the approximation of the friction slope model. The Corrector-phase does not primarily aim to improve numerical accuracy directly; rather, by enforcing the energy balance equation at a global level, it restores mass conservation. In this stage, observational data are incorporated into the formulation of an optimal control problem which, in its discrete version and after suitable manipulations, leads to a quadratic programming problem.

The remainder of this article is organized as follows. Section 2 presents the mathematical model and the physical considerations used to analyze the hydraulic problem. In this section, the reduced model associated with Manning's equation is developed within the underdetermined framework, and the criterion used for constructing the computational grid is introduced. The local and global models employed in each phase of the method are also presented. Section 3 describes the IPCDA methodology, detailing the expressions used in the computations. The numerical experiments are presented in Section 4, where two case studies are considered to evaluate the proposed methodology: a trapezoidal channel and a natural river reach. The first case represents a man-made channel, commonly found in agricultural or hydropower infrastructure, with known side slope and Manning's roughness coefficient. The second considers a natural river reach characterized by irregular cross-sectional geometry, spatially varying Manning's roughness coefficient, and non-uniform bed topography that cannot be adequately approximated by a trapezoidal section. In both cases, one-dimensional models were developed in HEC-RAS using known bathymetries and roughness coefficients. The results obtained from these simulations were subsequently extracted and used as input data for the IPCDA methodology. Section 5 presents comments, discussion, and potential directions for future work. Some technical aspects of the results are emphasized, highlighting the complexities of the model and presenting a proposal for generating recharge ranges based on the combination of results obtained for different values of the numerical parameters. Section 6 presents the overall conclusions of the work together with remarks related to future research directions. Finally, Appendix Section A presents the computational algorithms developed in this work.

2 | MATHEMATICAL MODEL FORMULATION

This section introduces the governing equations, the representation of the friction slope in terms of the flow variables and the computational grid used for the optimization problem. The formulation assumes steady state, incompressible flow with constant density. A subcritical regime is considered, where the hydrostatic approximation is valid and streamwise curvature effects are negligible, enabling a one dimensional description^{35,36}.

The method is designed to address cases with unknown boundary conditions and bed topography. The problem is first posed using the observable variables above the water surface, followed by a reformulation of the friction slope and then framed within a discrete optimal control framework.

2.1 | Governing equations

The Saint-Venant^{37,38,39} equations for continuity and momentum for a 1D flow in an open channel are defined by:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0 \quad (1)$$

$$\frac{1}{A} \frac{\partial Q}{\partial t} + \frac{1}{A} \frac{\partial}{\partial x} \left(\frac{Q^2}{A} \right) + g \frac{\partial h}{\partial x} - g (S_0 - (S_f + S_e)) = 0 \quad (2)$$

where Q is the discharge, A the cross-sectional area, g the gravitational acceleration, S_0 the bed slope and S_f and S_e denote the energy losses due to bed friction and contraction/expansion effects, respectively^{35,36}. In our study use the stationary regime ($\partial Q/\partial t = 0$ and $\partial A/\partial t = 0$) and we adopt the nonconservation form of the momentum equation, leading to:

$$\frac{U}{g} \frac{dU}{dx} + \frac{dh}{dx} = S_0 - (S_f + S_e) \quad (3)$$

where $U(x) = Q/A(x)$ is the cross sectional averaged velocity and $h(x)$ the water depth. We can write $S_0 = -\frac{dZ}{dx}$, with Z as the elevation of the channel bottom above a reference datum. Under this procedure, the method focuses on the integration procedure of the two energy loss terms in the right-hand side. In the present work, we consider the following representation for the rate of energy loss terms:

$$S_f = n^2 U^{10/3} \left(\frac{P_w}{Q} \right)^{4/3}, \quad S_e = \frac{K_e}{2g} \frac{dU^2}{dx}, \quad (4)$$

with n the Manning coefficient, P_w the wetted perimeter, and K_e a dimensionless contraction/expansion coefficient where $K_e > 0$ for contraction, $K_e < 0$ for expansion³⁷.

In a river segment $[x_{in}, x_f]$ in a subcritical regime, the boundary conditions can be written as:

$$U(x_{in})A(x_{in}) = Q, \quad h(x_f) = h_f, \quad (5)$$

where Q remains constant along the reach. In the present setting where we observe just some variables of the problems from above the water surface, none of $h(x_f)$, Q , or $Z(x)$ are known, which leads to a mathematical ill-posed problem. Consequently, the formulation requires coupling multiple approaches to derive and establish consistent relationships among the flow variables.

To partially address this issue, we reinterpret the bottom elevation Z and the water depth h using available remote sensing information. As illustrated in Figure 1(a) and following the work of Gessese et. al¹¹, both quantities can be combined into a single variable, the water surface elevation (WSE), defined as:

$$WSE = Z + h, \quad (6)$$

which, together with the depth-averaged velocity U , forms the set of observable state variables. In practice, WSE and U can be retrieved from remote sensing data (e.g., UAV imagery) and subsequently used as inputs for the inversion problem.

Using this new term and the representation for the energy losses in Eq. (4), we rewrite the Momentum Eq. in (3) as:

$$\frac{d}{dx} \left(\frac{U^2}{2g} + WSE \right) = -S_f - S_e \quad (7)$$

Figure 1 provides a schematic representation of these variables from three perspectives. In Panel (a), the observable quantities are the distance to the water surface, d , and the surface velocity, U^{sup} . At any location, WSE can be computed as the difference between a known reference datum above the water surface Z'_{dat} , where the remote vehicle operates and d . The depth-averaged velocity U can be estimated from U^{sup} using methods as the power law or the Maximum Entropy (see for example other studies^{40,2,4,5}). Panel (b) shows that these measurements can be collected at multiple points along a river reach in order to preserve the continuity of the equations. Panel (c) also shows an additional observable variable, the top width B , which can be related to the water depth h and the cross-sectional area A through geometric relationships, assuming a symmetric trapezoidal shape. These relationships are discussed in greater detail in the next subsection.

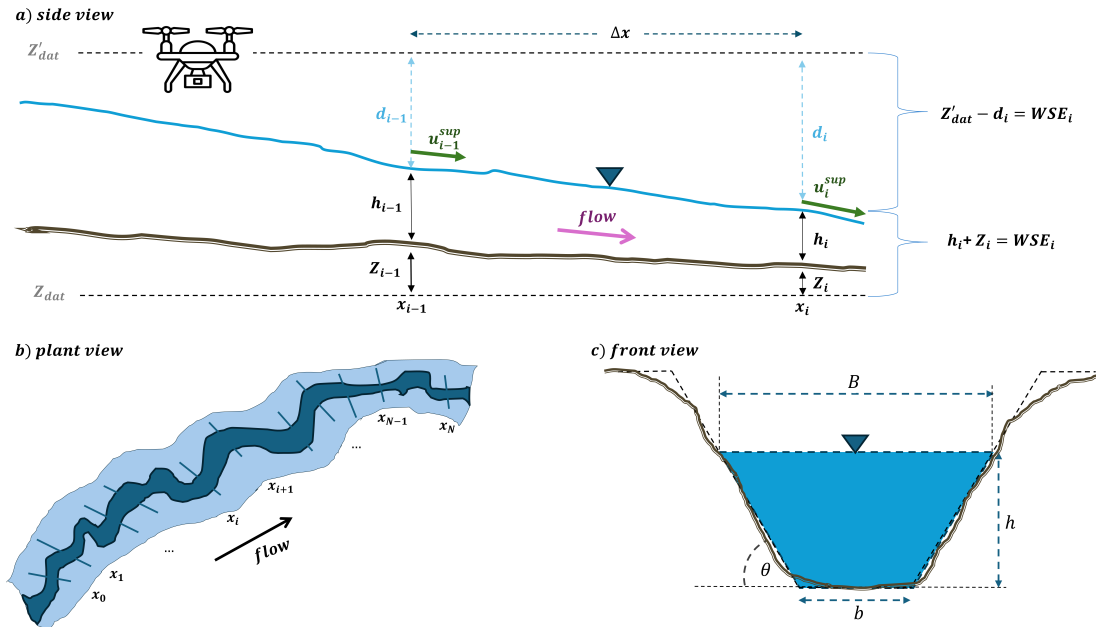


FIGURE 1 Schematic of open-channel flow and remotely observable variables: surface velocity and vertical distance from a reference height (adapted from Larnier (2021)³⁴).

In the subcritical regime considered here, the dominant balance in the momentum conservation equation arises between the water surface elevation gradient and the friction losses, leading to a balance of the form

$$\frac{dWSE}{dx} \sim -S_f, \quad (8)$$

also known as the diffusion-wave approximation (see Refs.^{35,37}). In the proposed approximations and numerical solution strategies, the friction term S_f is repeatedly involved; therefore, its accurate representation plays a fundamental role. The next subsection is devoted to constructing a suitable expression for this term. In particular, an explicit formulation of the ratio P_w/Q appearing in S_f is required, expressed in terms of $U(x)$, $WSE(x)$, $B(x)$, and Q . This step is essential since Q is the primary unknown of the inverse problem and directly governs the hydraulic relationships in the modified momentum equation (7).

Finally, the data obtained from observations are generally discrete in nature, with measurements of U^{obs} and WSE^{obs} available at specific locations. Therefore, Eq. (7) must be discretized over a computational grid. Specifically, assuming surface measurements at the nodes x_{i-1} and x_i , integration over the subinterval $\mathcal{I}_i := [x_{i-1}, x_i]$ yields

$$\frac{1}{2g} (U_i^2 - U_{i-1}^2) + (WSE_i - WSE_{i-1}) = -I[S_f]_i - I[S_e]_i, \quad i = 1, \dots, N, \quad (9)$$

where $I[S_f]_i$ and $I[S_e]_i$ denote the integrals of S_f and S_e over $\mathcal{I}_i$, respectively.

Subsequently, the discretization is developed by approximating the integral contributions through numerical quadrature, that we will rename as $I_{\text{num}}[S_f]_i$ and $I_{\text{num}}[S_e]_i$. The criterion is based on the structure of the left-hand side of (9), expressed in terms of flow variables that can be estimated from surface measurements. In this way, the formulation of the discrete models is constructed through manipulations of Eq. (7) together with the numerical treatment of the integral terms. To this end, the next steps are:

- (i) obtaining an explicit expression for Q through the treatment of S_f ,
- (ii) proposing a suitable computational grid design criterion, and
- (iii) simplifying the contributions of the term $I[S_e]_i$.

These issues are discussed in the following subsections.

2.2 | Reformulation of the friction slope based on the problem variables

The idea is to obtain an expression for S_f in terms of U , WSE , B , and Q , with the objective of using surface measurements to estimate the discharge Q and, consequently, determine the bathymetry $Z(x)$. The result presented here applies to channels where the aspect ratio $h(x)/B(x)$ is not sufficiently small. This point is addressed later in the section on numerical experiments.

We assume a trapezoidal cross-section with known lateral slope θ (see Fig. 1 (c)). Given the channel top width $B(x)$ as an available variable, the cross-sectional area and wetted perimeter are given by:

$$A = B h - \cot(\theta) h^2 \quad (10)$$

$$P_w = B + \frac{2}{\sin \theta} (1 - \cos(\theta)) h \quad (11)$$

We need an expression that can handle the unknown h , expressed in Equations (10) and (11). We can use the continuity equation $A = Q/U$ into Eq. (10) to generate a quadratic equation on h . Solving on the water depth, we obtain:

$$h = \frac{\sin(\theta)}{2 \cos(\theta)} B \left(1 - \sqrt{1 - \frac{4Q}{(B)^2 U} \left(\frac{\cos(\theta)}{\sin(\theta)} \right)} \right) \quad (12)$$

The approximation is derived using the Taylor expansion $\sqrt{1 - \eta} = \sum_{j=0}^{\infty} (-1)^j \binom{1/2}{j} \eta^j$, with:

$$\eta = \frac{4Q}{B^2 U} \left(\frac{\cos(\theta)}{\sin(\theta)} \right), \quad (13)$$

Thus, replacing into Eq. (12) we get:

$$h = -\frac{\sin(\theta)}{2\cos(\theta)}B \left(\sum_{j=1}^{\infty} (-1)^j \binom{1/2}{j} \left(\frac{4Q}{(B)^2 U} \left(\frac{\cos(\theta)}{\sin(\theta)} \right) \right)^j \right) \quad (14)$$

with $\binom{1/2}{j} = \binom{2j}{j} \frac{(-1)^{j+1}}{(2j-1)2^{2j}}$. Under the assumption $0 < \eta \ll 1$, from the expansion (14) follows:

$$h = \frac{Q}{UB} + \left(\frac{\cos(\theta)}{\sin(\theta)} \right) \frac{Q^2}{U^2(B)^3} + \mathcal{O}(\eta^3) \quad (15)$$

We propose the description of the water-depth from Eq. (15), considering the approximation:

$$h = \frac{Q}{UB} + \left(\frac{\cos(\theta)}{\sin(\theta)} \right) \frac{Q^2}{U^2(B)^3} \quad (16)$$

Thus, using Eq. (16) into Eq. (11) we obtain:

$$\left(\frac{P_w}{Q} \right) = \frac{B}{Q} + \frac{2}{\sin(\theta)}(1 - \cos(\theta)) \left(\frac{1}{UB} + \left(\frac{\cos(\theta)}{\sin(\theta)} \right) \frac{Q}{U^2(B)^3} \right) \quad (17)$$

Replacing into S_f in Eq. (4) we get the following approximate expression:

$$S_f = n^2 U^{10/3} \left(\frac{B}{Q} + \frac{2}{\sin(\theta)}(1 - \cos(\theta)) \left(\frac{1}{UB} + \left(\frac{\cos(\theta)}{\sin(\theta)} \right) \frac{Q}{U^2(B)^3} \right) \right)^{4/3} \quad (18)$$

i.e. $S_f = S_f(U, Q, B, n, \theta)$, then obtaining the desired dependence on U and Q , which are variables of the our mathematical models.

2.3 | Grid design criteria from observational data

Given that the measurements are generally available at discrete positions, instead of solving the continuous expression exactly, the scheme uses a quadrature rule to replace the integral by a sum. Thus, the criteria for the selection of the individual nodes for this quadrature play a crucial role in the problem formulation. The discretization takes into account the discrete positions of surface velocity, water surface elevation and the river width observations. We refer to this collection of points as the Observation Grid (OG), denoted by $x_0^{\text{obs}} < \dots < x_{N_{\text{obs}}}^{\text{obs}}$.

In our method, a subset of nodes is selected from the OG, which are referred to as the working nodes, and the corresponding grid is termed the Working Grid (WG). The selection of the WG is based on the technical limitations of the instruments or the measurements technics that are subject to multiple error sources. For *WSE*, radar altimetry achieves accuracies of 1–3 [cm] under favorable conditions^{41,42}, but wind-driven surface disturbances and rainfall can degrade these estimates^{43,42}. For surface velocity, LSPIV-based methods report mean absolute errors of 0.05–0.1 [m/s] at low flow and optimal flight altitude, increasing to 0.3–1.0 [m/s] at high flow or excessive flight height due to image resolution loss^{44,45}. Additional velocity errors arise at low flow velocities from disturbances by turbulent airflow generated downward by the drone's propellers, known as "propwash"⁴³, and from wind induced ripples that contaminate image based tracking⁴⁶. River width estimation from imagery carries pixel to meter conversion errors on the order of 3%⁴¹, which for channels of 30–50 [m] width translates to uncertainties of approximately 1 [m] near the banks. In practice all these issues can affect the measurements accuracy and must be considered in the WG selection process.

For two consecutive nodes to be considered in the WG, we adopted the condition that a minimum difference in *WSE* between them must be satisfied. This is related to the diffusion-wave model given in Eq. (8), where the frictional energy loss is assumed to be exactly balanced by gravity, see³⁷ for further details.

If we denote the tolerance ΔWSE^{tol} , then two nodes of the OG, x_n^{obs} and x_m^{obs} , will be selected for the WG as long as they satisfy the condition:

$$WSE(x_m^{\text{obs}}) - WSE(x_n^{\text{obs}}) \geq \Delta WSE^{\text{tol}} \quad (19)$$

with $m < n$, which is consistent with the monotonicity condition of *WSE* assumed in this work.

The WG construction starts with the most downstream node ($n = N_{\text{obs}}$). Then, the first upstream node m satisfying Eq. (19) is selected. This procedure is repeated iteratively, yielding the working nodes $x_1^{\text{wse}} < x_1^{\text{wse}} < \dots < x_N^{\text{wse}}$, with WSE_i denoting the water surface elevation at x_i^{wse} . Thus, denoting by WSE_i the value of WSE at x_i^{wse} , we have:

$$WSE_{i-1} - WSE_i \geq \Delta WSE^{\text{tol}}, \quad i = 1, \dots, N \quad (20)$$

The sketch of the WG selection procedure is presented in the Figure 2, where the work cell $[x_{i-1}, x_i]$ contain the observational nodes $x_{i_0}^{\text{obs}} < \dots < x_{i_6}^{\text{obs}}$.

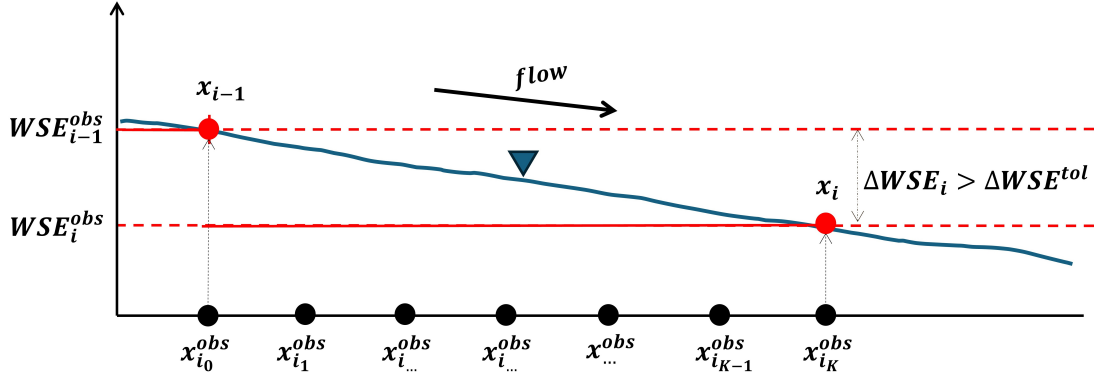


FIGURE 2 Sketch of the WG construction under the tolerance constraint criteria in Eq. (20). 1D Grid-Representation for the comparison between the Observational and the work Nodes with $x_{i-1}^{\text{wse}} = x_{i_0}^{\text{obs}}$.

Through the WG, the notation of the discretization for the variables is induced considering WSE_i , U_i and B_i to denote the values of $WSE(x)$, $U(x)$ and $B(x)$ at $x = x_i^{\text{wse}}$. For simplicity, in what follows we consider only the notation $x_0 < x_1 < \dots < x_{N-1}$ for the WG. It is important to note that is not necessarily that the OG to be uniform to be used in the WG creation.

2.4 | Simplification of the eddy loss contribution S_e

In general, and in accordance with the dominant balance expressed in Eq. (8), the contribution of $I_{\text{num}}[S_e]_i$ is not dominant in the total energy loss along the river or, in this case, within the i -th cell of the WG. Based on this observation, the first simplification criterion focuses on $I[S_e]_i$ by considering the approximation

$$I_{\text{num}}^{\text{obs}}[S_e]_i := \frac{1}{2g} \sum_{k=1}^{N_i^{\text{obs}}} K_e(\Delta U_{i_k}^{\text{obs}}) \left((U_{i_k}^{\text{obs}})^2 - (U_{i_{k-1}}^{\text{obs}})^2 \right) \quad (21)$$

In Eq. (21), N_i^{obs} represents the number of observational nodes included in the cell defined by $\mathcal{I}_i = [x_{i-1}, x_i]$, with $i(0) = i - 1$ and $i(N_i^{\text{obs}}) = i$, $U_{i_k}^{\text{obs}}$ denotes the value for the average cross section velocity observed and $K_e(\Delta U_{i_k}^{\text{obs}})$ the contraction/expansion coefficient, that depends on the differences of average velocities $U_{i_k}^{\text{obs}}$ and $U_{i_{k-1}}^{\text{obs}}$. Import to note that the Observational Grid (OG) is considered as an auxiliary grid, that contains the information to calculate the eddy loss contribution (see also Figure 2). We emphasize that this criterion will also be used in composite numerical quadrature, presented in the next sections. Now, onwards the variable $I_{\text{num}}^{\text{obs}}[S_e]_i$ is considered as a known value, calculated from the discrete observations of the OG.

2.5 | Mathematical models for IPCDA

The underdetermination associated with the bathymetry $Z(x)$ and the discharge Q leads to a lack of information both in the governing differential equation and in the associated boundary conditions. We address this issue by constructing simplified models in which the available surface information is incorporated. In this subsection, we describe the two approaches employed to derive the mathematical models used at each stage of the proposed algorithm.

2.5.1 | Local approach through algebraic models: first stage

These models are derived from the integration of the momentum balance equation (7). Accordingly, considering a partition of the river reach into subintervals $[x_{i-1}, x_i]$, the local models are obtained through the numerical treatment of the integral equation (9), which, by means of (18), takes the following form:

$$\frac{1}{2g}(U_i^2 - U_{i-1}^2) + (WSE_i - WSE_{i-1}) = -n^2 \int_{x_{i-1}}^{x_i} U^{10/3} \left(\frac{B}{Q} + \frac{2(1 - \cos(\theta))}{\sin(\theta)} \left(\frac{1}{UB} + \frac{\cos(\theta)}{\sin(\theta)} \frac{Q}{U^2 B^3} \right) \right)^{4/3} dx - \int_{x_{i-1}}^{x_i} S_e dx \quad (22)$$

Thus, for each subinterval of the considered partition, an equation is formulated to determine Q . In this way, local estimates of the discharge are obtained, from which a simple average is subsequently computed to generate a global estimate.

The main idea is to derive approximations that simplify both the resulting expressions and their dependence on the problem variables. Considering the approximation in Eq. (21), the equations are constructed by applying numerical quadrature to the right-hand side of (22), yielding

$$\frac{1}{2g}(U_i^2 - U_{i-1}^2) + (WSE_i - WSE_{i-1}) = -I_{\text{num}}[S_f]_i - I_{\text{num}}^{\text{obs}}[S_e]_i, \quad i = 1, \dots, N. \quad (23)$$

Here, the structure of the model is determined by the particular form adopted for $I_{\text{num}}[S_f]_i$, while the term $I_{\text{num}}^{\text{obs}}[S_e]_i$ is approximated according to Eq. (21) in order to avoid the explicit handling of the strong nonlinearity associated with this contribution. For simplicity in the overall numerical approach, Newton–Cotes quadrature is adopted.

Similar to Gessese et al.¹¹, we consider a direct numerical approach, although the calculations are developed in each working cell $\mathcal{I}_i = [x_{i-1}, x_i]$. This local approach is based on the inverse problem treatment, where the variables U_i and WSE_i are assumed to be known from the measurements, except the discharge.

Under this consideration, the expression $I_{\text{num}}[S_f]_i$ depends on Q and on the values of U and B at the nodes where the quadrature is evaluated. On the other hand, in order to formulate an equation for Q , we introduce the notation $I_{\text{num}}^{\text{direct}}[S_f]_i$ to denote the following direct evaluation:

$$I_{\text{num}}^{\text{direct}}[S_f]_i := -\frac{1}{2g}(W_i^{\text{obs}} - W_{i-1}^{\text{obs}}) - (WSE_i^{\text{obs}} - WSE_{i-1}^{\text{obs}}) - I_{\text{num}}^{\text{obs}}[S_e]_i \quad (24)$$

with $W_i = U_i^2$ and $I_{\text{num}}^{\text{obs}}[S_e]_i$ the fixed approximation presented in Eq. (21). Thus, in Eq. (24), a numerical value is provided for the approximation of the frictional contribution related to $I_{\text{num}}[S_f]_i$, while in $f_i(Q)$, an approximation has been introduced to derive an expression that depends solely on Q .

From Eq. (23), the first inverse approach is developed through the following data assimilation problem to solve Q in each $[x_{i-1}, x_i]$:

$$I_{\text{num}}[S_f]_i = I_{\text{num}}^{\text{direct}}[S_f]_i, \quad \text{for } i = 1, \dots, N - 1 \quad (25)$$

Thus, from this approach, for each $\mathcal{I}_i$ there is associated an discharge approximation namely Q_i^{inv} . The superscript *inv* is used to remark that the approximation is obtain under an inverse formulation approach.

On the other hand, the structure of the left-hand side of Eq. (25) is determined by the quadrature method used in the calculation. In this work, we consider the forward, backward, and trapezoidal rules, along with their corresponding composite rules.

The derivation of the expression for the left-hand side in Eq. (25) is discussed in the next section, where an additional expression is also presented. This expression can be considered as an algebraic data assimilation model for the discharge.

2.5.2 | Global approach based on an optimal control formulation: second stage

We quantify the discrepancy between the model response and the observations within the optimal control framework. Specifically, we define a weighted least-squares cost functional⁴⁷ in the minimization of the total discrepancy as:

$$\mathcal{F} = \gamma_1 \mathcal{F}_{vel} + \gamma_2 \mathcal{F}_{elev}, \quad \gamma_1 + \gamma_2 = 1 \quad (\text{with } \gamma_i > 0), \quad (26)$$

where

$$\mathcal{F}_{vel} = \int_{x_{in}}^{x_f} \left(\frac{1}{2g} U^2 - \frac{1}{2g} (U^{obs})^2 \right)^2 dx, \quad \mathcal{F}_{elev} = \int_{x_{in}}^{x_f} (WSE - WSE^{obs})^2 dx. \quad (27)$$

where the superscript obs denotes values obtained from measurements.

The objective function $\mathcal{F} = \gamma_1 \mathcal{F}_{vel} + \gamma_2 \mathcal{F}_{elev}$ (see Eq. (27)) was formulated in longitudinal units, consistently with the momentum balance equation. The coefficients in $\mathcal{F}$ may be used to emphasize either of the two variables, depending on the confidence assigned to the observations. In this same direction, the factor $1/2g$ in $\mathcal{F}_{vel}$ scales the velocity contribution to kinetic-energy units, consistently with the definition of WSE , so that both γ coefficients have the same physical units. Regarding the role of the parameters γ_i , they were introduced to analyze how each constraint contributes to the objective function. In this sense, they may be interpreted as a mechanism for combining a multi-objective problem into a single objective function. This point is further discussed in the numerical results section, see Section 4.2.

As a first approach, the problem of comparing the response of a physical-mathematical model with observations can be formulated within the framework of an optimal control problem as follows:

$$\begin{cases} \min_{U, WSE} \gamma_1 \mathcal{F}_{vel} + \gamma_2 \mathcal{F}_{elev}, \\ \text{subject to:} \\ \frac{d}{dx} \left(\frac{1}{2g} U^2 + WSE \right) = -S_f - S_e, \quad x_{in} < x < x_f. \end{cases} \quad (28)$$

The first remark concerning problem (28) is related to the manner in which the observational information is incorporated. As previously indicated, the measurements are available only at specific points, and therefore the representation given in Eq. (27) cannot be directly employed. In this direction, the need for a discrete formulation of problem (28) arises naturally.

In this regard, the problem constraints are formulated by integrating the momentum equation over each subinterval of the partition $[x_{i-1}, x_i]$, following the local approach described in Subsubsection 2.5.1. However, constructing the scheme from an integrated formulation of the form (23) is not convenient in the optimization context. In particular, the structure adopted by the constraints in the discrete problem significantly complicates the analysis. This constitutes the second remark concerning the formulation and motivates the main proposal of this work, namely, a linearization performed directly at the discrete integral level, thereby avoiding the functional-analytic framework commonly employed in variational approaches. We address this issue by returning to the momentum balance equation Eq. (7). In the direction of highlighting the advantage in handling the equation in a simplified manner, we introduce the function:

$$G_f(U, Q) = n^2 \left(B U + \frac{2(1 - \cos(\theta))}{\sin(\theta)} \left(\frac{Q}{B} + \frac{\cos(\theta)}{\sin(\theta)} \frac{Q^2}{U(B)^3} \right) \right)^{4/3} \quad (29)$$

through which Eq. (18) can be rewritten as:

$$S_f = \frac{1}{Q^{4/3}} G_f(U, Q) U^2, \quad (30)$$

Replacing (30) into (7) and multiplying by $Q^{4/3}$, we get an expression in terms of Q , U and WSE :

$$Q^{4/3} \frac{d}{dx} \left(\frac{U^2}{2g} + WSE \right) + \frac{Q^{4/3}}{2g} K_e(U) \frac{dU^2}{dx} = G_f(U, Q) U^2 \quad (31)$$

At first sight, this formulation does not appear to provide a significant advantage for the numerical treatment. However, it will be shown later that the manner in which the variables are coupled in this expression allows the dependencies to be decoupled, thereby leading to a derivative-free linearization procedure. This constitutes one of the main contributions of the present work.

With these considerations, we proceed to the formulation and discretization of the problem. To compare the variables in the optimization problem with the observations, we first consider the following quadrature-based approximation to obtain a discrete

objective function: $\int_a^b g(z)dz \approx \sum_{i=0}^{N-1} g(x_i)\Delta x_i$ with $\sum_{i=0}^{N-1} \Delta x_i = b - a$. Assuming a uniform partition, $\Delta x_i \approx \frac{b-a}{N-1}$, the discrete version of the cost function in Eq. (26) takes the form:

$$\mathcal{F}_{\text{num}} = \sum_{i=0}^{N-1} \left[\tilde{\gamma}_1 \left((U_i)^2 - (U_i^{\text{obs}})^2 \right)^2 + \tilde{\gamma}_2 \left(WSE_i - WSE_i^{\text{obs}} \right)^2 \right] \quad (32)$$

with $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ given by:

$$\tilde{\gamma}_1 = \frac{\gamma_1}{(2g)^2} \frac{(x_f - x_{in})}{(N-1)}, \quad \tilde{\gamma}_2 = \gamma_2 \frac{(x_f - x_{in})}{(N-1)} \quad (33)$$

Thus, the resulting optimization problem reads:

$$\begin{cases} \min & \sum_{i=0}^{N-1} \left[\tilde{\gamma}_1 \left((U_i^2) - (U_i^{\text{obs}})^2 \right)^2 + \tilde{\gamma}_2 \left(WSE_i - WSE_i^{\text{obs}} \right)^2 \right] \\ \text{subject to:} & \\ & Q^{4/3} \left(\frac{U_i^2}{2g} - \frac{U_{i-1}^2}{2g} + WSE_i - WSE_{i-1} \right) + Q^{4/3} I_{\text{num}}[S_e]_i = I_{\text{num}}[G_f(U, Q) U^2]_i \quad i = 1, \dots, N. \end{cases} \quad (34)$$

The finite dimensional optimization problem of Eq. (34) is treated in term of U_i^2 , WSE_i and Q , with quadratic nonlinearity in the cost function, while the nonlinear contribution in the equality constraints are given from the quadrature formula of the friction term $I_{\text{num}}[G_f(U, Q) U^2]_i$. Thus, to identify the mathematical structure of the optimization problem, it is necessary to express the formula arising from $I_{\text{num}}[G_f(U, Q) U^2]_i$ and/or the approximation criteria. This point is revisited in Subsection 3.2.1, where a linearized version of the problem is introduced obtained without performing the derivation process, thereby speeding up and simplifying the calculations, and leading to a new quadratic optimization formulation.

It is worth noting that the discharge Q , although being the main variable of interest, does not appear explicitly in the cost functional. This is not a limitation of the formulation, but a consequence of the observational setting: since Q is not directly accessible from surface measurements, incorporating it into the objective function would require either a prior estimate or a direct observation, neither of which is available in the problem as stated. Instead, Q is implicitly constrained through the friction slope $S_f = S_f(U, Q)$ in Eq. (18), which links the discharge to the observable variables U and WSE via the momentum equation. In this sense, the cost functional drives the optimization toward a state that is consistent with the surface observations, while the governing equations provide the physics through which Q is determined. This structure, however, introduces an additional difficulty, as in the absence of a downstream boundary condition $h(x_f)$, the problem remains underdetermined even after the observational constraints have been incorporated. Together, these two characteristics, the implicit role of Q in the functional and the missing boundary condition, motivate the two-phase algorithm described in the following section (Section 3), where a sequence of approximations and simplifications is introduced.

3 | METHODOLOGY: FORMULATION OF THE INVERSE PREDICTOR-CORRECTOR DATA ASSIMILATION (IPCD) ALGORITHM

This section is devoted to present the proposal of the numerical methodology for the discharge assessment, the Inverse Predictor-Corrector Data Assimilation (IPCD) algorithm. To address the issue related to the lack of sufficient information, as outlined in the previous section, the algorithm is structured in two phases: the Predictor-phase and the Corrector-phase.

The Predictor-phase aims to obtain a first estimate of the flow rate following the models presented in subsubsection 2.5.1. Calculations are performed locally at each work cell $\mathcal{I}_i = [x_{i-1}, x_i]$ of the WG, and the final result, considering all individual calculations, is regarded as the *a priori* estimate to be used in the Corrector-phase. The approximations Q_{predi} are obtained solving the nonlinear Eq. (23) in each work cell. The key objective of this step is to obtain an estimated range of values for Q , thereby addressing the issue arising from the absence of a boundary condition.

The Corrector-phase synthesizes the process by comparing the model results with the observations. The proposed approach consists of iteratively solving a quadratic approximation of the optimization problem derived from the manipulation of the problem (34). In this phase, the discharge Q_{corr} is estimated over the entire analysis region, thereby recovering the mass continuity principle that was not considered in the Predictor-phase. This is the main reason why this phase is regarded as a correction of the first phase.

3.1 | Predictor-phase: local data assimilation approach

3.1.1 | Deduction of a simplified scaling model

Once the quadrature rule has been selected, and the values of U and B at the computational nodes x_{i-1} and x_i are known, the expression $I_{\text{num}}[S_f]_i$ depends on the variable Q . Defining the function $f_i(Q)$ by

$$f_i(Q) = I_{\text{num}}[S_f]_i, \quad (35)$$

Equation (25) can then be written in the form:

$$f_i(Q) = -\frac{1}{2g} (W_i^{\text{obs}} - W_{i-1}^{\text{obs}}) - (WSE_i^{\text{obs}} - WSE_{i-1}^{\text{obs}}) - I_{\text{num}}^{\text{obs}}[S_e]_i, \quad \text{for } i = 1, \dots, N-1 \quad (36)$$

The next step is to present the expression for $f_i(Q)$, and therefore the proposal to the approximation. To do this, we use Eq. (4) along with (17) to formulate the equation in terms of Q . The expression is obtained using:

$$\left(\frac{P_w}{Q}\right)_i = \frac{B_i}{Q} + \frac{2(1 - \cos(\theta))}{\sin(\theta)} \left(\frac{1}{U_i^{\text{obs}} B_i}\right) + a(\theta) \frac{Q}{(U_i^{\text{obs}})^2 (B_i)^3} \quad (37)$$

with $a(\theta) = 2(1 - \cos(\theta)) \cos(\theta) / \sin^2(\theta)$. Thus, the function $f_i(Q)$ can be written in the form:

$$f_i(Q) = \left(\beta_1 (U_{i-1}^{\text{obs}})^{10/3} \left(\frac{P_w}{Q}\right)_{i-1}^{4/3} + \beta_2 (U_i^{\text{obs}})^{10/3} \left(\frac{P_w}{Q}\right)_i^{4/3} \right) n^2 \Delta x_i \quad (38)$$

where the parameters β_i take the values corresponding to the quadrature rule employed, as detailed in Table 1.

TABLE 1 Values of the coefficients β_1 and β_2 associated with different quadrature rules in Eq. (38).

	Backward	Forward	Trapezoidal
β_1	1	0	0.5
β_2	0	1	0.5

Substituting (38) into Eq. (36), the particular cases of backward and forward approximation becomes by solving:

$$(P/Q)_{i-1} = \left(\frac{I_{\text{num}}^{\text{direct}}[S_f]_i}{n_{i-1}^2 \Delta x_i (U_{i-1}^{\text{obs}})^{10/3}} \right)^{3/4}, \quad (P/Q)_i = \left(\frac{I_{\text{num}}^{\text{direct}}[S_f]_i}{n_i^2 \Delta x_i (U_i^{\text{obs}})^{10/3}} \right)^{3/4} \quad (39)$$

with $(P/Q)_i$ from Eq. (37) and $I_{\text{num}}^{\text{direct}}[S_f]_i$ given in Eq. (24). Similarly, the approximation from the trapezoidal integral leads to:

$$\frac{\Delta x_i}{2} \sum_{j=i-1}^i n_j^2 (U_j^{\text{obs}})^{10/3} \left(\frac{P_w}{Q}\right)_j^{4/3} = I_{\text{num}}^{\text{direct}}[S_f]_i \quad (40)$$

By substituting Eq. (37) into Eq. (39), and after multiplying by Q , the forward the backward solutions are explicitly obtained by solving:

$$B_m + \left(\frac{a(\theta)}{W_m^{\text{obs}} (B_m)^3} \right) Q^2 = \left[\left(-\frac{I_{\text{num}}^{\text{direct}}[S_f]_i}{n_m^2 \Delta x_i (W_m^{\text{obs}})^{5/3}} \right)^{3/4} - \frac{2(1 - \cos(\theta))}{\sin(\theta) U_m^{\text{obs}} B_m} \right] Q \quad (41)$$

where the backward is obtained at $m = i-1$ and the forward at $m = i$, and therefore the explicit solution follows for the closed form for quadratic equations. In both rectangular and trapezoidal cross section, the backward and forward solutions are obtained in closed-form. In that direction, and for sake of simplicity, the trapezoidal integration is calculated as the average from the backward and forward solution, so Q_i^{trap} is calculated as:

$$Q_i^{\text{trap}} = \frac{Q_i^{\text{bw}} + Q_i^{\text{fw}}}{2} \quad (42)$$

The composite cases were obtained using the OG in each control cell $\mathcal{I}_i$. The equations are formulated considering:

$$f_i^{\text{bw,c}}(Q) := \sum_{k=0}^{N_i-1} \left(\frac{P}{Q} \right)_{i_k}^{4/3} (U_{i_k})^{10/3} \Delta x_k^{\text{bw}}, \quad f_i^{\text{fw,c}}(Q) := \sum_{k=1}^{N_i} \left(\frac{P}{Q} \right)_{i_k}^{4/3} (U_{i_k})^{10/3} \Delta x_k^{\text{fw}} \quad (43)$$

and

$$f_i^{\text{trap,c}}(Q) := \frac{1}{2} \left(f_i^{\text{bw,c}}(Q) + f_i^{\text{fw,c}}(Q) \right) \quad (44)$$

with $\Delta x_k^{\text{bw}} = x_{i_{k+1}} - x_{i_k}$ and $\Delta x_k^{\text{fw}} = x_{i_k} - x_{i_{k-1}}$. Thus, the backward, forward and trapezoidal composite cases are treated by solving the local data assimilation in Eq. (36), substituting $f_i(Q)$ by $f_i^{\text{bw,c}}(Q)$, $f_i^{\text{fw,c}}(Q)$, and $f_i^{\text{trap,c}}(Q)$, respectively.

3.1.2 | Comments and results on the rectangular case

In the particular case when $\theta = \pi/2$ (rectangular) we have $a(\theta) = 0$, and therefore from Eq. (41), the backward and forward approximations are directly calculate by substitution as follows:

$$Q_i^{\text{bw}} = B_{i-1} \left((P/Q)_{i-1}^{\text{obs}} - \frac{2}{U_{i-1}^{\text{obs}} B_{i-1}} \right)^{-1}, \quad Q_i^{\text{fw}} = B_i \left((P/Q)_i^{\text{obs}} - \frac{2}{U_i^{\text{obs}} B_i} \right)^{-1} \quad (45)$$

where $(P/Q)_m^{\text{obs}} = \frac{I_{\text{num}}^{\text{direct}}[S_f]_i}{n_m^2 \Delta x_i (U_m^{\text{obs}})^{10/3}}$ with $m = i-1$ or $m = i$. Consequently, a global estimate, denoted by $Q^{(0)}$, can be obtained, for example, by taking the average of all local approximations.

We would like to point out that the calculation of $Q^{(0)}$ under the rectangular assumption (or effectively condition $\theta = \pi/2$) can be useful from different perspectives. Specifically, it can serve as an initial reference in computational procedures within other inversion algorithms, such as those presented in other articles^{34,20}, or also as a reference in the implementation of a formulation of the problem Eq. (34) using numerical optimization strategies.

In our case, when $\theta \neq \pi/2$, $Q^{(0)}$ derived from the values obtained from Eq. (45), is then used as an initial value when applying a solver, as the Newton-Raphson method or similar, to solve Eq. (36). This approximation is employed in Subsection 4.2, which presents numerical experiments for a 7.5 [km] river reach, where the aspect ratio h/B is of the order of 10^{-2} .

3.1.3 | Estimation of the Predictor discharge, Q_{pred}

Each calculation is developed on control cell $\mathcal{I}_i = [x_{i-1}, x_i]$ to get Q_i^{pred} . Grouping all the values of Q_i^{pred} by some filtering criteria or by a point-wise estimator, the discharge from the Predictor-phase is estimated and passed to the Corrector-phase. For example using the average value of all the local discharges, the discharge estimation of the Predictor-phase, Q_{pred} , gets the form of:

$$Q_{\text{pred}} = \frac{1}{(N-1)} \sum_{i=1}^{N-1} Q_i^{\text{pred}} \quad (46)$$

It is important to point out that the treatment is performed on each control cell in the context of a local approximation. Unlike the approach used in other articles³⁴, where a time-dependent problem is addressed and the initial approximation is derived from an approximation obtained from another procedure (see Appendix A.2 in that article), in our proposal, the problem is solved through a self-consistent approximation process, with the initial approximation being obtained from the consideration of a rectangular channel configuration. See Algorithm 2 in Section A of the Appendix.

3.2 | Corrector-phase: global data assimilation

At this stage the optimization problem in Eq. (34) is addressed. As the restrictions are formed by discretizing the balancing equation considering a specific section and different sub intervals, unlike the previous stage, this treatment is then of a global nature. Thus, the purpose behind this approach is the treatment of the continuity of mass (discharge) in the process of transporting momentum.

The treatment of the optimization approach is considering an approximation of the term associated with the contribution S_f . The proposal is based on a reformulation of the problem, considering a linearization of the constraints (balance equations) and therefore the formulation of a quadratic programming problem.

3.2.1 | Linear constraint approximation via direct method

We now present the proposed method for linearizing the constraints without resorting to Taylor series truncation.

Let $\mathcal{I}^j = [x_{i_0}^{wse}, x_{i_f}^{wse}]$ the j -th channel section analyzed, and N the number of total nodes in that section. Thus, the discrete problem is formulated on $\mathbb{R}^{N_{\text{dim}}}$ with $N_{\text{dim}} = 2N + 1$.

The formulation of the optimization problem is worked in term of the following variables:

$$\tilde{W} = W - W^{\text{obs}}, \quad \widetilde{WSE} = WSE - WSE^{\text{obs}}, \quad \lambda = \left(\frac{Q}{Q^{\text{data}}} \right)^{4/3} \quad (47)$$

with $W = U^2$ and Q^{data} a discharge estimation obtained through a previous calculation. It is important to note that the optimization problem is solved iteratively. Therefore, the symbol Q^{data} indicates that the result obtained in the previous iteration is used as input for the new iteration, i.e., in the k -th iteration we have $Q^{\text{data}} = Q_{k-1}^{\text{ite}}$. In the first iteration, the initial estimate is taken as the value obtained in the Predictor-phase $Q_0^{\text{ite}} = Q_{\text{Total}}^{\text{pred}}$ (see definition in Eq. (46)).

Under this variable substitutions, the cost function (34) take the form:

$$\mathcal{F}_{\text{dat}} = \sum_{i=0}^{N-1} \left[\tilde{\gamma}_1 \left(\tilde{W}_i \right)^2 + \gamma_2 \left(\widetilde{WSE}_i \right)^2 \right] \quad (48)$$

with $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ presented in Eq. (33). Introducing Eq. (47) into the constraint in Eq. (34), and grouping the observable terms, we obtain the following relationship:

$$Q^{4/3} \left(\frac{1}{2g} \left(\tilde{W}_i - \tilde{W}_{i-1} \right) + \widetilde{WSE}_i - \widetilde{WSE}_{i-1} \right) - Q^{4/3} \left(I_{\text{num}}^{\text{direct}}[S_f]_i \right) = -\Delta_i \left(G_f(U, Q) W \right) \quad (49)$$

with $I_{\text{num}}^{\text{direct}}[S_f]_i$ introduced in (24). To address the nonlinearity shown in the first term of the previous equation, we propose substituting the term $Q^{4/3}$ with $(Q^{\text{data}})^{4/3}$, so the first linear approximation is developed into the left-hand side (LHS_i) in Eq. (49) using the following consideration $LHS_i \approx LHS_i^{\text{Lapp}}$, with:

$$LHS_i^{\text{Lapp}} := (Q^{\text{data}})^{4/3} \left(\frac{1}{2g} \left(\tilde{W}_i - \tilde{W}_{i-1} \right) + \widetilde{WSE}_i - \widetilde{WSE}_{i-1} \right) - Q^{4/3} \left(I_{\text{num}}^{\text{direct}}[S_f]_i \right) \quad (50)$$

Obtaining a linearization by substitution. The purpose of this decomposition is the superposition of two linear contribution: the first in terms of the variables $\tilde{W}$ and $\widetilde{WSE}$, and the second in terms of Q . This last point also is connect with the introduced variable $\lambda = (Q/Q^{\text{data}})^{4/3}$.

The next step is to derive some favorable linear approximation of the right-hand side in Eq. (49) using prior information. The goal for the linear formulation is to incorporate the necessary information from the new contribution and establish a sufficiently stable relationship with the computations carried out in the previous iterations.

The second linearization is developed on the right-hand side (RHS_i) of Eq. (49). The quadrature is performed using the trapezoidal rule. The approximation criterion is formulated as $RHS_i \approx RHS_i^{\text{Lapp}}$ with:

$$RHS_i^{\text{Lapp}} := \frac{\Delta x_i}{2} \left(G_f(U_i^{\text{data}}, Q^{\text{data}}) + G_f(U_{i-1}^{\text{data}}, Q^{\text{data}}) \right) \left(\tilde{W} + W^{\text{obs}} \right) \quad (51)$$

where the leading term corresponds to the trapezoidal quadrature of G_f evaluated at $(U^{\text{data}}, Q^{\text{data}})$, with U^{data} denoting the velocity field obtained from a previous approximate procedure.

Thus, through these linear approximation presented for each term of the integrate momentum equation, the constraints in the discrete optimization problem are expressed by:

$$LHS_i^{\text{Lapp}} = RHS_i^{\text{Lapp}} \quad (52)$$

for $i = 1, \dots, N - 1$. The formulation of the quadratic programming is presented in the next section as well as the criteria used to the construction of the discharge numerical approximation.

3.2.2 | Mathematical formulation of the optimization problem

We present the matrix formulation of the optimization problem. For simplicity we continue using N to denote the number of work nodes used in the analysis, $N_{\text{dim}} = 2N + 1$ the optimization problem dimension, and the nodes subscript are writing by $i = 0, \dots, N - 1$. The box brackets $[A_{N'}]$ are used to denote a square matrix of order N' .

Considering $\mathbf{Z} \in \mathbb{R}^{N_{\text{dim}}}$ the column with block-entries $\tilde{\mathbf{W}}$, $\tilde{\mathbf{WSE}}$ both in $\mathbb{R}^N$ along with $\lambda \in \mathbb{R}$, the objective function is expressed as follows:

$$\mathcal{F}_{\text{data}} = \frac{1}{2} [\mathbf{Z}]^t [\mathcal{Q}_{N_{\text{dim}}}^{\gamma_1, \gamma_2}] [\mathbf{Z}] \quad (53)$$

where the quadratic form is determine by the diagonal matrix $[\mathcal{Q}_{N_{\text{dim}}}^{\gamma_1, \gamma_2}]$ with $[\mathcal{Q}]_{i,i} = \tilde{\gamma}_1$ for $i = 1, \dots, N$, $[\mathcal{Q}]_{i,i} = \tilde{\gamma}_2$ for $i = N + 1, \dots, 2N$, and $[\mathcal{Q}]_{N_{\text{dim}}, N_{\text{dim}}} = 0$.

The first type of constraints is derived from the integration of the momentum equation. Let $[\mathcal{M}_N] = \text{tri}_N[0, -1, 1]$ the tri-diagonal matrix used to write the direct integration. The contribution derived from the friction term is represent by $[\mathcal{G}_N^{\text{data}}] = \text{tri}_N[0, G_{i-1}^{\text{data}}, G_i^{\text{data}}]$, with $G_i^{\text{data}} = \frac{\Delta x_i}{2} \left(Q^{\text{data}} \right)^{-4/3} G_f(U_i^{\text{data}}, Q_i^{\text{data}})$. Under this notations, the total unknown contributions associated to the velocity and water elevation can be obtained from the action of the following matrices:

$$[\mathcal{A}_N]_{\text{vel}}^{\text{data}} := \frac{1}{2g} [\mathcal{M}_N] + [\mathcal{G}_N^{\text{data}}], \quad [\mathcal{A}_N]_{\text{elev}} := [\mathcal{M}_N] \quad (54)$$

The column containing the inhomogeneous contributions is given by:

$$\mathbf{b}_{\text{obs}}^{\text{data}} := [\mathcal{G}_N^{\text{data}}] \mathbf{W}^{\text{obs}} \quad (55)$$

with $\mathbf{W}^{\text{obs}}$ denoting the column with entries W_i^{obs} with $i = 0, \dots, N - 1$. Thus, the equality constraint are grouped as follows:

$$[\mathcal{A}]_{\text{obs}}^{\text{data}} \cdot \mathbf{Z} = \mathbf{b}_{\text{obs}}^{\text{data}} \quad (56)$$

with $[\mathcal{A}]_{\text{obs}}^{\text{data}}$ the block matrix defined from Eq. (54) as follows:

$$[\mathcal{A}]_{\text{obs}}^{\text{data}} = \begin{bmatrix} [\mathcal{A}_N]_{\text{vel}}^{\text{data}} & [\mathcal{A}_N]_{\text{elev}} \end{bmatrix} \mathbf{I}[\mathbf{S}_f]^d \quad (57)$$

and $\mathbf{I}[\mathbf{S}_f]^d$ representing the column with entries $\mathbf{I}_{\text{num}}^{\text{direct}}[\mathbf{S}_f]_i$, see Eq. (50) third term, and Eq. (24) for the definition.

The second type of constraints is derived from the fact that $W = U^2$, WSE and Q are positive. Through those considerations, the following positive constraints are posed:

$$\tilde{\mathbf{W}} + \mathbf{W}^{\text{obs}} \geq \mathbf{0}, \quad \tilde{\mathbf{WSE}} + \mathbf{WSE}^{\text{obs}} \geq \mathbf{0} \quad (58)$$

along with $\lambda > 0$. This last constraint is controlled using tol_λ an arbitrary positive parameter of the numerical scheme.

Thus, introducing $\mathbf{c}_{\text{num}}^{\text{obs}} = (\mathbf{W}^{\text{obs}} \quad \mathbf{WSE}^{\text{obs}} \quad \text{tol}_\lambda)^t$ to represent Eq. (58), and from the all previous consideration, the quadratic optimization problem (QP) is formulated as:

$$\begin{cases} \text{Minimize} & \frac{1}{2} \mathbf{Z}^t [\mathcal{Q}_{N_{\text{dim}}}^{\gamma_1, \gamma_2}] \mathbf{Z} \\ \text{subject to:} & \\ & [\mathcal{A}]_{\text{obs}}^{\text{data}} \mathbf{Z} + \mathbf{b}_{\text{obs}}^{\text{data}} = \mathbf{0}, \\ & \mathbf{Z} + \mathbf{c}_{\text{num}}^{\text{obs}} \geq \mathbf{0} \end{cases} \quad (59)$$

with $\mathbf{c}_{\text{num}}^{\text{obs}} \geq \mathbf{0}$ and $[\mathcal{Q}_{N_{\text{dim}}}^{\gamma_1, \gamma_2}]$ is positive semidefinite. Through the formulation in Eq. (59), an optimization problem is defined using a simplified model that incorporates the results obtained in the previous iteration. These results are embedded in the definition of matrix $[\mathcal{A}]_{\text{obs}}^{\text{data}}$. The iterative procedure is summarized in the next subsection (see also in Algorithm 3 in Appendix section). The stopping criterion (Step 4) is based on the comparison of discharge, namely tol . The requirements refer to satisfying conditions on the difference.

Finally, we note that the discrete formulation (59) leads to a convex quadratic optimization problem, in which the dependence on the friction slope S_f is incorporated through a direct linearization based on dominant-balance arguments and substitution into the energy balance equation (see Eq. (49) and Eq. (50)). This framework differs substantially from standard approaches to

inverse problems in fluid mechanics. Sequential quadratic programming methods typically rely on local Taylor expansions of nonlinear constraints, whereas variational data assimilation (VDA) introduces Lagrange multipliers, leading to an adjoint system coupled with the state equations. As a consequence, such approaches generally involve nonlinear or saddle-point problems and often require multiple discretization stages. In contrast, the present approach preserves a convex quadratic structure at the discrete level, enabling efficient solution strategies while avoiding adjoint equations and reducing the overall computational cost. This feature is particularly relevant for the strategy adopted in this work, which relies on the iterative computational procedure described in the following subsection.

3.3 | Inverse Predictor-Corrector Data Assimilation (IPCDA) algorithm

In the previous subsections, the background for the formulation of each phase of the proposed inversion process with data assimilation has been presented. To conclude this section, we summarize the proposed numerical procedure for discharge and bathymetry estimation in the following algorithm:

- **Step 1** Compute $Q^{(0)}$.
- **Step 2** Apply the predictor approach and obtain Q_i^{pred} in each cell $\mathcal{I}_i$. Use these approximations to compute Q_{pred} . Set $Q^{\text{data}} = Q_{\text{pred}}$ and $U^{\text{data}} = U^{\text{obs}}$.
- **Step 3** Compute $[A]_{\text{obs}}^{\text{data}}$, $b_{\text{obs}}^{\text{data}}$, and $c_{\text{num}}^{\text{obs}}$, and then compute Q_{corr} and U .
- **Step 4** If

$$|Q_{\text{corr}} - Q^{\text{data}}| < \text{tol},$$

then the final result is given by Q_{corr} . Otherwise, set $Q^{\text{data}} = Q_{\text{corr}}$ and $U^{\text{data}} = U$, and return to Step 3.

It is important to note that the linearization approach shown in Eq. (30) aimed to decompose the nonlinear term S_f into a contribution from velocity and another from flow rate, with the latter contribution reflected in Eq. (49) through multiplication. The linearization can be carried out in different ways, and the formulation presented in Eq. (57) was chosen to facilitate iterative computations.

Some details regarding the algorithms employed in each phase are presented in the Appendix section.

4 | NUMERICAL EXPERIMENTS

To test the IPCDA algorithm, we used the results from two synthetic 1D models developed in HEC-RAS, with known bathymetries and roughness coefficient. HEC-RAS software (US Army Corps of Engineers, 2021) is a widely accepted hydraulic model for river reach simulations in one and two dimensions. The concept of using synthetic models to test related methodologies has also been used in other studies (see for example Ref. ^{8,30}). Important to note that HEC-RAS calculates and presents as a result the cross sectional average velocity. In these test of the IPCDA method, we assume that the average velocity in every cross section is known or can be estimated from the surface velocity (similar to the assumption presented in Refs. ^{11,12,40,4}), although this will be further discussed.

Two case studies were simulated: a prismatic trapezoidal channel and a reach of the Biobío River in Chile. The first case represents a man-made channel, commonly found in agricultural or hydropower infrastructure, with known side slope and roughness coefficient, whereas the second considers a natural reach with irregular cross-sectional geometry, spatially varying Manning's n and non uniform bed topography not well approximated by a trapezoid (Ref. ⁴⁸). From a hydraulic perspective, the trapezoidal channel is sufficiently narrow relative to its flow depth that a rectangular approximation is not appropriate, whereas this simplification can be reasonably adopted for the much wider natural river reach. Together, these cases illustrate the capabilities and limitations of the IPCDA method for discharge estimation.

For most of the simulations, the following parameters were used: (i) the expansion and contraction coefficients were set equal to the default values used in HEC-RAS ($K_e = -0.3$ for expansions and $K_e = 0.1$ for contractions), (ii) The weighting coefficients in the objective function were set to $\gamma_1 = 0.5$ and $\gamma_2 = 0.5$, (iii) the iterative tolerance was fixed at $\epsilon = 10^{-5}$, (iv) for cases with spatially variable roughness, such as the Biobío River case, the Manning's roughness coefficient assigned to each assessed reach was defined as the median value (the 50th percentile) of the Manning's coefficients within that reach, (v) the maximum number of iterations was set to $N_{\text{max}} = 100$. The experiments were performed using the three integration methods together with their

corresponding composite rules, resulting in a total of six integration schemes. For a sensibility analysis we run also the method changing the values of γ_1 , γ_2 and n , and those results will be further analyzed in the Discussion section (5).

4.1 | Trapezoidal channel

The trapezoidal channel case is designed to evaluate the IPCDA method under controlled and well characterized hydraulic conditions. The channel has a bottom width of 30 [m], side slopes defined by $\theta = \pi/4$, and a Manning roughness coefficient of $n = 0.022$ [s/m^{1/3}], representative of an old concrete-lined or earthen channel. It is divided into four prismatic reaches with distinct bed slopes for a total length of 2,600 [m] (Table 2), producing a longitudinal profile with both mild and stronger slope sections and an intermediate bump, which results in a varied distribution of flow depth and surface width along the reach. The model was simulated in HEC-RAS under subcritical flow conditions using normal depth as the downstream boundary condition with a slope of 2.50×10^{-4} [m/m], and the output was sampled at 1 [m] intervals, serving as a quasi-continuous synthetic dataset analogous to what could be obtained from drone or other remote sensing devices under favorable conditions, defined as the Observation Grid (OG) for the IPCDA algorithm. Three independent segments (S.1, S.2, S.3) and the full channel length are analyzed separately to assess the sensitivity of the method to the spatial extent and hydraulic characteristics of the analysis domain. The segments represent distinct hydraulic behaviors: and a mild uniform flow reach (S.1), a short steep transition reach (S.2) and a gradually backwatered profile with mild *WSE* gradient (S.3), providing a systematic basis for evaluating the performance of both the Predictor and Corrector phases under varying grid resolution, discharge magnitude, and hydraulic regime.

TABLE 2 Hydraulic Parameters for the different reaches of the designed channel in HEC-RAS.

Variable	Reach 1	Reach 2	Reach 3	Reach 4
Length [m]	1,000	200	400	1,000
Bottom Slope [m/m]	5.00×10^{-5}	-1.00×10^{-4}	5.50×10^{-4}	2.50×10^{-4}

We run the HEC-RAS model using values of ΔWSE^{tol} from 1 [cm] to 10 [cm] and discharges from 1 to 100 [m³/s] to assess the IPCDA methodology under different configurations. Figure 3 presents the HEC-RAS output for a reference discharge of $Q = 1$ [m³/s], which is used as input to the IPCDA algorithm. The figure shows the water surface elevation, the top width and the mean velocity along the 2,600 [m] reach.

4.1.1 | Input data and grid design criteria

The selection criterion of the independent segments was based on characteristics of the channel width $B(x)$ and averaged velocity $U(x)$, considering the monotonicity in the values of *WSE* with respect to x (Table 3). Using the information of the different segments, it is possible to define the Working Grid (WG) as a function of the tolerance value ΔWSE^{tol} .

TABLE 3 Definition of the computational reaches used to evaluate the IPCDA method. As the OG has $\Delta x = 1$ [m], the length of each segment corresponds to the number of cells in the OG.

Segment	Start [m]	End [m]	Length [m]	Hydraulic characteristics
S.1	50	1150	1100	M2 curve. Mild gradually varied flow at the beginning with a strong contraction to the end, generating a rapid variation of U and B .
S.2	1175	1550	375	M1 curve. Short segment with a mild expansion, generating an increase in B and a decrease in U , stronger to the end.
S.3	1600	2550	950	Also M1 curve. Mildly increasing $B(x)$ and decreasing $U(x)$, also stronger to the end
Full reach	50	2550	2500	Longest segment. Non-monotonic $B(x)$ and $U(x)$ across segment boundaries

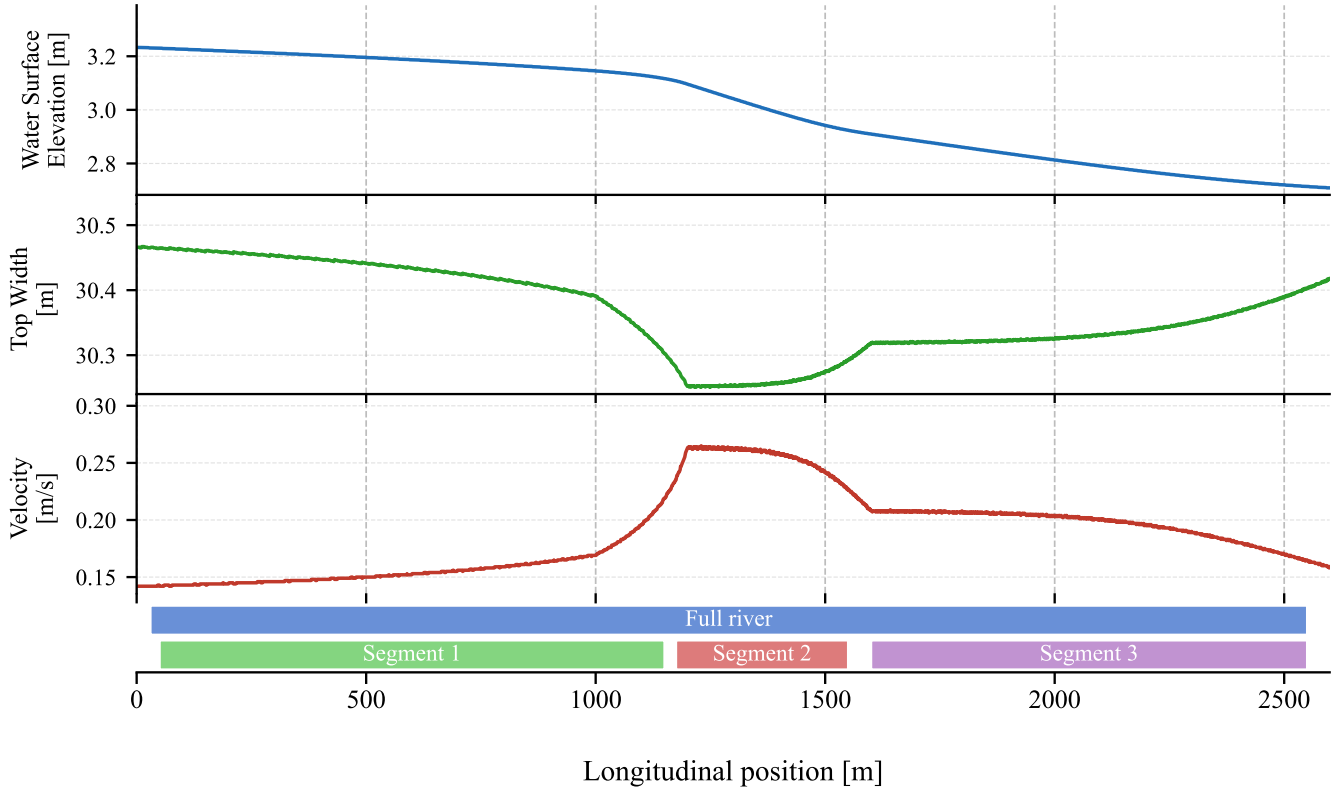


FIGURE 3 Input variables to the IPCDA algorithm from the HEC-RAS simulation of the trapezoidal channel ($Q = 1 \text{ [m}^3/\text{s]}$), at 1 [m] spacing: WSE (top), top width B (middle), and cross-sectional mean velocity U (bottom). Colored boxes highlight the three independent segments S.1, S.2, and S.3. Abrupt changes in B and U mark boundaries between reaches of different bed slope (Table 2).

TABLE 4 Summary of the Working Grid properties for different ΔWSE^{tol} values, reference discharges Q_{ref} , and segments.

ΔWSE^{tol} [cm]	Segment 1			Segment 2			Segment 3			Full river		
	$Q = 1$	$Q = 10$	$Q = 100$	$Q = 1$	$Q = 10$	$Q = 100$	$Q = 1$	$Q = 10$	$Q = 100$	$Q = 1$	$Q = 10$	$Q = 100$
<i>Working grid number of cells</i>												
1	9	17	12	16	7	3	17	10	8	47	39	27
5	1	2	1	2	— ^a	— ^a	2	1	1	9	7	5
10	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	4	3	2
<i>Cell average length [m]</i>												
1	113.7	61.4	81.7	20.8	39.7	84.0	48.4	74.7	98.0	50.7	59.4	87.2
5	632.0	335.5	397.0	94.0	— ^a	— ^a	207.5	322.0	449.0	237.3	264.4	403.8
10	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	476.0	505.7	779.0

Discharge values of Q_{ref} are expressed in $[\text{m}^3/\text{s}]$.

^a Symbol “—” indicates that no result was obtained for this reach.

Table 4 summarizes the number of WG cells and the average cell length for each segment, across tolerances $\Delta WSE^{\text{tol}} = 1, 5, 10 \text{ [cm]}$ and reference discharges $Q_{\text{ref}} = 1, 10 \text{ and } 100 \text{ [m}^3/\text{s}]$. As expected, the number of WG cells decreases monotonically with increasing tolerance, since the cell count in each segment is approximately $\sim \Delta WSE_{\text{segm}} / \Delta WSE^{\text{tol}}$, where the term $\Delta WSE_{\text{segm}} = WSE_a - WSE_b$ denotes the total water surface elevation drop over the segment $[a, b]$. Consequently, for segments with a limited total WSE drop, coarser tolerances may prevent the formation of even a single WG cell, as the required minimum elevation difference between consecutive nodes cannot be satisfied. This is illustrated by the short individual segments S.1, S.2, and S.3 under $\Delta WSE^{\text{tol}} = 10 \text{ [cm]}$, where no valid WG could be constructed, and even for the full river reach only two cells

are obtained under this tolerance. These results highlight a fundamental practical constraint of the methodology: the minimum achievable ΔWSE^{tol} is bounded below by the accuracy of the remote sensing instruments, while the maximum number of WG cells is bounded above by the total WSE .

It can be noted that in general the number of WG cells decreases with the reference discharge Q_{ref} , but this is not the case for Segment 1. A possible explanation is that the spatial distribution of the total head drop depends on Q . At low discharges, segments 2 and 3 take the bulk of the roughly 50 [cm] drop in WSE , but Segment 1 just 11 [cm]; at moderate discharges, ΔWSE_{S1} peaks at roughly 20 [cm], meanwhile the downstream segments shorten their adjustment, so Segments 2 and 3 shrink in WSE . Then at higher discharges ($Q = 50$ and 100 [m³/s]) the flow is close to normal depth almost everywhere, as the profile is roughly parallel to the bed with a small backwater tail. Total ΔWSE on the whole river decreases so Segment 1 drops back again down to 15 [cm], dominated now by the fixed bed-slope contribution plus a small friction component.

The cell average length as expected also decreases with increases in ΔWSE^{tol} , similarly to the number of cells but depending on the kind of flow that is represented is its relation to the discharges. In most of cases the length tend to increase with increasing discharges except for the case of Segment 1. It is interesting to see that an average cell in the case with $Q = 100$ [m³/s] and $\Delta WSE^{tol} = 10$ [cm] is in the order of 780 [m], whereas with $Q = 1$ [m³/s] and $\Delta WSE^{tol} = 1$ [cm], in the order of 50 [m].

4.1.2 | Predictor-phase analysis

Table 5 show the Predictor results for all the segments evaluated using the values of ΔWSE^{tol} and reference discharges used in Table 4 using the Trapezoidal integration rule. The first part of the table show the ratio Q_{pred}/Q_{ref} between the Predictor value and the reference discharge. The global estimate Q_{pred} is computed as the mean of the local cell discharges after discarding values below the 5th and above the 95th percentile, a mild outlier filter that retains the central 90% of the distribution while reducing the influence of cells where the local inversion is poorly conditioned. In general, all values show good match comparing to the reference values, with the highest error in less than 6%. No appreciable differences exists when using coarse values of ΔWSE^{tol} , although as shown in Table 4, in most cases the WG could not be formed when using $\Delta WSE^{tol} = 10$ [cm].

The second part of Table 5 show the contributions of the singular losses S_e , through approximations $I_{num}^{obs}[S_e]_i$ shown in Eq. (21). This was done to verified the assumption that the frictional losses control the total losses and the simplification in the calculation of the singular losses was correct or at least, generate small errors. Let $I[S_e]^{obs}$ the vector of values $I_{num}^{obs}[S_e]_i$ in each working cell, then R is the relation between the maximum value of the singular losses from all cells in reach and the forced WSE drop given by $\Delta WSE^{tol}/\Delta x$ as (see balance in Eq. (8)):

$$R = \frac{\|I[S_e]^{obs}\|_{\infty}}{\Delta WSE^{tol}/\Delta x} \quad (60)$$

From the obtained values of R shown in the lower part of Table 5, it can be observed that the anticipated balance is satisfied in each of the reaches, and therefore the computation of the nonlinear term link to singular losses yields a max relation of less than 5% compared the forced drop in WSE for each cell. Consequently, considering this approximation through data assimilation, rather than treating it as a term in the governing equation, is advantageous in the calculations and reduces the complexity of the nonlinearity.

Table 6 presents the Predictor values calculated by six quadrature rule, including the composite quadrature cases, using $\Delta WSE^{tol} = 5$ [cm] for the full river. This value was selected for the scheme comparison in Table 6 as it represents a threshold consistent with the accuracy limits of current UAV-based WSE measurement technology commented above. It can be seen that the results are similar to 1, showing an almost perfect match with the reference discharges. The the difference between integration rules are negligible, so from now on we will display results just using the Trapezoidal rule for the Predictor phase. Although not presented here, the values of $Q^{(0)}$ are also relatively similar to the Predictor results, using a much simpler calculation procedure that can also be used in other methodologies as a seed value or initial guess (see Algorithm 1).

4.1.3 | Corrector-phase analysis

Table 7 reports the results of the IPCDA applied to each of the previously described segments, including the entire reach, using the grids $\Delta WSE^{tol} = 1, 5$, and 10 [cm], using the trapezoidal integration scheme in the Predictor-phase. The number of iterations required to converge to the final Corrector-phase discharge is also presented alongside the corresponding relative results for

TABLE 5

Summary of the predictor discharge ratio $Q_{\text{pred}}/Q_{\text{ref}}$ and the indicator R for different ΔWSE^{tol} values and segments using reference discharges $Q = Q_{\text{ref}}$.

ΔWSE^{tol} [cm]	Segment 1			Segment 2			Segment 3			Full river		
	$Q = 1$	$Q = 10$	$Q = 100$	$Q = 1$	$Q = 10$	$Q = 100$	$Q = 1$	$Q = 10$	$Q = 100$	$Q = 1$	$Q = 10$	$Q = 100$
$Q_{\text{pred}}/Q_{\text{ref}}$												
1	0.998	1.001	0.999	0.991	0.991	0.991	0.997	0.996	0.994	0.996	0.997	0.997
5	1.006	0.999	0.993	0.943	— ^a	— ^a	0.990	0.985	0.993	0.996	0.996	0.999
10	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	1.000	1.010	0.976
$R = \frac{\ I[S_e]^{\text{obs}}\ _{\infty}}{\Delta WSE^{\text{tol}}/\Delta x}$												
1	0.3%	0.6%	0.7%	1.2%	2.4%	4.3%	0.9%	1.4%	2.1%	0.9%	1.3%	1.8%
5	0.2%	0.6%	0.6%	1.1%	— ^a	— ^a	0.9%	1.4%	2.1%	0.9%	1.2%	1.7%
10	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	0.8%	1.1%	1.6%

Discharge values of Q_{ref} are expressed in [m³/s].

^a Symbol “—” indicates that no result was obtained for this reach.

TABLE 6 Predictor discharge ratio $Q_{\text{pred}}/Q_{\text{ref}}$ over the full river (50–2550 [m]) for the six quadrature schemes: backward (B), forward (F), trapezoidal (T) and their composite counterparts (BC, FC, TC).

Q_{ref} [m ³ /s]	B	F	T	BC	FC	TC
1	1.036	0.979	0.996	0.997	0.997	0.997
10	1.007	0.976	0.996	0.996	0.996	0.996
100	0.999	0.999	0.999	0.999	0.999	0.999

B, F, and T denote the backward, forward, and trapezoidal quadrature schemes, respectively. BC, FC, and TC denote their corresponding composite versions.

each case. The final discharge estimates of Q_{corr} are obtained by applying the procedure outlined in Subsection 3.3; see also Algorithm 4 in Section A of the Appendix.

TABLE 7 Corrector discharge ratio $Q_{\text{corr}}/Q_{\text{ref}}$ and number of iterations N_{ite} for different ΔWSE^{tol} values, reference discharges Q_{ref} , and segments.

ΔWSE^{tol} [cm]	Q_{ref} [m ³ /s]	Segment 1		Segment 2		Segment 3		Full river	
		$Q_{\text{corr}}/Q_{\text{ref}}$	N_{ite}	$Q_{\text{corr}}/Q_{\text{ref}}$	N_{ite}	$Q_{\text{corr}}/Q_{\text{ref}}$	N_{ite}	$Q_{\text{corr}}/Q_{\text{ref}}$	N_{ite}
1	1	0.923	14	0.990	7	0.991	6	0.993	3
	10	1.001	5	0.989	4	0.994	5	0.997	6
	100	0.998	3	0.990	3	0.994	3	0.997	5
5	1	0.062	12	0.915	13	0.990	1	0.858	12
	10	1.000	6	— ^a	— ^a	0.985	2	0.995	8
	100	0.993	2	— ^a	— ^a	0.993	2	0.998	5
10	1	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	0.875	10
	10	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	0.952	7
	100	— ^a	— ^a	— ^a	— ^a	— ^a	— ^a	0.973	5

Q_{ref} denotes the reference discharge, Q_{corr} the corrected discharge estimate, and N_{ite} the number of iterations required by the corrector stage.

^a Symbol “—” indicates that no result was obtained for this reach.

In general errors are relatively small, mostly near 1%, although an exception to the generally stable behavior of the Corrector-phase is observed for $Q_{\text{ref}} = 1$ [m³/s] in Segment 1, where the error increases sharply relative to the Predictor result, from ~1% from the Predictor-phase to 94% after 12 iterations. This deterioration is attributed to the combined effect of three factors that become critical at low discharge and using just 1 or 2 cells of the WG. First, the sensitivity of the friction slope to discharge, $\partial S_f / \partial Q \sim -\frac{4}{3} S_f / Q$, grows inversely with Q , so small errors in the linearization of S_f around Q^{data} are amplified significantly at low flow. Second, at low velocities the velocity head $U^2/2g$ is of the same order as ΔWSE^{tol} , reducing the signal available to the quadratic programming and increasing the condition number of the constraint matrix $[\mathbf{A}]_{\text{obs}}^{\text{data}}$. Third, the velocity contribution to

the objective function weighted by $\tilde{\gamma}_1$ through $W = U^2$, becomes negligible at low flow, so the optimization effectively minimizes only the WSE discrepancy, losing the physical coupling between velocity and discharge that anchors the inversion. Together, these effects cause the Corrector to converge to a solution that satisfies the linearized constraints but is physically inaccurate. For low-discharge cases, the Predictor estimate can be a more reliable output of the IPCDA algorithm.

4.1.4 | Channel bed reconstruction

Using Eq. (16), the water depth can be calculated using the following relation

$$h_i^{\text{inv}} = \frac{Q_{\text{inv}}}{U_i^{\text{obs}} B_i} + \frac{\cos(\theta)}{\sin(\theta)} \frac{(Q_{\text{inv}})^2}{(U_i^{\text{obs}})^2 B_i^3} \quad (61)$$

with $Q_{\text{inv}} = Q_{\text{pred}}$ or $Q_{\text{inv}} = Q_{\text{corr}}$. The inverse estimation for the bathymetry follows from the relationship:

$$Z_i^{\text{inv}} = WSE_i^{\text{obs}} - h_i^{\text{inv}} \quad (62)$$

Hence, after completing the calculation procedure and obtaining the global estimate of Q_{corr} , the relation $Z = WSE - h$ can be applied in general, where $h = h(Q_{\text{corr}})$ is computed as in Eq. (61), using the discharge obtained from the Corrector-phase and $WSE = WSE^{\text{obs}}$ for all nodes in the working grid (WG). Figure 4 presents the bathymetry reconstruction Z^{inv} obtained from (62) for the reference cases $Q = 1 \text{ [m}^3/\text{s]}$, considering Q_{corr} calculated by IPCDA and $\Delta WSE^{\text{tol}} = 1, 5$ and 10 [cm] . The results are compared against the exact description implemented in HEC-RAS (black line), where the piecewise-linear elevation function $Z^{\text{HEC-RAS}}(x)$ is derived from the specific bottom elevation values $\{3, 2.95, 2.97, 2.75, 2.5\} \text{ [m]}$, corresponding to the reach indicated in Table 3.

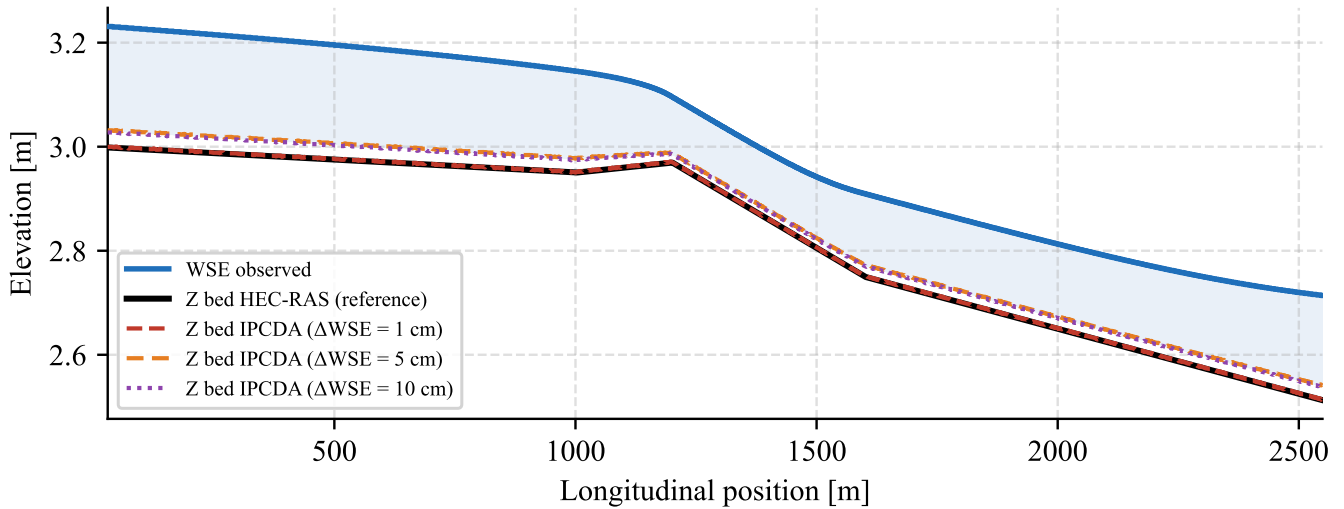


FIGURE 4 Reconstruction of the bed elevation $Z(x)$. Results obtained for $Q_{\text{ref}} = 1 \text{ [m}^3/\text{s]}$. The thick black curve represents the exact (synthetic) bathymetry and the dashed lines, the reconstruction using $\Delta WSE^{\text{tol}} = 1, 5, 10 \text{ [cm]}$.

The results obtained with different grids show similar behavior, indicating that the bathymetry approximation is consistently reliable although some errors show when using coarser grids. We can analyze the impact of Q_{ref} on the calculations of Z and the physical significance of the approximations made. To illustrate this point, we rewrite Z_{inv} in Eq (61), with $\theta = \pi/4$, as follows:

$$h_i^{\text{inv}} = \left(\frac{Q_{\text{inv}}}{Q_{\text{ref}}} \right) \frac{Q_{\text{ref}}}{U_i^{\text{obs}} B_i} + \left(\frac{Q_{\text{inv}}}{Q_{\text{ref}}} \right)^2 \left(\frac{(Q_{\text{ref}})^2}{(U_i^{\text{obs}})^2 B_i^3} \right) \quad (63)$$

This representation, together with the ratio $Q_{\text{inv}}/Q_{\text{ref}}$ shown in Table 7, provides insight into the discrepancies of Z_{inv} with respect to the reference value observed in Figure 4. Specifically, under the assumption that $B_i \ll h_i$, the expression in Eq. (63) can be approximated by retaining only the first term on its right-hand side, yielding:

$$Z_i^{\text{HEC-RAS}} - Z_i^{\text{inv}} \approx \left(\frac{Q_{\text{inv}}}{Q_{\text{ref}}} - 1 \right) h_i^{\text{HEC-RAS}} \quad (64)$$

From the relation above, the sensitivity of the bathymetry reconstruction results is proportional to both the water depth and the ratio $Q_{\text{corr}}/Q_{\text{ref}}$. Considering the case shown in Figure 4 and the corresponding ratios reported in Table 7, *WSE* tolerances of 5 and 10 [cm] yield a ratio of approximately 0.85, resulting in a reconstructed channel bottom of 3.03 [m] compared to the 3 [m] value prescribed in the HEC-RAS model. It seems a small difference but actually is a 15% error compared from the *WSE* value. For the case of the reference discharge of $Q = 100 \text{ [m}^3/\text{s]}$, the maximum error occurs at $\Delta WSE^{\text{tol}} = 10 \text{ [cm]}$, with an absolute difference of 12.9 [cm] relative to a water depth of 3.14 [m] (not shown).

$$\Delta Z_{\text{RMSE}} = \sqrt{\frac{1}{N} \sum_{k=1}^N [Z_{\text{HEC-RAS}}(x_k) - Z^{\text{inv}}(x_k)]^2} \quad (65)$$

Another source of discrepancy lies in the approximation of h developed through the truncation with respect to η , as presented in Eq. (13) ($0 < \eta \ll 1$). To complement the arguments used to explain the differences that arise in Z_{inv} with respect to Q_{ref} , we present Table 8. The minimum and maximum values of η are reported, as well as the RMSE between the HEC-RAS original values and the reconstructed.

TABLE 8 Representation of errors associated to the bathymetry reconstruction in the full river by discharge and *WSE* tolerances.

$Q_{\text{ref}} \text{ [m}^3/\text{s]}$	$\Delta WSE^{\text{tol}} \text{ [cm]}$	$n_{\text{min}} \text{ [-]}$	$n_{\text{max}} \text{ [-]}$	$\Delta Z_{\text{RMSE}} \text{ [m]}$	$(\Delta Z/h^{\text{HEC-RAS}})_{\text{max}} \text{ [%]}$	$h_{\text{mean}}^{\text{HEC-RAS}} \text{ [m]}$
1	1	0.016	0.030	0.001	1.50	0.182
	5	0.016	0.029	0.006	4.29	
	10	0.014	0.027	0.023	13.2	
10	1	0.074	0.101	0.003	0.60	0.720
	5	0.074	0.101	0.003	0.71	
	10	0.073	0.100	0.015	2.37	
100	1	0.305	0.328	0.054	1.91	3.14
	5	0.305	0.328	0.052	1.83	
	10	0.298	0.321	0.129	4.29	

As Table 8 shows, the η values increase consistently with discharge, ranging from approximately 0.015-0.030 for $Q_{\text{ref}}=1 \text{ [m}^3/\text{s]}$ to 0.298-0.328 for $Q_{\text{ref}}=100 \text{ [m}^3/\text{s]}$, and notably remain nearly insensitive to the tolerance value of the *WSE* drop for a given discharge, suggesting the approximation of η is robust to grid resolution even when the reconstructed bathymetry degrades. Regarding bathymetry reconstruction accuracy, ΔZ_{RMSE} increases monotonically with ΔWSE^{tol} across all discharge cases, with the sharpest deterioration occurring between $\Delta WSE^{\text{tol}} = 5$ and 10 [cm], where it grows by a factor of 2.5 to 5 depending on the discharge. This abrupt change is consistent with the sharp reduction in the number of WG cells reported in Table 4. The worst absolute error corresponds to $Q_{\text{ref}} = 100 \text{ [m}^3/\text{s]}$ with $\Delta WSE^{\text{tol}} = 10 \text{ [cm]}$ ($\Delta Z_{\text{RMSE}} = 0.129 \text{ [m]}$), yet its maximum relative error remains modest at 4.29% owing to the large mean water depth of 3.14 [m]. Conversely, the worst relative error (13.2%) occurs at $Q_{\text{ref}} = 1 \text{ [m}^3/\text{s]}$ with the coarsest grid, where the shallow mean depth of 0.182 [m] amplifies even small absolute errors, highlighting that the combination of low discharge and coarse *WSE* tolerance represents the most critical scenario for the method. For the finest grid ($\Delta WSE^{\text{tol}} = 1 \text{ [cm]}$), all discharge scenarios yield relative errors below 2%, which can be considered acceptable for practical engineering applications, and a tolerance of $\Delta WSE^{\text{tol}} \leq 5 \text{ [cm]}$ appears sufficient for intermediate and high discharges. Finally, the intermediate discharge $Q_{\text{ref}} = 10 \text{ [m}^3/\text{s]}$ exhibits the most stable reconstruction across all tolerances, with relative errors below 2.4% even for the coarsest grid, suggesting the existence of an optimal discharge range where the method performs most reliably.

4.2 | Biobío river at its rivermouth

To apply the IPCDA method under more complex conditions, results from a calibrated HEC-RAS model were used for a 7.5 [km] reach of the Biobío river in Chile, near its mouth with the Pacific Ocean. The hydraulic model that represents the base condition of the river was developed to assess the impacts of a new bridge⁴⁸. The model development used usual elements required for 1D flow simulation, including surveyed cross sections and calibration for several simulation discharges. Upstream and downstream boundary conditions were defined using normal-depth slopes of 8.8×10^{-4} [m/m] at the upstream end and 5.7×10^{-4} [m/m] at the downstream end. The model included 30 cross sections distributed between the Binimellis sector in Chiguayante and the area downstream of the Chacabuco Bridge in the city of Concepción, each of them separated every 1 [km] approximately. The topobathymetric surveys were carried out with RTK GNSS equipment and ADCP measurements. Calibration was performed against average discharges recorded by the DGA (Dirección General de Aguas, Chile) gauging station during the topobathymetric survey campaign, with observed discharges ranging between 689 and 716 [m³/s]⁴⁸. Manning's roughness coefficients were adjusted reach by reach until the difference between simulated and observed water-surface elevations was less than 5 [cm], yielding calibrated values of n between 0.013 and 0.035 [s/m^{1/3}]⁴⁸. For the present study, the original 30 cross sections were interpolated at 25 [m] spacing to increase longitudinal resolution, retaining the n values assigned to each reach in the calibrated model.

Figure 5 shows the input variables to IPCDA, similar to the one presented for the trapezoidal channel. In this case we simulated the full length of the river using $Q = 2000$ [m³/s], a value close to a exceedance probability of 90% of maximum annual discharge. We assumed a constant lateral slope of $\theta = \pi/2$ as the width of the river is almost 2 [km] and the expected water depth is in the order of less than 10 [m]. We used the median values for all the cross sectional roughness coefficient n of the river, as the ones used in the HEC-RAS model.

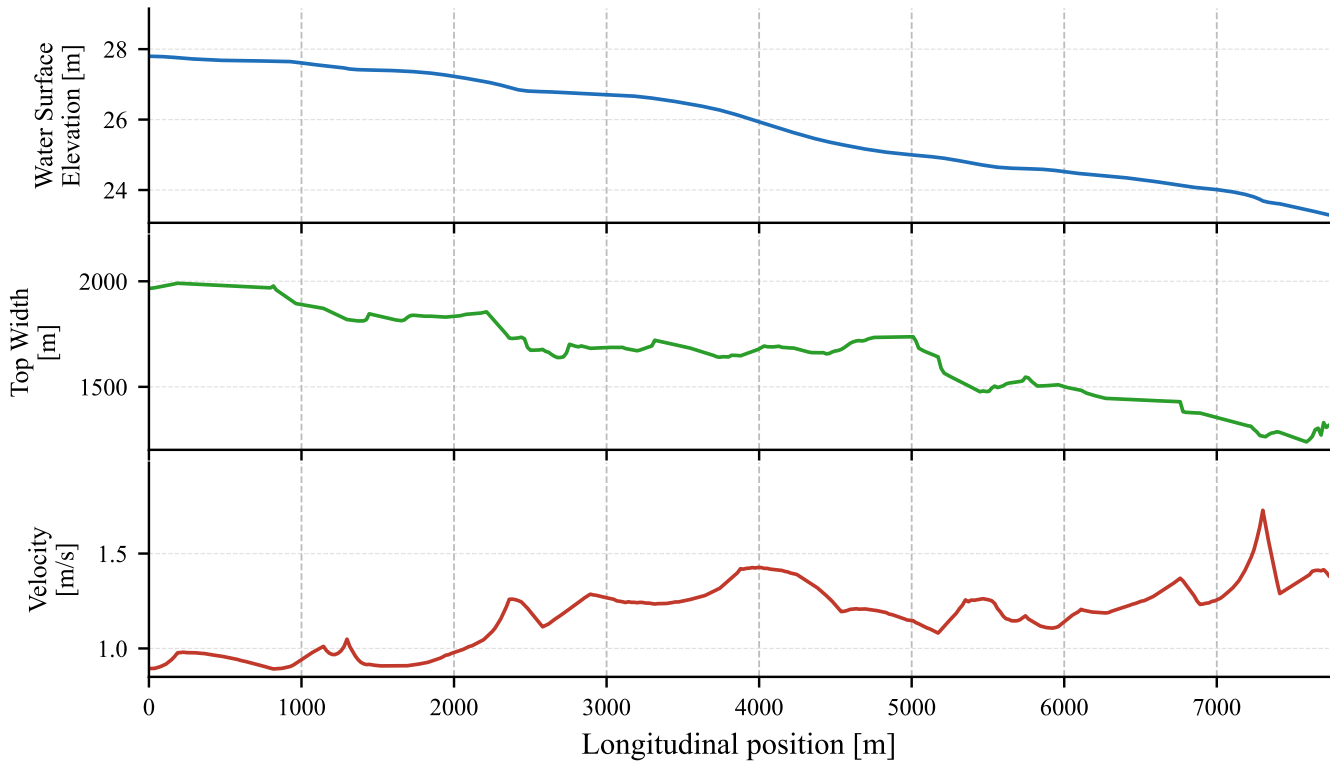


FIGURE 5 IPCDA input variables for the 7.5 [km] Biobío reach ($Q = 2000$ [m³/s]) from the calibrated HEC-RAS model at 25 [m] spacing: WSE (top), top width B (middle), and mean velocity U (bottom). Irregular profiles reflect natural channel geometry and spatially varying Manning roughness.

Figure 6 presents the local discharge estimates obtained from the Predictor phase along the Biobío reach for $\Delta WSE^{tol} = 5$ [cm]. The individual cell estimates Q_i show a considerable dispersion around the reference value of $Q_{ref} = 2000$ [m³/s], particularly the contiguous cells in the upper part of the reach (stations 0–3000 [m]), where the longer cell lengths reflect the mild WSE gradient and the flow is closer to uniform conditions. In contrast, the region near station 4000 [m] exhibits shorter cells and reduced dispersion between closer cells, consistent with a steeper local WSE gradient. The last section of the river, above position 5500 [m] show consistent values near 2500 [m³/s]. The boxplots on the right confirm that all six integration schemes produce comparable median estimates, consistent with the findings reported for the trapezoidal channel in Table 6. This robustness across integration schemes suggests that the Predictor-phase is not sensitive to the quadrature choice when applied to a natural river reach, and that the dispersion is primarily driven by the spatial variability of the hydraulic conditions rather than by numerical artifacts.

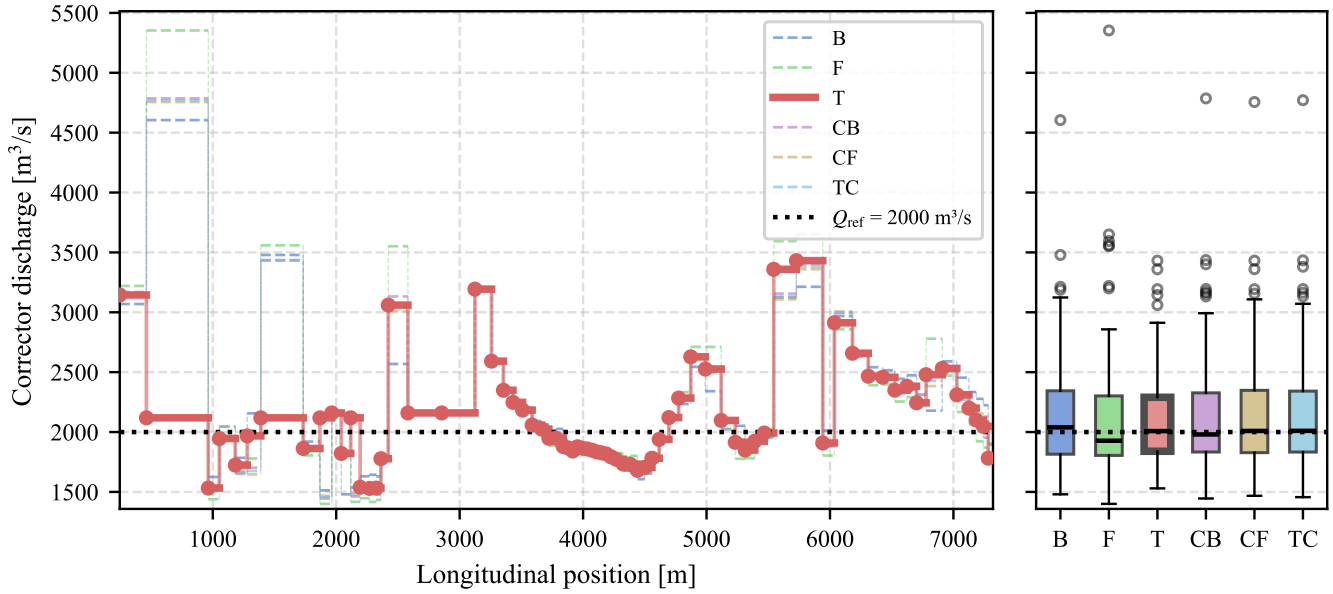


FIGURE 6 Predictor-phase normalized discharge estimation Q_i/Q_{ref} along the Biobío reach ($Q_{ref} = 2000$ [m³/s], $\Delta WSE^{tol} = 5$ [cm]), where $Q_i = (Q_{bw,i} + Q_{fw,i})/2$. The six integration schemes are superimposed; the trapezoidal rule is highlighted in red. Box plots on the right summarize the dispersion of each scheme across all evaluated ΔWSE^{tol} thresholds.

Once a Q_{pred} has been obtained from each integration rule, the value is iterated in the Corrector phase until a final discharge is reached, as in the trapezoidal channel case. Figure 7 shows the convergence of the trapezoidal corrector for the Biobío reach (full river, $\Delta WSE^{tol} = 5$ [cm]) under three weighting configurations of the objective function: a WSE dominated case ($\gamma_1/\gamma_2 = 0.2/0.8$), a balanced case ($\gamma_1/\gamma_2 = 0.5/0.5$) and a U^2 dominated case ($\gamma_1/\gamma_2 = 0.8/0.2$). All three curves share the same predictor seed of approximately 2120 [m³/s] and converge to Q_{corr} values within 1 and 7 % of Q_{ref} , although the numbers of iterations to get to the final value in each case changes from 1 iteration in the case of the WSE dominated model to 6 in the case of U dominated case (see Table 9).

Figure 8 illustrates the bathymetry reconstruction for a representative cross section of the Biobío river at RM 26.70 (Longitudinal position ≈ 1018 [m], $WSE = 27.60$ [m]), comparing the irregular HEC-RAS geometry with the IPCDA reconstruction obtained under the rectangular section assumption ($B(x) = P_w$). This simplification is well-justified for the Biobío reach, where the channel width approaches 2 [km] and the expected water depth is below 10 [m], so the wetted perimeter is dominated by the channel width and the hydraulic radius reduces to $R_h = A/P_w \approx h$, which is the standard wide-channel approximation widely adopted in hydraulic engineering for wide natural channels^{49,50,51}. The figure superimposes the HEC-RAS reference trapezoid built from the local hydraulic radius $R_h = 1.11$ [m] (solid red line) and three IPCDA reconstructions, one per objective-function weighting: $h = 1.10$ [m] for $\gamma_1/\gamma_2 = 0.2/0.8$, $h = 1.04$ [m] for $\gamma_1/\gamma_2 = 0.5/0.5$, and $h = 1.08$ [m] for $\gamma_1/\gamma_2 = 0.8/0.2$ (dashed lines). The three IPCDA depths span only ≈ 0.06 [m] across the full range of γ_1/γ_2 tested, and all three sit within ± 6 % of the HEC-RAS reference R_h at this section, confirming that the rectangular reconstruction successfully captures the mean depth of an irregular natural cross section. Combined with the equally narrow spread of the converged corrector discharges (Fig. 7), this

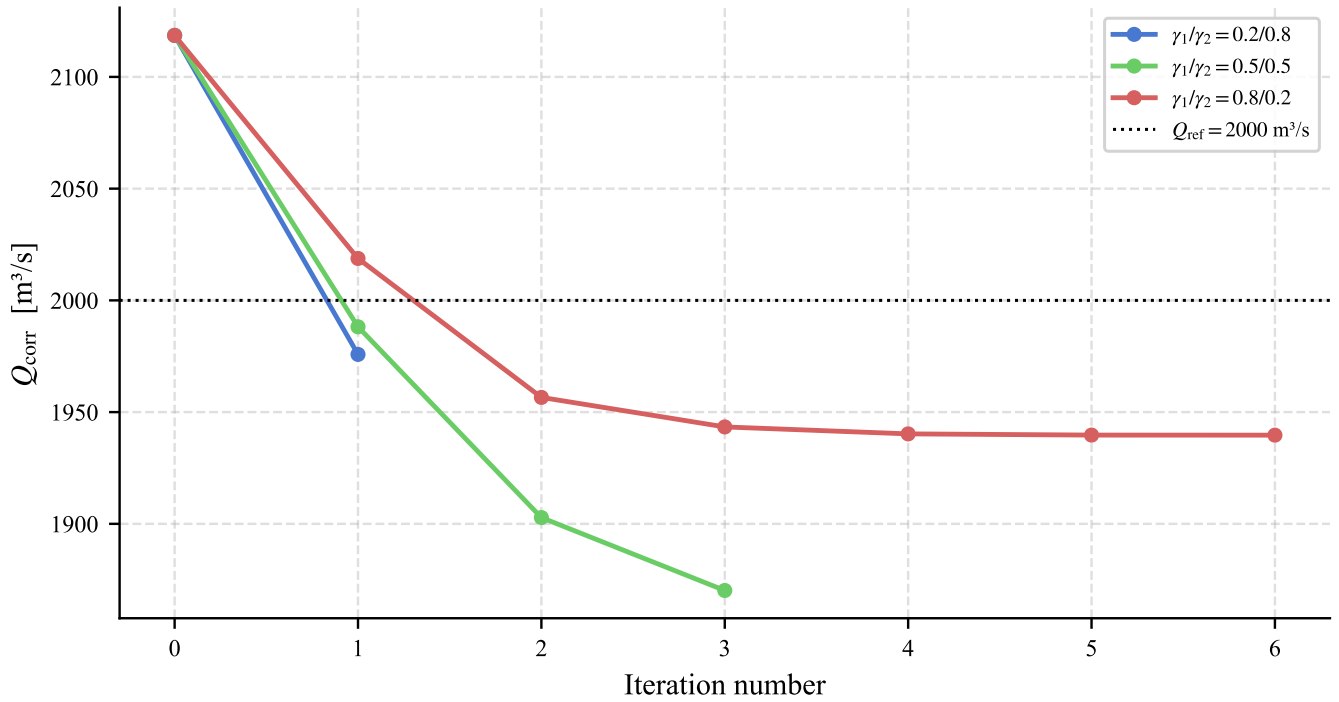


FIGURE 7 Corrector-phase convergence for the Biobío reach ($Q_{\text{ref}} = 2000$ [m³/s], full river, $\Delta WSE^{\text{tol}} = 5$ [cm]), evaluated with the trapezoidal quadrature for three weighting configurations of the objective function γ_1/γ_2 . The three curves share the predictor seed Q_{pred} .

result shows that both the global flow rate and the locally inferred bed depth are essentially insensitive to the choice of γ_1/γ_2 in the cost functional. For wide natural rivers, then, the rectangular assumption represents a practical and effective trade-off between geometric fidelity and model simplicity, and the IPCDA method recovers physically meaningful estimates of Q and h without requiring careful tuning of the weighting parameters.

5 | DISCUSSION AND REMARKS

The results presented in Section 4 are discussed here from three complementary perspectives. Subsection 5.1 addresses the practical conditions under which the method can be deployed, focusing on the measurement setup and the Working Grid construction. Subsection 5.2 examines the modeling choices and approximations that determine the mathematical structure of the algorithm, including the treatment of energy losses and the validity of the geometry expansion. Finally, Subsection 5.3 analyzes the numerical behavior of each phase of the algorithm and introduces a framework based on the sensitivity of parameters for estimating a plausible range of discharge values in the absence of a reference measurement. Taken together, these three perspectives provide a basis for assessing both the current capabilities and the known limitations of the IPCDA method, and for identifying the conditions under which each phase of the algorithm should be preferred as the primary output.

5.1 | IPCDA implementation: practical considerations

Onsite measurements conditions

The first assumption made is that data in a sparse ways is available, considering a long river reach. In our study cases, we used two river reaches of 2.6 [km] and 7.5 [km] long, although the IPCDA method can be applied using shorter reaches. In these cases we used the results of HEC-RAS simulations as input data for the Observation Grid (OG), having an almost continuous and regular grids, although in the first case we used a 1 [m] detailed mesh while in the second, an sparser grid with data separated every 25 [m]. In real cases, a drone can take images and videos flying along river maintaining its horizontality in order to keep

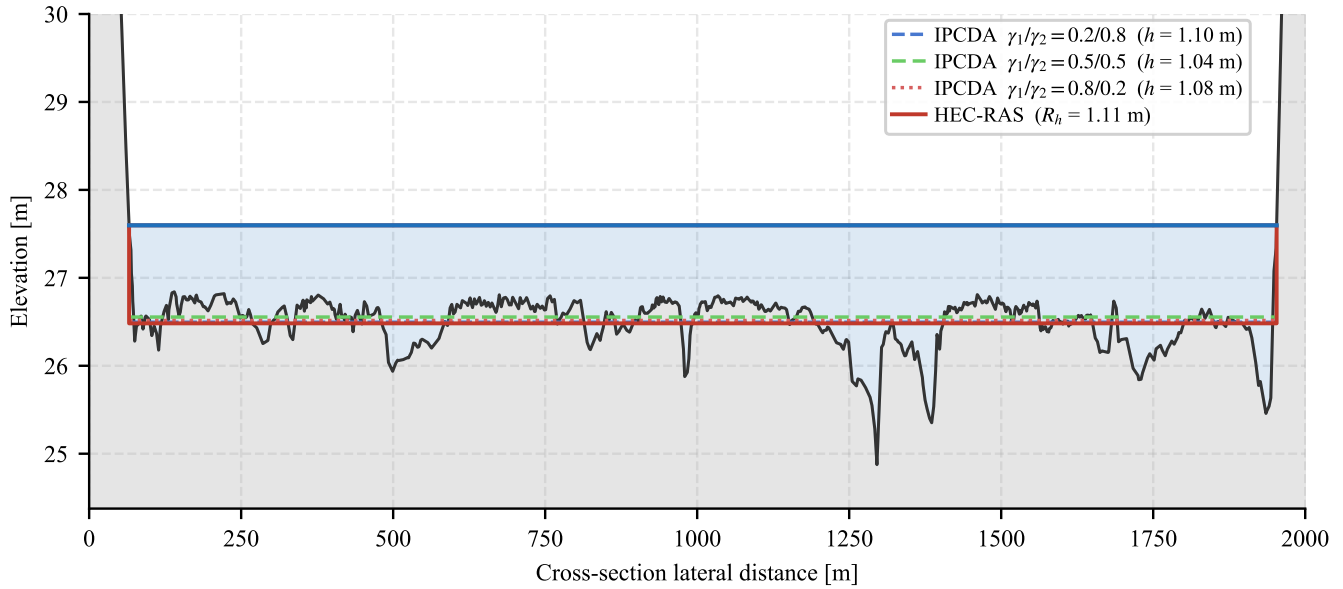


FIGURE 8 Cross-section comparison at RM 26.70 (Longitudinal position ≈ 1018 [m]) of the Biobío reach, $WSE = 27.60$ [m]. The HEC-RAS cross-section profile is shown in black with a gray fill, and the observed water surface is the blue line. The HEC-RAS reference trapezoid (solid red line, depth equal to the hydraulic radius $R_h = A/P$) is overlaid on the IPCDA rectangular reconstructions (dashed lines), one per objective-function weighting γ_1/γ_2 , with the corresponding inverted depths h reported in the legend. The closeness of all four trapezoids highlights the limited sensitivity of the IPCDA bed estimate to γ_1/γ_2 for wide natural rivers.

the known datum to calculate later different variables. To calculate the water surface elevation, topographic processing is needed to construct the elevation of the water surface similarly to the construction of digital elevation models, the surface velocity can be estimated using image velocity tracking or similar methodologies and the river width observation is a process that may need also human supervision to correct automatic estimations in the different reaches.

Lastly, in our IPCDA method we assume that the Manning's roughness coefficient and the lateral slope of the river are known parameters. This first can be estimated by text books depending on the characteristics of the riverbed, vegetation, etc., and the second, by visualization but also assuming that in rivers wide enough, the section could be set as rectangular. Although some uncertainty may arise in this sense, future works on this method should consider a range of possible values for these parameters. More comments on n values in Subsubsection 5.3.

Working Grid and water surface elevation

The Working Grid design criterion was based on technical limitations related to real world measurements. We tested detailed (1 [cm]) to coarser tolerances (10 [cm]) in the WSE drop value to assess the differences in the results. As expected, detailed grid permitted river segments with more cells to apply the methodology as in cases with tolerances of 5 [cm] or more, no cells could form.

It is worth noting that various techniques for generating surface velocity fields are available and can, in principle, be used as a basis to guide the spatial mesh design in this work. On the other hand, the balance indicated in Eq. (8) highlights the subcritical regime in which this work is developed, as well as the importance of accurately describing the WSE measurements. In this direction, the proposal presented in Subsection 2.3 is consistent with this dominant balance for the regime under consideration, where the extraction of information from the WSE variable provides a more robust basis for defining approximation criteria. In our case, this directly informs the discretization and, consequently, the numerical integration.

5.2 | Model formulation and simplifications

Energy losses representation

The nonlinearity in the discretization shown in Eq. (23) is treated by introducing suitable approximations for the integrals arising from the energy loss terms defined in Eq. (4). The criteria adopted in each case are guided by both modeling considerations and practical requirements. In this regard, the contributions associated with S_e are computed as fixed quantities using only surface information from the available observations, denoted in this work as OG (see Subsection 2.3). This approximation is introduced in Eq. (21). The rationale for neglecting the dependence on the flow variables lies in commonly reported results, which indicate that these contributions are typically bounded by at most 10% of the total energy losses. Furthermore, the use of OG is motivated by the dependence of these contributions on the coefficient K_e which varies with changes in the flow gradients. From a practical standpoint, the aim is to incorporate as much information as possible over extended channel reaches, since when only a limited number of nodes is considered, variations in the expansion coefficient may not be adequately resolved.

In the subcritical regimen, the contribution of the friction losses, $S_f = S_f(U, Q)$ in Eq. (18), is dominant (see Eq. (8)). Therefore, the approximation strategy follows that the dependence on the flow variable is explicitly retained. This criterion is applied both in the algebraic models of the predictor stage (see Subsection 3.1.1) and in the linearization procedure of the corrector stage (see Subsection 3.2.1).

Although the expressions for this contribution are obtained through numerical quadrature, the proposed approximation allows the term S_f to be expressed directly in terms of the model variables U and Q , while avoiding a strong dependence on the flow depth. The resulting expression is constructed using a truncated Taylor series expansion in terms of the dimensionless variable η (see Eq. (13)). Moreover, the proposed approximations and linearizations are consistent with the relative contribution of the terms in the governing balance equations. It is important to note that the approximation presented in Eq. (18) was derived from the expansion $\sqrt{1 - \eta} \approx 1 - \frac{\eta}{2} - \frac{\eta^2}{8}$, which was used in the representation of the flow depth in Eq. (16). Therefore, by retaining a larger number of terms in the truncated series expansion of $\sqrt{1 - \eta}$, higher-order contributions can be systematically incorporated. Although these terms are asymptotically smaller, they may prove beneficial in specific applications for improving the accuracy of the results, as they introduce additional degrees of freedom into the model. We are currently working on this type of problem as part of ongoing research, where we analyze the extension of the methodology to configurations that also treat the Manning coefficient as a calibration parameter. Further discussion is provided below.

Simplified algebraic model for the estimation of discharge Q and bathymetry $Z(x)$

These low complexity models were primarily formulated to address the Predictor-phase. This was useful to derive mathematical relations in terms of the discharge Q such that, by using surface information, it becomes possible to generate local estimates, i.e., approximations within a control cell $[x_{k-1}, x_k]$. A significant portion of Subsection 3.1 is devoted to this purpose.

In connection with the previous discussion regarding S_f , it is important to emphasize that the expressions derived to compute the estimates of Q are constructed by considering the approximation of the friction term given in Eq. (18). The resulting form of the equations depends both on the geometry of the cross-section and on the type of numerical integration employed within the cell $[x_{k-1}, x_k]$. To the best of our knowledge, and based on the literature reviewed so far, no approximation of S_f of the type proposed here has been reported. Therefore, all these estimates can be regarded as incremental contributions within the framework of low complexity models. Therefore, the steps presented in Section 2.2 allow for the generation of various estimates, as well as the improvement of the approximations by including a larger number of terms in the Taylor expansion (as discussed in the previous section).

Another point worth emphasizing is that the flow depth h in Eq. (16) depends on Q . Consequently, an estimate of this value yields an approximation of h , which, together with the relation in Eq. (6) and under the assumption that the water surface elevation (WSE) is known, implies that the bathymetry Z can also be estimated (see Fig. 4).

We further emphasize that an initial estimate can be obtained using the approximation derived for a rectangular channel configuration as a first guess, i.e., by employing $Q^{(0)}$ in each formula. In this sense, the proposed approach to solve the general local problem can be interpreted as a self-consistent approximation procedure, where the initial guess $Q^{(0)}$ is computed assuming a rectangular channel (see Subsubsection 3.1.2). Closed-form expressions for the discharge are presented in Eq. (45), and the results obtained are sufficiently accurate within the precision ranges typically expected in engineering applications. Further details are provided in Table 9 (presented in the following subsection), which highlights this point.

According to the simplified model approach, and in connection with the relevance of approximating P_w/Q as presented in Eq. (17), different pointwise approximations can be derived from the reduced model obtained by balancing the dominant terms shown in Eq. 8. This observation can be related to the analyses presented in Refs.^{30,34}, where estimations are developed using

low-complexity algebraic flow models. In this context, and in order to highlight the advantage of having the expression in Eq. (18), the following pointwise estimate is derived from Eq. (8):

$$-\left(\frac{Q}{n^{3/2}}\right)\left(\frac{1}{U^2}\frac{\Delta WSE}{\Delta x}\right)^{3/4} = BU + \frac{2(1 - \cos(\theta))}{\sin(\theta)}\left(\frac{Q}{B} + \frac{\cos(\theta)}{\sin(\theta)}\frac{Q^2}{UB^3}\right) \quad (66)$$

where the approximation is subject to the numerical derivative $\Delta WSE/\Delta x$ introduced in the left-hand side. From the simplified point-wise model in equation above, it is possible to observe the dependence of the expression on the ratio $(Q/n^{3/2})$, which highlights the need for additional relations that may assist in coupling models for the parameter adjustment problem, particularly with respect to Manning's coefficient. We are currently progressing with results in this direction for a forthcoming applications-oriented work.

In connection with the motivation outlined in the introduction, particularly regarding the interest in addressing a problem of apparent simplicity, the results obtained in the predictor stage provide reference values that may be useful for other inversion algorithms, such as those presented in Refs.^{34,20}.

Derivative-free linearization

Another approach that we wish to highlight for the construction of simplified models is the one developed in the Corrector-phase. This proposal was introduced in Subsubsection 3.2. It is worth reiterating that computational approaches for inverse problems in hydraulics, such as variational data assimilation (VDA) algorithms, typically require solving a system of nonlinear equations in order to formulate the adjoint problem associated with the Lagrange multipliers, which in turn introduces an additional system of differential equations. Although the resulting optimization problem is addressed using powerful numerical techniques, such as the L-BFGS algorithm⁵², it generally involves two discretization steps, thereby adding an extra layer of mathematical complexity (see Ref.¹⁷ and the references therein). In contrast, the quadratic programming formulation adopted in this work can be interpreted as a modified approach that provides a sufficiently consistent framework for tackling more complex problems. Therefore, the proposal presented in Subsubsection 3.2.1 constitutes one of the main contributions of this work.

We emphasize that this development is once again based on the approximation of S_f , highlighting the importance of constructing a model capable of linking the flow variables and parameters. Through this expression, together with the auxiliary representation in Eq. (29) and the identification of the dominant terms, it was possible to systematically manage (see Eqs. (31) and (49)) the decomposition of the terms for the formulation of expressions in terms of the variables of the optimization problem.

5.3 | Numerical remarks and implementation

Remarks on the roles of the Predictor and Corrector phases

As discussed above, the simplified models used in both phases are closely linked to the representation of the friction term S_f given in Eq. (18). However, their implementation differs significantly between phases, reflecting their distinct objectives.

The predictor approach aims to generate local point-wise estimates that provide preliminary bounds for the discharge Q . These values are obtained by solving algebraic equations in terms of Q , while the remaining terms are approximated using observational data. In contrast, the corrector approach does not necessarily enhance numerical accuracy directly; rather, by enforcing the energy balance equation at a global level, it ensures mass conservation. In this case, the proposal is to use observational data both to linearize the problem and to perform comparisons. This constraint is partially relaxed in the predictor phase in order to obtain efficient preliminary estimates.

Moreover, in many cases, the approximation $Q^{(0)}$, obtained by considering only the rectangular channel configuration, provides sufficiently accurate results within typical engineering ranges, with errors below 20% for this type of highly underdetermined problems. From the results presented, it can be readily observed that the estimates obtained using the predictor approach are generally quite satisfactory. In most cases, they achieve accuracies with errors below 10%, and in several instances they even outperform those obtained with the corrector approach. See, for example, Tables 5 and 7 for the trapezoidal channel experiment.

We reinforce the previous remark by presenting the flow-rate estimates obtained under the assumption of rectangular cross sections in Table 9 for the experiment analyzed in Subsection 4.2.

The results also show the influence of the weighting factors used to measure the contributions associated with the comparisons against the surface velocities and the water surface elevation. This highlights the role of fitting the computations to the observed WSE data, consistently with the balance analysis presented in Eq. (8).

TABLE 9 Summary of the IPCDA pipeline at the full river of the Biobío river for $Q_{\text{ref}} = 2000 \text{ [m}^3/\text{s]}$, $\Delta WSE^{\text{tol}} = 5 \text{ [cm]}$, by objective-function weighting γ_1/γ_2 . Discharge values are normalized by Q_{ref} .

γ_1/γ_2	$Q^{(0)}/Q_{\text{ref}}$	$Q_{\text{pred}}/Q_{\text{ref}}$	$Q_{\text{corr}}/Q_{\text{ref}}$	N_{iter}	OF [m]
0.2/0.8	1.00	1.06	0.99	1	7.21×10^{-5}
0.5/0.5			0.94	3	3.54×10^{-4}
0.8/0.2			0.97	6	4.40×10^{-4}

We reiterate that the objectives of the two approaches are different. While both aim to produce an estimate, the corrector phase has fewer degrees of freedom and therefore does not necessarily improve the approximation in terms of numerical accuracy. Instead, its role is to enforce consistency with the continuity equations. Therefore, the proposed approach is to treat each computational stage as having a distinct objectives, each providing satisfactory estimates on its own. Their combination, however, establishes a more reliable framework, as the results are obtained through different procedures and by employing diverse simplification strategies. This, in turn, enhances the robustness of the computed values. The above remarks are further detailed in the next section using the numerical results presented in Section 4.

A proposal for generating a discharge range estimation using the IPCDA

Giving that in a real case, we would not have a reference value of the discharge, we propose a framework to assess the mathematical and technological uncertainties to generate a range of possible discharges estimated by the IPCDA method, where the 50% percentile range over the median is taken.

Most of the results presented above were obtained using the same weights in the objective function, with $\gamma_1 = \gamma_2 = 0.5$, so that both observed variables, U^2 and WSE , contributed equally to the optimization process. The Manning's roughness coefficient was prescribed from the HEC-RAS model, using the corresponding value for the trapezoidal channel and the median value within each assessed reach for the Biobío River case. This assumption represents a case of perfect knowledge of the channel roughness characteristics along the flow domain. To assess the sensitivity of the method to these parameters, additional simulations were performed by varying the values of γ_1 and γ_2 and a sensitivity analysis over n . This analysis is the base for the discharge range estimation proposal.

Following the same approach used for the trapezoidal channel, the Biobío reach was also divided into three independent segments plus the full river length. In both cases, the boxplots pool results across all ΔWSE^{tol} values from 1 to 10 [cm], retaining only those thresholds for which a valid Working Grid could be formed in the corresponding segment. Figure 9 presents four subplot with three boxplots each, corresponding to the ratio $Q_{\text{corr}}/Q_{\text{ref}}$ for the trapezoidal channel (panels (a) and (c)) and the Biobío river (panels (b) and (d)), changing the γ values (upper) or the n Manning's values (lower) under the combinations of segment configurations and WSE tolerances. In each subplot, the middle boxplot represents the case with γ and n presented on the results and the left and right ones, changes in those original values (γ_1/γ_2 equals to 0.2/0.8 and 0.8/0.2 and $n=0.9n_0$ and $1.1n_0$).

For the case of the trapezoidal channel, a total of 9 and 21 cases fall below 0.8 when changing the γ 's and n respectively, and only five out of 99 simulations yield values close to zero, which are interpreted as failed runs (not shown by the zoomed plot). Notably, almost no estimates exceed the reference discharge, indicating a systematic tendency toward underestimation, an issue will be studied in future research. For the Biobío case, 40 and 34 cases over 120 are below and over the values of 0.8 and 1.2, showing a stronger dispersion of the results.

Considering all the data shown in Figure 9, the resulting discharge intervals at the interquartile range shown in Table 10 are 0.86-0.99 [m³/s] and 1772-2240 [m³/s], with median values of 0.96 [m³/s] and 2008 [m³/s] respectively. In both cases, the Manning roughness coefficient n exerts the strongest influence on the discharge estimates, producing a substantial amplification of the errors. In particular, the error increases by approximately a factor of 1.5 relative to the baseline case (check the derivatives in the lower row). Further works should be done over this parameter, to decrease the uncertainties in the results, but also the segment definition could play an important role on the uncertainty of the results.

Despite the dispersion introduced by parametric uncertainty in γ_1/γ_2 and specially by n , the results shown in Figure 9 and Table 10 confirm that the IPCDA methodology performs robustly across the tested configurations, with median estimates consistently close to Q_{ref} in both case studies. The interquartile range of the pooled distribution provides a practical measure of this uncertainty and its width reflects the combined sensitivity of the method to the weighting parameters and the roughness coefficient, while its central value offers a reliable point estimate of the discharge in the absence of a reference measurement.

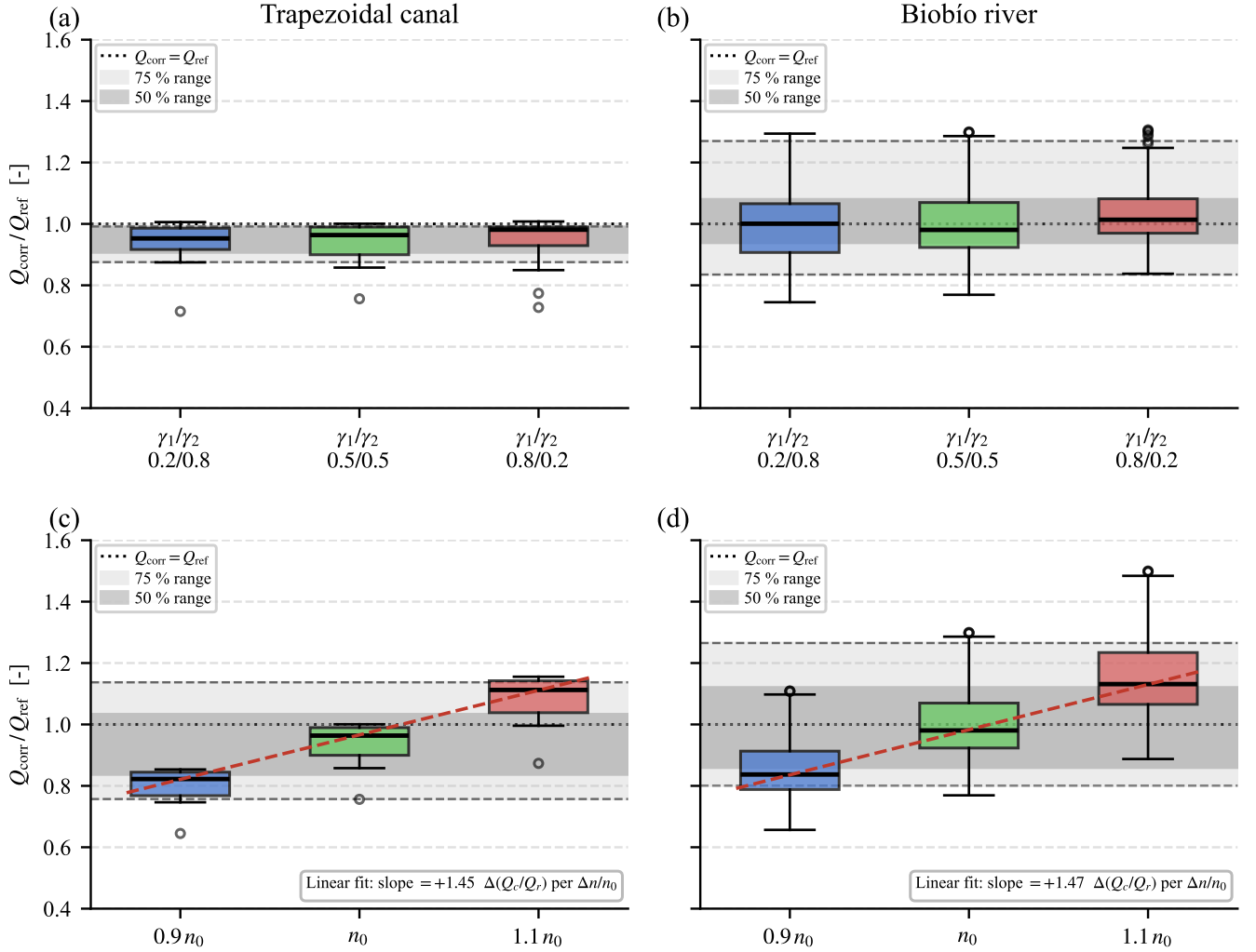


FIGURE 9 Sensitivity of the trapezoidal corrector $Q_{\text{corr}}/Q_{\text{ref}}$ to the objective-function weighting and to the Manning roughness, for the two test cases. Top row (panels (a) and (b)): weighting sensitivity using three configurations of the cost-function weights $(\gamma_1, \gamma_2) = (0.2/0.8, 0.5/0.5, 0.8/0.2)$; bottom row (panels (c) and (d)): Manning sensitivity using a multiplicative perturbation $\{0.9, 1.0, 1.1\} n_0$ of the baseline per-segment roughness n_0 . Each boxplot pools all valid combinations of the four segments (S.1, S.2, S.3 and full river) and the ΔWSE^{tol} from 1 to 10 [cm] thresholds. The light- and dark-grey bands enclose the central 75 % ($P_{12.5}$ – $P_{87.5}$) and 50 % (P_{25} – P_{75}) of the pooled distribution within each panel. The black dotted line marks $Q_{\text{cor}} = Q_{\text{ref}}$. In the bottom row, the red dashed line is the least-squares fit through the three box medians; its slope $\partial(Q_{\text{cor}}/Q_{\text{ref}})/\partial(n/n_0)$ is annotated in each panel. (a) Trapezoidal canal, $Q_{\text{ref}} = 1 \text{ [m}^3/\text{s]}$, γ 's weighting sensitivity. (b) Biobío reach, $Q_{\text{ref}} = 2000 \text{ [m}^3/\text{s]}$, γ 's weighting sensitivity. (c) Trapezoidal canal, Manning n sensitivity. (d) Biobío reach, Manning n sensitivity.

6 | CONCLUSIONS

The estimation of river discharge and bed topography in the absence of conventional gauging infrastructure remains a relevant open problem in hydraulic engineering. This work has shown that 1D surface observations of velocity, water surface elevation, and channel width contain sufficient information to recover both Q and $Z(x)$ through a structured inverse algorithm, provided that the underdetermination of the problem is explicitly acknowledged and addressed at each stage of the numerical procedure. The IPCDA method formalizes this idea, and the results obtained for both a controlled trapezoidal channel and a natural river reach confirm its potential for applications in data-limited environments.

The proposed two-phase Predictor and Corrector strategy of the IPCDA method demonstrated both robustness and accuracy. The Predictor-phase provides preliminary local discharge estimates from surface observations, whereas the Corrector-phase

TABLE 10 Percentile ranges of $Q_{\text{corr}}/Q_{\text{ref}}$ over the pooled distribution (3 sensitivity boxes $\times$ 4 segments $\times$ all valid ΔWSE^{tol} thresholds) for the trapezoidal canal ($Q_{\text{ref}} = 1 \text{ [m}^3/\text{s]}$) and the Biobío reach ($Q_{\text{ref}} = 2000 \text{ [m}^3/\text{s]}$). The Manning sensitivity perturbs n_0 by $\pm 10\%$; the γ_1/γ_2 sensitivity uses three weighting configurations (0.2/0.8, 0.5/0.5, 0.8/0.2) of the objective function. Last two columns give the number of cases outside the [0.8, 1.2] band.

Sensitivity	N	$P_{12.5}$	P_{25}	P_{50}	P_{75}	$P_{87.5}$	50 % width	75 % width	$N_{<0.8}$	$N_{>1.2}$
<i>Trapezoidal canal · $Q_{\text{ref}} = 1 \text{ [m}^3/\text{s]}$</i>										
$n (\pm 10\%)$	99	0.757	0.837	0.925	1.033	1.137	0.196	0.380	21 (21%)	0
γ_1/γ_2	99	0.876	0.907	0.970	0.990	0.992	0.082	0.116	9 (9%)	0
<i>Biobío river · $Q_{\text{ref}} = 2000 \text{ [m}^3/\text{s]}$</i>										
$n (\pm 10\%)$	120	0.801	0.860	1.007	1.120	1.265	0.261	0.464	14 (12%)	20 (17%)
γ_1/γ_2	120	0.835	0.939	1.001	1.080	1.270	0.141	0.435	10 (8%)	30 (25%)

restores the global consistency of the hydraulic model through an optimization-based formulation. In particular, the hypothetical setting considered in this study, where only surface data are available, revealed the intrinsic difficulties associated with underdetermined inverse problems, even under simplified assumptions. Despite the apparent simplicity of the 1D formulation, discharge and bathymetry estimation remains a highly challenging ill-posed engineering problem requiring the integration of physical modeling, numerical analysis, and observational data.

The numerical experiments indicate that the IPCDA algorithm achieves discharge estimation errors generally below 5%, highlighting its potential applicability in river environments with data limitations. Although the methodology was developed within a 1D framework and validated using synthetic experiments, the obtained results provide a solid basis for future extensions toward more realistic hydraulic configurations.

From a methodological perspective, the proposed approach introduces an alternative inversion strategy compared with previously reported methods. In particular, the linearization procedure does not rely on standard Taylor nor adjoint based formulations. Instead, it is constructed through factorization and substitution directly within the energy balance equation, inspired by semi implicit treatments of stiff terms in evolutionary problems. As a result, the associated quadratic programming problem simplifies part of the inversion structure while preserving consistency for more complex flow configurations. Furthermore, the Predictor and Corrector decomposition avoids the direct treatment of strongly coupled nonlinear systems, providing a flexible framework for incorporating observational data into the hydraulic reconstruction process.

Finally, this study opens several directions for future research. Ongoing work includes the incorporation of 2D synthetic models derived from real river data and the extension of the IPCDA framework to calibrated river models. Future developments will explore the sensitivity of the methodology to measurement errors and sensor placement, its performance under different hydraulic regimes and bathymetric configurations and roughness coefficient values. In addition, the integration of satellite-based observations, such as those provided by the SWOT mission, is expected to enhance the applicability of the proposed methodology in realistic hydrological contexts.

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APPENDIX

A COMPUTATIONAL ALGORITHM

The calculations performed in this first stage of preliminary results are shown in the following algorithm:

Algorithm 1 Local Direct-Numerical Inverse Problem: Rectangular Case

```

for  $i = 1$  to  $N - 1$  do
    Calculate  $I_{\text{num}}^{\text{obs}}[S_e]_i$  from Eq. (21)
    Calculate  $I_{\text{num}}^{\text{direct}}[S_f]_i$  from Eq. (24)
    Calculate  $Q_i^{(0)}$  by substitution from Eq. (45) and store the result
end for
Output: Global estimation  $Q^{(0)}$ , and arrays  $I[S_e]^{\text{obs}}$ ,  $I[S_f]^d$  containing  $I_{\text{num}}^{\text{obs}}[S_e]_i$  and  $I_{\text{num}}^{\text{direct}}[S_f]_i$ , respectively

```

The algorithm of the calculation generated in the Predictor-phase is given by:

Algorithm 2 Local Direct-Numerical Inverse Problem

```

Input: Values  $I_{\text{num}}^{\text{obs}}[S_e]_i$ ,  $I_{\text{num}}^{\text{direct}}[S_f]_i$ , and  $Q^{(0)}$ ; numerical quadrature
for  $i = 1$  to  $N - 1$  do
    Obtain  $Q_i^{\text{pred}}$  by solving Eq. (36) and store the result
end for
Output: Array containing the local discharge approximation  $[Q^{\text{pred}}]$ 

```

The iterative procedure in the Corrector-phase is summarized as follows:

Finally, the total algorithm is given by:

Algorithm 3 Corrector Iterative Procedure

Input: Columns WSE^{obs} , W^{obs} , B , Q_{Total}^{pred}
 Define W^{data} , WSE^{data} , and Q^{data}
 Calculate $[A]_{obs}^{data}$ in Eq. (57) and b_{obs}^{data} in Eq. (55)
 Solve Eq. (59) and store Q_{it}^{corr}
 Comparison step
if results satisfy the requirement **then**
 Output
else
 Return to Step 2
end if
Output: $[Q^{corr}]$, array of all Q_{it}^{corr} obtained in the iterative procedure

Algorithm 4 IPCDA Algorithm

Input: Columns WSE^{obs} , W^{obs} , B
 Call **Algorithm 1** ▷ Predictor-Rectangular phase
 Call **Algorithm 2** ▷ Predictor-Phase
 Calculate a global estimate Q_{Total}^{pred} (e.g., using Eq. (46))
 Call **Algorithm 3** ▷ Corrector-Phase
Output: Predicted and corrected discharges Q_{Total}^{pred} , Q^{corr} , and bottom elevation $Z(x)$

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