A Non-Linear Ohm’s Law – A Phenomenological Approach

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Abstract

We deduce from the second Newton’s law the medium constitute Ohm’s law for an electrical flow in a medium with square power law for the conductor resistivity.

**Key words:** Non-Linear Ohm’s Law.
The elementary and phenomenological discussions of the electrical conductivity in mediums without recourse to Quantum Mechanics always have been made in a context where the effective medium resistance has an effective behavior of a linear damping term proportional to the flow’s velocity ([1],[2]). In this pedagogical comment we present an elementary approach on electrical conductivities in a medium with a non-linear quadratic polynomial damping term on the electron flow velocities [3].

Let us start our analysis from the second Newton’s law applied to a one-dimensional electron flow through a conductor with a square power law for the conductor resistance under the presence of a steady electric field (electron mass \( m_e = 1 \))

\[
\frac{dV}{d\tau} = eE - \nu V^2 \quad (1)
\]

\[
V(0) = 0 \quad (2)
\]

here \( V(\tau) \) is the one-dimensional electron velocity and \( I(\tau) = \rho AV(\tau) \) is the current flow where \( A \) is the normal cross section of the conduction and \( \rho \) is the electron charge density.

The solution of the Riccati equation (1) can be obtained through the substitution

\[
V(t) = \frac{1}{\nu} \frac{dy(t)/dt}{y(t)} \quad (3)
\]

with \( y(t) \) satisfying the linear equation

\[
\frac{d^2y(t)}{dt^2} - (\nu eE)y(t) = 0
\]

\[
y(0) = y_0; \quad y'(0) = 0 \quad (4)
\]

the solution of eq.(4) is easily given by

\[
y(t) = c_1 e^{(\sqrt{\nu eE})t} + c_2 e^{-(\sqrt{\nu eE})t} \quad (5)
\]

By performing the inverse transformation in eq.(3) we obtain the physical electron flow velocity

\[
V(\tau) = \sqrt{\frac{eE}{\nu}} \left( \frac{1 - e^{-2(\sqrt{\nu eE})t}}{1 + e^{-2(\sqrt{\nu eE})t}} \right) \quad (6)
\]

In a steady-state processes (the asymptotic \( t \to \infty \) limit), we get:

\[
V_s = \sqrt{\frac{eE}{\nu}} \quad (7)
\]
and leading, thus, to the non-linear Ohm’s law (note that $E = \hat{V}/d$), namely

$$I = \rho A \sqrt{\frac{e\hat{V}}{\nu d}}$$  \hspace{1cm} (8-a)$$

or

$$\hat{V} = I^2 \bar{R}$$  \hspace{1cm} (8-b)$$

with the effective electrical resistance

$$\bar{R} = \frac{\nu d}{\rho^2 A^2 e}$$ \hspace{1cm} (9)$$

At this point and just for completeness let us analyze the more general case with the microscopic friction law $\alpha V + \nu V^2$ ($\alpha$ and $\nu$ positive constants):

$$\frac{dV(\tau)}{d\tau} = eE - \alpha V - \nu V^2$$ \hspace{1cm} (10)$$

$$V(0) = 0$$ \hspace{1cm} (11)$$

By performing the Riccati transformation (3) in eq.(9) we obtain

$$\frac{d^2 y(\tau)}{d\tau^2} + \alpha \frac{dy(\tau)}{d\tau} - (e\nu) y(\tau) = 0$$ \hspace{1cm} (12)$$

with $y(0) = y_0$ and $y'(0) = 0$.

As a consequence we have the result

$$V(\tau) = \frac{\alpha}{2\nu} (\gamma - 1) \left[ 1 - A e^{\alpha \gamma \tau} \right]$$ \hspace{1cm} (13)$$

Here $\gamma = \left( 1 + \frac{4\nu E}{\alpha^2} \right)^{1/2}$ and $A$ is a constant. The steady solution is thus given by the $\tau \rightarrow \infty$ limit in the above equation

$$V_c = \frac{\alpha}{2\nu} (\gamma - 1) = \frac{1}{\nu} \left[ -\frac{\alpha}{2} + \frac{\alpha}{2} \sqrt{1 + \frac{4\nu E}{\alpha^2}} \right]$$ \hspace{1cm} (14)$$

and thus, giving the associated non-linear Ohm’s law

$$\bar{I} = \frac{\rho A \alpha}{\nu} \left[ -1 + \sqrt{1 + \frac{4\nu E}{\alpha^2}} \right]$$ \hspace{1cm} (15)$$
or to the generalization of eq(8-b):

\[ \hat{V} = \left( \frac{\alpha d}{\bar{ho}Ae} \right) \bar{I} + \left( \frac{\nu d}{\bar{ho}^2A^2e} \right) \bar{I}^2 \]  

(16)

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**References**

