A Non-Linear Ohm's Law – A Phenomenological Approache

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Abstract

We deduce from the second Newton's law the medium constitute Ohm's law for an electrical flow in a medium with square power law for the conductor resistivity.

Key words: Non-Linear Ohm's Law.

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The elementary and phenomonogical discussions of the electrical conductivity in mediums without recourse to Quantum Mechanics always have been made in a context where the effective medium resistence has an effective behavior of a linear damping term proportional to the flow's velocity ([1],[2]). In this pedagogical comment we present an elementary approach on electrical conductivities in a medium with a non-linear quadratic polinomial damping term on the electron flow velocities [3].

Let us star our analysis from the second Newton's law applied to a one-dimensional electron flow through a conductor with a square power law for the conductor resistence under the presence of a steady electric field (electron mass $m_e = 1$)

$$\frac{dV}{d\tau} = eE - \nu V^2 \tag{1}$$

$$V(0) = 0 \tag{2}$$

here $V(\tau)$ is the one-dimensional electron velocity and $I(\tau) = \rho A V(\tau)$ is the current flow where A is the normal cross section of the conduction and ρ is the electron charge density.

The solution of the Riccati equation (1) can be obtained through the substitution

$$V(t) = \frac{1}{\nu}\,\frac{dy(t)/dt}{y(t)} \tag{3}$$

with $y(t)$ satisfying the linear equation

$$\begin{aligned} &\frac{d^2y(t)}{dt^2} - (\nu eE)y(t) = 0 \\ &y(0) = y_0;\ y'(0) = 0 \end{aligned} \tag{4}$$

the solution of eq.(4) is easily given by

$$y(t) = c_1\, e^{\left(\sqrt{e\nu E}\right)t} + c_2\, e^{-\left(\sqrt{e\nu E}\right)t} \tag{5}$$

By performing the inverse transformation in eq.(3) we obtain the physical electron flow velocity

$$V(\tau) = \sqrt{\frac{eE}{\nu}}\,\left(\frac{1 - e^{-2\left(\sqrt{e\nu E}\right)t}}{1 + e^{-2\left(\sqrt{e\nu E}\right)t}}\right) \tag{6}$$

In a steady-state processes (the asymptotic $t \to \infty$ limit), we get:

$$V_s = \sqrt{\frac{eE}{\nu}} \tag{7}$$

and leading, thus, to the non-linear Ohm's law (note that $E = \widehat{V}/d$), namely

$$I = \rho A \sqrt{\frac{e\widehat{V}}{\nu d}} \tag{8-a}$$

or

$$\widehat{V} = I^2 \bar{R} \tag{8-b}$$

with the effective electrical resistance

$$\bar{R} = \frac{\nu d}{\rho^2 A^2 e} \tag{9}$$

At this point and just for completeness let us analyze the more general case with the microscopic friction law $\alpha V + \nu V^2$ (α and ν positive constants):

$$\frac{dV(\tau)}{d\tau} = eE - \alpha V - \nu V^2 \tag{10}$$

$$V(0) = 0 \tag{11}$$

By performing the Riccati transformation (3) in eq.(9) we obtain

$$\frac{d^2y(\tau)}{d\tau^2} + \alpha \frac{dy(\tau)}{d\tau} - (eE\nu)y(\tau) = 0 \tag{12}$$

with $y(0) = y_0$ and $y'(0) = 0$.
As a consequence we have the result

$$V(\tau) = \frac{\alpha}{2\nu}(\gamma - 1)\left[\frac{1 - Ae^{\alpha\gamma\tau}}{1 + Ae^{\alpha\gamma\tau}}\right] \tag{13}$$

Here $\gamma = \left(1 + \frac{4e\nu E}{\alpha^2}\right)^{1/2}$ and A is a constant. The steady solution is thus given by the $\tau \to \infty$ limit in the above equation

$$V_c = \frac{\alpha}{2\nu}(\gamma - 1) = \frac{1}{\nu}\left[-\frac{\alpha}{2} + \frac{\alpha}{2}\sqrt{1 + \frac{4e\nu E}{\alpha^2}}\right] \tag{14}$$

and thus, giving the associated non-linear Ohm's law

$$\bar{I} = \frac{\rho A}{\nu}\frac{\alpha}{2}\left[-1 + \sqrt{+1 + \frac{4e\nu E}{\alpha^2}}\right] \tag{15}$$

or to the generalization of eq(8-b):

$$\widehat{V} = \left(\frac{\alpha d}{\rho A e}\right) \bar{I} + \left(\frac{\nu d}{\rho^2 A^2 e}\right) \bar{I}^2 \tag{16}$$

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References

[1] Luiz C.L. Botelho, "Ohm's Law: a Path-Integral Study at Classical and Quantum Level", Notas de Física.

[2] M.A.L. Omar, "Elementary Solid State Physics", World Student Series Edition, 1975.

[3] Luiz C.L. Botelho, "Methods of Bosonic Path Integrals Representations – Random Systems in Classical Physics", Nova Science Publisher, N.Y., 2006.