

Eulerian Coherent Structure Sub-Grid Model (ECSM) for Large Eddy Simulation (LES)

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Abstract

This paper introduces a new mathematical formulation to Large Eddy Simulation (LES) sub-grid scale (SGS) modeling: the Eulerian Coherent Sub-Grid Model (ECSM). The presented formulation attempts to replace the instantaneous, algebraic calculation of eddy viscosity found in classical models with a deterministic Eulerian scalar field that tracks sub-grid flow intermittency. By treating sub-grid coherent structures as a transportable volume fraction (ϕ), the ECSM introduces flow memory, allowing turbulence to naturally advect, diffuse, and decay based on bulk flow history. Production is driven by the second invariant of the velocity gradient tensor (the Q -criterion) to intrinsically preserve attached laminar boundary layers without empirical wall-damping functions. Destruction is formulated by translating the Lagrangian geometric collapse trajectory of a coherent structure under local strain into a Eulerian volumetric drain rate. The model baseline empirical constants have been calibrated against a 0D volume-averaged Taylor-Green Vortex, establishing a foundation for future 3D spatial testing.

Nomenclature

ϕ	Dimensionless Eulerian volume fraction of sub-grid coherent structures
Q	Second invariant of the velocity-gradient tensor (Q -criterion)
P_ϕ	Production term for the coherent-structure scalar field
D_ϕ	Diffusion term for the coherent-structure scalar field
$ S $	Magnitude of the resolved strain-rate tensor
Δ	Characteristic LES filter width
ν_t	Sub-grid eddy viscosity
Sc_t	Turbulent Schmidt number
C_d	Collapse or destruction calibration constant
C_p	Production calibration constant
C_v	Baseline viscosity calibration constant

Introduction

When fluid equations are solved on a discrete numerical grid considering the LES framework, turbulent energy naturally cascades from large fluid swirls down towards smaller ones. If a grid cannot resolve the smallest eddies,

energy builds up artificially at the grid cutoff, leading to non-linear computational instability where the simulation fails or produce errors. Classical LES models, including the foundational Smagorinsky approach [1], bypassed the tracking of small-scale eddies by introducing a model that represents the collective mixing effect of small-scale eddies, effectively increasing the fluid's effective viscosity with an additional term- known as eddy viscosity. Foundational models such as Smagorinsky, algebraically predict the eddy viscosity based on the instantaneous magnitude of the resolved strain rate tensor. The amount of added sub-grid viscosity depends on two factors: grid mesh size (Δ) and local deformation rate ($|S|$). While practically robust, it is well documented that this localized algebraic assumption lacks flow memory and famously over-dampens physical shear layers [2, 3]. This particularly originates from utilizing the model constant such as C_s in Smagorinsky model which is treated as a single fixed number. In doing so, the model can apply too much artificial viscosity in smooth laminar zones, near solid walls, or during laminar-to-turbulent transition. Another modelling limitation of such model is the one-way energy drain which can only drain energy from the grid (acting as a sink term) and cannot represent backscattering issues known to be present in turbulent flows. The latter limitation is not addressed by the ECSM model, and it remains a pressing problem in turbulence modelling.

Over the decades of turbulence research and modelling efforts within LES framework, several dynamic sub-grid models have been developed. Examples include: the model proposed by Germano et. al. (1991) which allowed model coefficients to update itself in real time based on local flow state [2]. Similarly, the model proposed by Nicoud et.al (1999), famously known as Wall Adapting Local Eddy-viscosity (WALE) model attempts to address the limitations of Smagorinsky and Germano's dynamic model by introducing purely local, algebraic sub-grid model that can naturally adapt to solid walls and laminar-to-turbulent transition [3]. This model was a big practical leap in LES turbulence modelling and has been widely adapted in LES studies. Regardless of these significant improvements in sub-grid models, the models still came with limitations: grid sensitive constants, lack of adaptation to rotating flows and no guarantee of zero-viscosity condition in a pure shear flow. To address some of these challenges, topological approaches like Kobayashi's Coherent Structure Sub-Grid Model (CSSM) [4] was developed. Instead of relying on pure shear condition to ensure zero eddy viscosity, CSSM relied on the physical existence of coherent fine scale eddies that governs the energy dissipation in turbulence.

The model utilized a Coherent Structure Function (CSF) and successfully demonstrated that identifying coherent structures using the second invariant of the velocity gradient tensor (Q) can intrinsically shut off eddy viscosity in pure shear and strictly vanishes at the walls. The model was also extended using Coherent Structure Kinetic-Energy Model (CKM) using sub-grid kinetic energy which provided a unified formulation for both compressible and incompressible flows [4]. The developed topological sub-grid models which have been an improvement over previously developed algebraic and statistical models, however, lacked one critical aspect: tracking of a full life cycle of these coherent structures, e.g. the birth, life and death of these coherent structures (flow memory) which can allow to dynamically update the eddy viscosity based on the asymptote of its full life cycle.

ECSM framework attempts to bridge the gap between topological identification and the flow memory discussed above. Instead of instantaneous switching with designed function such as CSF, ECSM utilizes a continuous Eulerian scalar field (ϕ) to modulate turbulence, actively tracking the birth (rotational production), life (advection and diffusion), and death (strain-driven volumetric drain) of sub-grid coherent structures.

2. Physical Reasoning and Model Development

2.1 The Volumetric Drain Rate (The Sink Term)

The foundational innovation of the presented ECSM is its destruction term (ϵ_s). Initially, the model concept was synthesized by considering the Lagrangian distribution of coherent structures onto a background canonical TG vortex and tracking their geometric evolution as a function of local strain. Figure 1 demonstrates an example of the collapsing of coherent structures in the high shear region (red) and maintaining its size in the low shear region (blue).

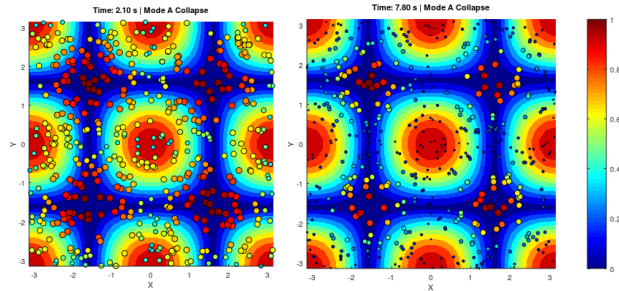


Figure 1: Demonstration of Geometric Evolution of Coherent Structures in a TG Vortex with Lagrangian Tracking at Two Different Time Steps: 2.2 seconds (left) and 7.8 seconds (right).

Classical mechanics dictates that a vortex filament's core radius (R) shrinks in direct proportion to the localized normal strain ($|S|$) acting upon it [5,6]:

$$\frac{dR}{dt} = -\beta|S|R$$

Where β is a kinematic proportionality constant bounding the geometric alignment of the vortex to the strain field. $|S|$ is defined as the characteristic magnitude of the resolved strain rate magnitude, defined conventionally as $|S| = \sqrt{2S_{ij}S_{ij}}$.

If the volumetric shrinkage rate of these coherent structures scales proportionally to shrinkage of area with respect to time as:

$$\frac{1}{V} \frac{dV}{dt} = \frac{1}{A} \frac{dA}{dt}$$

This assumption can leverage a concept of *volumetric energy drain* (the core idea of the ECSM model), which is introduced to move beyond 1D framework core radius shrinkage rate described for Burger vortex above.

To achieve this, the cross-sectional area of the coherent filament ($A = \pi R^2$) is evaluated. Applying the chain rule to the Lagrangian rate of change yields:

$$\frac{dA}{dt} = 2\pi R \left(\frac{dR}{dt} \right)$$

Substituting the radial trajectory into this equation provides the exact geometric collapse rate of the structure's area:

$$\begin{aligned} \frac{dA}{dt} &= 2\pi R(-\beta|S|R) = -2\beta|S|(\pi R^2) \\ \frac{dA}{dt} &= -2\beta|S|A \end{aligned}$$

Through scaling assumption:

$$\frac{dV}{dt} = -2\beta|S|V$$

Mathematically, the above relation shows that the destruction of a vortex filament's geometry is strictly proportional to its current size and the surrounding local strain.

2.2 The Eulerian Translation

For obvious reasons, tracking the discrete volumes of millions of moving chaotic vortex filaments is computationally unfeasible for industrial LES. Therefore, the ECSM translates this discrete Lagrangian framework into a continuous Eulerian one.

Let us define the continuous volume of these coherent structures as a dimensionless scalar field ($\phi \in [0,1]$). This variable ϕ represents the volume fraction of coherent structures within any specific grid cell. Combining the proportionality constant (2β) into a single empirical collapse calibration constant (C_c), the Lagrangian collapse equation translates to a continuous Eulerian volumetric sink term:

$$\epsilon_s = C_c |S| \phi$$

By integrating this formulated sink term, the CFD solver can actively deplete the scalar field (ϕ) along the geometric slope of the energy cascade, applying the physical viscous limit right before the sub-grid structure completely dissipates.

3. The Eulerian Transport Framework

The spatial and temporal evolution of the sub-grid scalar field is governed by an advection-diffusion transport equation:

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (u\phi) = \nabla \cdot (\Gamma_\phi \nabla \phi) + P_\phi - \epsilon_s$$

3.1 Production (P_ϕ) and Intrinsic Transition

Assuming that sub-grid structures are physically originated by macroscopic rotational shear and building upon the foundational logic of Kobayashi [4] including the kinematic and diagnostic framework introduced by Hunt et al. [6], the production term must be designed in a way such that it can distinguish between true swirling eddies and pure laminar wall shear. ESCM model considers the production of coherent structure is driven by the positive threshold of the second invariant of the velocity gradient tensor (Q -criterion), bounded by a saturation limit ($1 - \phi$) to prevent unphysical scalar accumulation in a Eulerian grid:

$$P_\phi = C_p \sqrt{\max(Q, 0)} (1 - \phi)$$

The square root is considered for dimensional consistency in the transport equation. The physical relevance of the defined production term is: in a purely attached, 2D laminar boundary layer, shear strain and rotation balance

perfectly, results in $Q \leq 0$. Consequently, $P_\phi = 0$. This ensures that the current capabilities of SG model are preserved in the developed model and the solver becomes intrinsically aware of laminar states, leaving attached boundary layers perfectly untouched without the need for empirical wall-damping functions.

3.2 Sub-grid Eddy Viscosity

Adapting the conventional scaling of classical SGS models, ECSM updates the sub-grid eddy viscosity by dynamically scaling it as a function of the localized concentration (volume fraction) of the transported coherent structure field:

$$\nu_s = C_v \Delta^2 |S| \phi$$

Where C_v is the baseline viscosity constant, Δ is the characteristic LES filter width, and $|S|$ is the resolved strain rate magnitude. If $\phi = 0$ (such as in a stable boundary layer), the eddy viscosity evaluates strictly to zero, natively preserving the physical flow. This simple idea can track the evolving eddy viscosity based on the state of the flow.

3.3 Diffusion (Γ_ϕ)

The diffusion of the sub-grid intermittency is modeled using a standard turbulent Schmidt number (Sc_t) approach, where:

$$\Gamma_\phi = \nu_{laminar} + \frac{\nu_s}{Sc_t}.$$

While sub-grid coherent structures are active topological features rather than passive scalars, adopting a turbulent Schmidt number analogy (with $Sc_{t,\phi} \approx 0.7 - 1.0$) provides a robust, computationally efficient closure consistent with established 1-equation transport frameworks.

4. Calibration Methodology

To establish baseline empirical constants (C_c, C_p, C_v), a 0-D volume-averaged surrogate framework was utilized, modeling the canonical Taylor-Green Vortex (TGV) transition at $Re = 1600$. The calibration was performed using an Octave script.

By integrating the scalar transport equation against the prescribed macroscopic strain and rotation fields of the TGV, the constants were optimized to minimize the L_2 error norm against Direct Numerical Simulation (DNS) spectral energy decay benchmarks established by Brachet et al. (1983) [7] as shown in Figure 2. The model successfully reproduced the exact peak dissipation rate of $\epsilon \approx 0.016$ at dimensionless time $t \approx 9$.

Calibrated baseline parameters for initial testing are:

- $C_c \approx 0.80$ (Collapse Constant)
- $C_p \approx 0.15$ (Production Constant)
- $C_v \approx 0.10$ (Viscosity Constant)

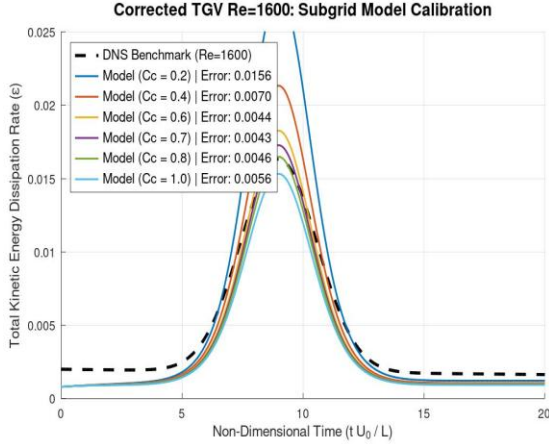


Figure 2: Calibration of Model Coefficient Considering TKE Dissipation Evaluated with 0-D DNS Surrogate TG Vortex

5. Discussion, Transparency, and Next Steps

It is imperative to state that the ECSM is currently a mathematical framework calibrated against 0-D surrogate canonical TGV. While the underlying physical description can mathematically resolve historic LES limitations, specifically by introducing flow memory and utilizing the Q -criterion for native transition, the model has explicit limitations that must be addressed during grid-scale implementation:

1. **Backscattering:** The current framework assumes a strict forward energy cascade. Backscattering (the inverse transfer of energy from sub-grid scales back to resolved scales) is neglected. The destruction term is strictly positive.
2. **3D Spatial Discretization:** The calibration constants were derived in a 0D flow environment. When implemented in a 3D discretized mesh, grid filter commutation errors and spatial truncation will likely alter the delicate balance between production and destruction. It is highly likely that the constants (C_c, C_p, C_v) will require recalibration against 3D DNS data.
3. **Numerical Bounding:** While the mathematics bound ϕ between 0 and 1, numerical solvers will require

explicit clipping at these limits to prevent unphysical scalar divergence (undershoots and overshoots) during highly transient iterations.

Future Work:

The model framework is now ready for spatial testing. The immediate next step involves implementing the ECSM via User-Defined Scalars (UDS) and User-Defined Functions (UDF) within the Ansys Fluent framework to evaluate its performance on a 3D transitional boundary layer grid and further tuning required. As with any turbulence model moving from theory to the discretized mesh, the model is expected to go through rigorous process of tuning and validation. The model presented here is in the early stage of development.

References

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