

The Moment Method for the Seismic Inverse Problem

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Abstract

We propose a new moment technique for the inverse problem of Wave Propagation in an inhomogeneous (half – space) medium in the acoustical Wave field Born Approximation by using the Analytical Regularization Scheme of Bollini, Giambiagi and Dominguez.

Key-words: Seismic; Inverse Problem.

One of the most challenging problem in the inverse methods for the Geophysical problem of imaging an inhomogeneous medium described by a depth – variable refraction index $M(z)$ ($0 \leq z < \infty$) from the backscattered acoustical wave field in the Born – approximation is the problem of its ill – posedness ([1] – pag. 323). In this Brief Report we will propose a solution of this difficulty by considering the Analytical Regularization Scheme previously used in Quantum Field Theory ([2]) to reduce the imaging of the medium to the well-know problem of determining a function by knowing its associated moments ([3] – Theorem 15.26).

Let us start our analysis by considering the corresponding Fredholm integral equation relating the medium refraction index ($M(z)$ ($0 \leq z < \infty$)) to the Born approximated backscattered acoustical Wave Field $U_S(\xi, \xi; w)$ on the medium boundary ($\xi = (x, y) \in R^2$, $-0 < z < +\infty$) and written in the frequency domain

$$U_S(\xi, w) = (w)^2 \cdot \int_{-\infty}^{+\infty} d^2\eta \int_0^{\infty} dz M(z) \cdot \exp \left[2iw \sqrt{(\eta - \xi)^2 + z^2} \right] \left((\eta - \xi)^2 + z^2 \right)^{-1}. \quad (1)$$

Unfortunately, the direct solution of the Fredholm Integral Equation eq.(1) to evaluate $M(z)$ is an ill - conditioned mathematical problem since its kernel is not square integrable and thus, defining a non – bounded operator in the Square Integrable functions which by its turn make this operator loses its inverse continuity in this function space.

In order to overcome such mathematical problem, we follow an usual procedure borrowed from similar distribution multiplication problem in quantum Field Theory ([2]). We propose to consider an analytically continued kernel in eq.(1) where the problem of ill – posedness is absent and at the ed of the calculation, we make an analytical continuation of the result to the Physical Problem as first proposed by Bollini, Giambiagi and Dominguez in 1964. Let us, then, study the analytically continued problem

$$U_S^\alpha(\xi, \xi; w) = (w)^2 \cdot \int_{-\infty}^{+\infty} d^2\eta \int_0^{\infty} dz m(z) \cdot \exp \left[2iw \sqrt{(\eta - \xi)^2 + z^2} \right] \left((\eta - \xi)^2 + z^2 \right)^{-\alpha/2} \quad (2)$$

where α is chosen in such way that it guarantee its square integrability. In this region of α values, it is possible to interchange the $\eta - z$ integration order by the use Fubini Theorem and evaluate explicitly the η -integration for $z > 0$ ([2]).

$$-\frac{U_S^{(\alpha)}(\xi, \xi, w)}{w_2} = \int_0^{\infty} dz m(z) \cdot \left[\sum_{m=0}^{\infty} \frac{(2i)^m \cdot (w)^m}{m!} \times \left(\pi \frac{\Gamma \left(\frac{\alpha-m}{2} - 1 \right)}{\Gamma \left(\frac{\alpha-m}{2} \right)} \cdot (z^2)^{+(1-\frac{\alpha-m}{2})} \right) \right] \quad (3)$$

where we have used the result

$$\int_{-\infty}^{+\infty} d^2p (p^2 + \beta^2)^{\frac{m-\alpha}{2}} = \pi \frac{\Gamma\left(\frac{\alpha-m-2}{2}\right)}{\Gamma\left(\frac{\alpha-m}{2}\right)} (\beta^2)^{-\left(\frac{\alpha-m-2}{2}\right)}. \quad (4)$$

By considering now the existence of power expansion associated to the analytically continued Back-Scattered Acoustic Wave Field (which is mathematically the correct assumption at least for large time), since $w \rightarrow 0^+$ in low frequency regime ([4])

$$\frac{-U_S^{(\alpha)}(\xi, \xi, w)}{w^2} = \sum_{m=0}^{\infty} \frac{U_m^{(\alpha)} \cdot w^m}{m!} \quad (5)$$

and comparing with the power expansion of eq.(3), we get the moment problem relationship

$$U_m^{(\alpha)} = \frac{(2i)^m}{\Gamma\left(\frac{\alpha-m}{2}\right)} \cdot \pi \cdot \Gamma\left(\frac{\alpha-m-2}{2}\right) \left(\int_0^{\infty} dz \cdot m(z) \cdot z^{m+2-\alpha} \right). \quad (6)$$

At this point we implement the Physical Limit of $\alpha \rightarrow 2$ (see eq.(1) and taking into account that $U_0^{(\alpha-2)}$ has an infinite piece

$$\begin{aligned} U_0^{(\alpha-2)} &= \int_0^{\infty} dz \cdot m(z) \cdot \lim_{\alpha \rightarrow 2} \left(\frac{2}{\alpha-2} \right) \cdot z^{2-\alpha} \\ &= -2 \left(\int_0^{\infty} dz m(z) \right) \lim_{\alpha \rightarrow 2} \left(\frac{1}{\alpha-2} \right) - 2 \int_0^{\infty} dz \cdot m(z) \lg z \end{aligned} \quad (7)$$

The infinite piece is disregarded in our approach by supposing that the “medium área” $\int_0^{\infty} m(z) dz$ vanishes identically. Otherwise eq.(7) must be understood as a finite-part prescription.

By grouping together eq.(6)–eq.(7) we reduce the problem of imaging the refraction index $m(z)$ to the solution of the well-know Moment Problem of determining a function from its moment of order p

$$U_{(p>0)}^{(2)} = (2i)^p \pi \overbrace{\int_0^{\infty} dz \cdot m(z) \cdot z^p}^{:=m(p)} \quad (8.a)$$

$$U_{(0)}^{(2)} = -2 \cdot \int_0^{\infty} dz m(z) \lg(z) dz \quad (8.b)$$

In the case that $m(z)$ has a Fourier Transform which is an analytical function around the origin in the Fourier Domain (for instance when $m(z)$ has compact support), we can solve formally eq.(8.a). The validity of this claim is a result of the equation below

$$\tilde{m}(k) = \frac{1}{2\pi} \int_0^{\infty} dz \cdot e^{ikz} m(z) = \frac{1}{2\pi} \sum_{p=0}^{\infty} \frac{(i)^p k^p}{p!} m^{(p)} \quad (9.a)$$

with

$$\begin{aligned}
 U_{(m>0)}^{(2)} &= (2i)^m \pi \int_0^\infty dz \cdot z^m \left(\int_{-\infty}^{+\infty} dx e^{ikz} \tilde{m}(k) \right) \\
 &= (2i)^m \pi \cdot \frac{1}{(i)^m} \cdot \left(\frac{d^m \tilde{m}(k)}{dk^m} \right)_{k=0}
 \end{aligned}
 \tag{9.b}$$

As a consequence, we have the final result connecting the coefficients of eq.(9.a) and eq.(5).

$$U_{(m>0)}^{(2)} = 2^m \cdot \pi \cdot m_{(m)}.
 \tag{10}$$

Finally, we remark that the above mathematical operations hold true at least if $|m_{(m)}| \leq e^{-m} \cdot C_m$, where C_m are the Taylor coefficients of an analytical real function with the property of $\Sigma C_m^2 < \infty$.

References

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