Some Exact Formulae for Performing Wave Field Extrapolation in Constant-Velocity and Depth-Dependent Mediums

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Abstract

We deduce well-posed formulae for wave field extrapolation in depth-dependent mediums.

1 Introduction

The basic starting point in the subject of wave field extrapolation and its application to seismic inverse procedures is the derivation of the acoustic pressure field $U(\vec{r}, z, t)$, developing in a medium (upper half-space $z > 0$) with a constant refraction index, from a known pressure field $U(\vec{r}, z = 0, t)$ at its surface. In section 2 of this short comment, we improve mathematically some of those results of ref. [1] by considering now the well-posedeness formulation of the problem.

2 The Depth-Extrapolation Prblem for a Medium with a Constant Refraction Index

Let us consider the acoustic wave field equation for a pressure field developing in a medium (upper half-space $z > 0$) defined by a constant refraction index and from a known pressure field data $U(\vec{r}, z = 0, t)$ but *added with depth derivative* at the surface, namely:

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2} - (k_x^2 + k_y^2)\right] U_{\vec{k}}(z; t) = 0 \tag{1}$$

$$U_{\vec{k}}(z,t)|_{z=0} = f_{\vec{k}}(t) \tag{2}$$

$$\frac{\partial U_{\vec{k}}}{\partial z}(z,t)|_{z=0} = g_{\vec{k}}(t) \tag{3}$$

where

$$U(\vec{k},z,t) = \int_{-\infty}^{+\infty} dk_x dk_y \exp(i\vec{k}\cdot\vec{r})U_{\vec{k}}(z;t). \tag{4}$$

Let us analyze firstly the case of non-evanescent waves defined by the condition $|\vec{k}|^2 < \frac{w^2}{v^2}$ (with $w \geq 0$). Here

$$U_{\vec{k}}(z;t) = \int_{0}^{\infty} dw \exp(iwt)U_{\vec{k}}(z;w). \tag{5}$$

The time-domain Fourier transformed field solution of eq.(1)–eq.(3) is explicitly given by

$$U_{\vec{k}}(z;w) = F_{+}(w;\vec{k})\exp\left[\left(i\sqrt{\frac{w^2}{v^2}-(\vec{k})^2}\right)z\right] + F_{-}(w;\vec{k})\exp\left[\left(i\sqrt{\frac{w^2}{v^2}-(\vec{k})^2}\right)z\right] \tag{6}$$

with

$$F_{+}(w;\vec{k}) = \frac{1}{2}\left(\hat{f}_k(w) - i\frac{\hat{g}_k(w)}{\sqrt{\frac{w^2}{v^2}-(\vec{k})^2}}\right) \tag{7}$$

$$F_{-}(w;\vec{k}) = \frac{1}{2}\left(\hat{f}_k(w) + i\frac{\hat{g}_k(w)}{\sqrt{\frac{w^2}{v^2}-(\vec{k})^2}}\right) \tag{8}$$

and

$$\hat{f}_k(w) = f_0^{\infty} dz \exp(iwt) f_k(t)dt \tag{9}$$

$$\hat{g}_k(w) = f_0^{\infty} dz \exp(iwt) g_k(t)dt. \tag{10}$$

Note that only on this situation of non-evanescent case, one could claim to obtain well-posed extrapolating formulae with our new condition eq.(3), opposite to those somewhat differents results presented in ref. [1] without this condition.

Let us thus introduce the following variable change in the Fourier-Integral eq.(4) for the acoustic pressure field: $w' = w$; $k_x = p_x w$; $k_y = p_y w$ ([1]). We thus, have the following result:

$$U(t,z;\vec{r}) = \int_{-\infty}^{+\infty} d^2\vec{k}e^{i\vec{k}\cdot\vec{r}} \int_0^{\infty} dw\, e^{iwt}(U_{\vec{k}}(z;w)) =$$

$$\int_{-1/c}^{1/c} dp_x \int_{-1/c}^{1/c} dp_y \int_0^{\infty} dw \cdot e^{iwt}(w^2)e^{iw(\vec{p}\cdot\vec{r})}$$

$$\left\{\left(e^{+iw\left(\sqrt{\frac{1}{c^2}-(\vec{p})^2}\right)z}\right)F_+(w,\vec{p}w) + \left(e^{-iw\left(\sqrt{\frac{1}{c^2}-(\vec{p})^2}\right)z}\right)F_-(w,\vec{p}w)\right\}. \tag{11}$$

The above expression by its turn is the sum of four Fourier integrals which are going to be analyzed. the first one is exactly given by the following expression (here $\hat{f}_{\vec{k}}(w) \equiv \hat{f}(w,\vec{k})$) [1]

$$U^{(1)}(t,z;\vec{r}) = \int_{-1/c}^{1/c} dp_x \int_{-1/c}^{1/c} dp_y \int_0^{\infty} dw \cdot w^2 e^{iw\left[t+\vec{p}\cdot\vec{r}+\left(\sqrt{\frac{1}{v^2}-(\vec{p})^2}\right)\cdot z\right]}\frac{1}{2}\hat{f}(w,\vec{p}w). \tag{12}$$

As in ref. [1], we can re-write eq.(12) as a depth-extrapolator integral operator (by means of a t-convolution integral)

$$U^{(1)}(t,z;\vec{r}) = \int_{-1/c}^{1/c} dp_x \int_{-1/c}^{1/c} dp_y R^{(1)}(z,t,\vec{r};p)(*)_t\tilde{U}(z=0,t,p) \tag{13}$$

where the surface observed pressure field is given by

$$\tilde{U}(z=0,t,p) = \int_0^{\infty} dwe^{+iwt}\hat{f}(w,\vec{p}w) \tag{14}$$

and the depth-extrapolation Kernel is written explicitly as ([2])

$$R_1(z,t,x,p) = \int_0^{\infty} dwe^{iwt}e^{iw\left(\vec{p}\cdot\vec{r}+\left(\sqrt{\frac{1}{v^2}-(\vec{p})^2}\right)z\right)}w^2$$

$$= -\pi\left(\frac{d^2}{d^2\xi}\left(\delta(\xi)+\frac{i}{\pi\xi}\right)\right)\Big|_{\xi=t+\vec{p}\cdot\vec{r}+\left(\sqrt{\frac{1}{v^2}-(\vec{p})^2}\right)z}. \tag{15}$$

The other part of the pressure field in eq.(11) corresponding to the knowledgement of the depth-derivative surface data eq.(3) and needed to turn the extrapolation problem a well-posed mathematical problem is given by an analogous Fourier integrals formulae

$$U^{(2)}(t,z,\vec{r}) = -i\int_{-1/c}^{1/c} dp_x \int_{-1/c}^{1/c} dp_y \int_{-\infty}^{+\infty} dw \cdot w^2 e^{iw\left[t+\vec{p}\cdot\vec{r}\left(\sqrt{\frac{1}{v^2}-(\vec{p})^2}\right)z\right]}\left(\frac{\hat{g}(w,\vec{p}w)}{w\sqrt{\frac{1}{v^2}-(\vec{p})^2}}\right)$$

$$= \int_{-1/c}^{1/c} dp_x \int_{-1/c}^{1/c} dp_y R^{(2)}(z,t,\vec{r};p)(*)_t\tilde{U}_t(z=0,t,\vec{p}) \tag{16}$$

[1] $\int_0^{\infty} e^{\pm ibt}dt = \pi\delta(b) \pm i\mathbb{P}\{\frac{1}{b}\}$

with

$$\tilde{U}_t(z=0,t,\vec{p}) = \int_0^\infty dw\, e^{iwt}\hat{g}(w,\vec{p}w)$$

and

$$R^{(2)}(z,t,\vec{r},\vec{p}) = \frac{1}{\sqrt{\frac{1}{v^2}-(\vec{p})^2}}\,\frac{\pi}{i}\left(\frac{d}{d\xi}\left(\delta(\xi)+\frac{i}{\pi\xi}\right)\right)\Big|_{\xi=t+\vec{p}\cdot\vec{r}+(\sqrt{\frac{1}{v^2}-(\vec{p})^2})z}. \tag{17}$$

The other two integrals are obtained by just changing $z \to -z$ in the above obtained formulae.

In the case of evanescent waves ([1]) defined by the condition $k_x^2 + k_y^2 \geq \frac{w^2}{v^2}$, the associated well-posed problem is governed by the following initial and boundary conditions imposed on eq.(1)

$$U_{\vec{k}}(z,t)\Big|_{z=0} = f_{\vec{k}}(t) \qquad \text{and} \qquad \lim_{z\to\infty} U_{\vec{k}}(z,t) = 0. \tag{18}$$

The solution takes, now, the following form

$$\begin{aligned} U^{(3)}(z,t,\vec{r}.\vec{p}) &= \int_{|\vec{p}|\geq\frac{1}{c}} d^2\vec{p}\int_0^\infty dw\, e^{iwt}\, w^2\, e^{iw(\vec{p}\cdot\vec{r})}\exp\left(-w\left(\sqrt{(\vec{p})^2-\frac{1}{v^2}}\right)z\right)\times\tilde{f}(w,\vec{p}w) \\ &= \int_{|\vec{p}|\geq\frac{1}{c}} d^2R^{(3)}(z,t,\vec{r}.\vec{p})(*)_t\tilde{U}_t(z=0,t,\vec{p}) \end{aligned} \tag{19}$$

with the evanescent depth-extrapolating Kernel ([2]), ([3]).

$$R^{(3)}(z,t,\vec{r}.\vec{p}) = -\pi\left(\frac{d^2}{d\xi^2}\left[\frac{i}{\pi\xi}\right]\right)_{\xi=t+\vec{p}\cdot\vec{r}+i\left(\sqrt{(\vec{p})^2-\frac{1}{v^2}}\right)z} \tag{20}$$

References

[1] M. Tygel and P. Hubral, "Constant velocity migration in the various guises of plane-wave theory", Surveys in Geophysics, 331–348, 1989.

[2] Luiz C.L. Botelho and R. Vilhena, Phys. Rev. E, vol 49, n.2, R1003–R1004, 1994.

[3] D.E. Miller, M. Oristaglio and G. Beylkin, Geophysics 52, 943–964, (1987).