

Choosing a Simple Stress-Conductivity Relationship for Shale

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Abstract. A fitted stress-conductivity curve can describe a loading branch accurately while remaining unreliable outside the fitted interval. This note compares exponential and power-law forms using a deliberately synthetic calibration point. Both reproduce the selected point but diverge at higher stress, illustrating why goodness of fit alone does not establish extrapolation quality. A practical model-selection procedure is proposed that separates loading history, exposure, specimen identity, and validation. No new experimental dataset or universally preferred correlation is claimed.

1. Purpose and candidate relationships

A simple correlation may be useful for interpolation, sensitivity analysis, or a reservoir-model input. These uses require different levels of evidence. Mechanistic contact modeling offers a richer description of closure [1], while non-local models address interactions across rough surfaces [2]. An empirical curve can approximate their output over a limited interval without reproducing the underlying state dependence.

Let $R = C/C_0$ and $x = (s_{\text{eff}} - s_0)/S$, where S is a stated positive stress scale and s_0 is the reference stress. Consider $R_E = \exp(-a x)$ and $R_p = (1 + b x)^{-n}$, with positive parameters. These forms are comparison candidates chosen for this note. They are not equations attributed to the preserved-shale experiments.

Choose the synthetic condition $R = 0.5$ at $x = 1$. Setting $a = \ln(2)$ fits the exponential. Setting $b = 1$ and $n = 1$ fits the power law. Both also give $R = 1$ at $x = 0$. Yet at $x = 4$ the exponential gives 0.0625 and the power law gives 0.20, a factor of 3.2 difference. The calculation is exact for these selected functions.

Kakkar's nitrogen-loading summary gives a specified stress interval and incomplete recovery [3]. The 2017 study includes fluid and cyclic effects [4], and the related 2019 paper confirms their importance for unproped behavior [5]. These findings motivate restricting any fitted relation to an identified branch and exposure state. They do not supply calibration for the synthetic curves in Table 1.

Table 1. Synthetic curves sharing $R(0) = 1$ and $R(1) = 0.5$.

Normalized stress x	Exponential: 2^{-x}	Power law: $1/(1+x)$
0	1.0000	1.0000
1	0.5000	0.5000
2	0.2500	0.3333
4	0.0625	0.2000

2. Parameterization and model flexibility

2.1 Stress scale and reference state

The normalized variable x uses a chosen stress scale S . If S is changed, the numerical coefficient in an exponential written as $\exp(-a x)$ must change to preserve the same dimensional relationship. A coefficient without its stress definition and scale is therefore incomplete. The quantity that controls a given dimensional stress increment is the ratio a/S , not a alone.

For illustration, let $S = 1,000$ psi and $a = \ln(2)$. The curve has retention 0.5 after a 1,000 psi increment. If S becomes 2,000 psi, the corresponding coefficient must become $2 \ln(2)$ to preserve that curve. Keeping a fixed while changing S would describe a different model. This is a unit-and-scaling example, not a recalibration to laboratory data.

The same care applies to the reference conductivity and reference stress. A model normalized at an already damaged state answers a different question from a model normalized before exposure. Preserve the reference measurement and its history when comparing coefficients across studies. Without that context, apparent differences in stress sensitivity may partly reflect different choices of normalization rather than different underlying behavior.

A reference-stress form makes the amplitude easier to interpret. For an exponential model, write conductivity as the measured-state scale multiplied by an exponential of the stress increment from a named reference. The amplitude then corresponds to that reference stress, rather than to zero stress far outside the observations. Centering does not add information or change the model family, but it can make the parameters and their uncertainty more directly connected to the experimental range.

For illustration, let $C = C_{\text{ref}} \exp[-a(s - s_{\text{ref}})]$, with a in inverse MPa when stress is in MPa. If C_{ref} is 0.5 mD-ft at 10 MPa and a is 0.05 per MPa, the predicted value at 20 MPa is approximately 0.303 mD-ft. This is a selected mathematical example. It does not calibrate shale behavior or establish that an exponential relationship remains valid beyond the fitted stress interval.

The reference value may be fitted or fixed to a measured observation; state which choice is made. Fixing it to one noisy observation can transmit that observation's error to every normalized point. Fitting it jointly introduces parameter uncertainty that should be retained. Stress-sensitive preserved-shale results motivate empirical relationships [3, 4, 5], but they do not remove the need to define the reference state, phase, and exposure history represented by the fitted amplitude.

2.2 Residual conductivity and identifiability

A model with a positive residual term can represent a curve that approaches a nonzero value at high stress. That flexibility may improve interpolation, but a fitted asymptote is not necessarily identified by data collected before the curve visibly flattens. A proposed comparison should examine whether different residual values produce nearly indistinguishable predictions over the measured range while diverging strongly outside it. If they do, the residual is an extrapolation assumption as much as a data-constrained parameter.

Consider the illustrative relationship $C = C_{\text{res}} + A \exp[-a(s - s_{\text{ref}})]$. When the exponential term remains much larger than C_{res} throughout the observations, the data may constrain their sum better than the residual alone. Conversely, measurements near an apparent plateau can still be limited by bypass flow or detection thresholds. Before interpreting the plateau physically, show that it lies above the relevant measurement limits and is reproducible under the stated test conditions.

Contact and rough-flow models motivate finite residual pathways under some configurations [1, 6], but they do not prove that every fitted positive asymptote represents a persistent connected channel. Compare the empirical residual with independent geometric or hydraulic observations where available. If such evidence is absent, label it as a model parameter and report a range of extrapolated outcomes rather than converting it into a uniquely measured high-stress fracture capacity.

One observed retention value cannot uniquely determine both b and n in the proposed power law. Many parameter pairs can reproduce the same point. A fitting routine may still return a pair, but that numerical result does not establish that both parameters are constrained by the observations. Report whether the data span enough of the curve to distinguish changes in shape.

A proposed diagnostic fixes one parameter over a plausible range and refits the other, then examines how much the predictions change inside and outside the calibration interval. If many pairs give nearly identical fits but divergent extrapolations, the extrapolation is poorly identified. Additional measurements should target stress regions where those candidate curves differ and where the experiment can resolve the difference.

Mechanistic models can impose additional structure [1, 2], but they also introduce quantities that require constraints. Their complexity should be justified by the intended prediction and available observations. The choice is not between an automatically reliable physical model and an automatically unreliable empirical model. It is between descriptions whose assumptions, parameter support, and predictive limits can be examined explicitly.

3. Calibration and uncertainty

3.1 Fitting the experimental question

First identify which specimens, fluids, and loading branches belong together. Keep repeated publications of the same experiment in one data group. Second, retain absolute C_0 alongside the normalized observations. Third, choose an error model consistent with measurement uncertainty; logarithmic residuals emphasize relative deviations, whereas linear residuals emphasize absolute deviations. Report that choice and check behavior near the instrument's lower limit.

Use independent specimens or an intentionally withheld stress interval for validation. A random split of adjacent readings from the same loading curve can give an optimistic result because the observations share preparation and history. Barnett laboratory measurements provide an external experimental context [7], while rock-property experiments motivate conditioning on material and support descriptors [8]. Shale measurement procedures also matter when assembling comparable data [9].

A mechanistic propped-fracture calculation based on a population balance equation is an alternative modeling precedent [10]. It should not be transferred unchanged to an unpropped path. Likewise, the ideal cubic relation [11] and rough-fracture flow theory [6] constrain physical interpretation without selecting a unique empirical stress law.

3.2 Error models and measurement limits

A fitted stress-conductivity relationship depends on how discrepancies are measured. Minimizing squared errors in conductivity emphasizes absolute deviations, while minimizing squared errors in its logarithm emphasizes relative deviations for positive values. Neither objective is automatically preferable. A proposed analysis should select the error representation from the measurement behavior and prediction purpose, then state it alongside the fitted equation. Otherwise, two analysts can obtain different coefficients from the same observations without actually disagreeing about the underlying data.

For a synthetic comparison, predicting 0.2 instead of 0.1 mD-ft creates an absolute error of 0.1, whereas predicting 2 instead of 1 creates an error of 1. Both predictions are a factor of two high. A fit in ordinary conductivity treats the second discrepancy as much larger under squared

error; a logarithmic fit gives the same log discrepancy to both. The example shows why a visually good high-conductivity fit may perform poorly near the low-conductivity endpoint.

Values at or below a measurement limit require separate treatment because the logarithm of zero is undefined and an arbitrary positive replacement changes the objective. Preserve the limit and use a fitting approach consistent with that information, or report a bounded evaluation without claiming a fully resolved fit. The proposed model-selection record should include sensitivity to reasonable error assumptions so that apparent superiority does not depend entirely on one undocumented transformation.

4. Validation across specimens and histories

4.1 Independent specimens

Measurements along one loading curve share a specimen, preparation history, baseline, and apparatus sequence. Randomly distributing neighboring points between training and validation sets can leave these shared influences in both sets. The resulting prediction error then measures interpolation along a familiar trajectory more than transfer to a new specimen. That may be useful, but it should be labeled correctly.

A stronger proposed test withholds entire specimens, or an entire exposure condition if transfer across that condition is the intended claim. Report both absolute-conductivity and normalized-retention errors. A model can predict the shape of a normalized curve while missing its baseline capacity, or predict absolute values adequately over a narrow interval while misrepresenting the stress trend.

Inspect residuals against stress, time, and specimen identity. A systematic pattern can reveal an omitted branch or state variable. The preserved-shale evidence motivates such variables [3, 4, 5], while independent laboratory procedures provide context for measurement comparability [7, 9]. Validation should examine the prediction actually needed, not simply maximize a fit statistic on the observations already used to construct the model.

4.2 Loading branches and representation limits

A single-valued function of stress cannot represent different loading and unloading conductivities at the same stress unless another state variable is added. Fluid exposure may require a separate branch or state parameter; water-damage experiments provide a relevant example [12]. Preservation history should remain a dataset attribute [13]. Joint behavior models offer a broader framework for path dependence [14].

When the intended use is field interpretation, compare the admissible parameter range with independent constraints such as diagnostic fracture injection test analysis [15]. Do not treat agreement with one pressure transient as unique identification of conductivity. Report an

extrapolation envelope by comparing plausible forms, and show the actual calibration interval on every prediction plot. Table 1 demonstrates how form uncertainty can survive an apparently perfect small-data fit.

A single-valued function of current stress predicts the same conductivity whenever the same stress is revisited. A loading-unloading dataset provides a direct test of that assumption. Compare observations at matched stress and documented hold times, retaining the order of the measurements. If the return branch differs beyond measurement uncertainty, a current-stress-only model cannot reproduce both branches without an additional history variable or separate branch definitions.

For selected values, suppose conductivity at 10 MPa is initially 0.5 mD-ft and becomes 0.2 mD-ft after a higher-stress excursion and return. A function assigning one value to 10 MPa cannot fit both exactly. Adding more curvature to the same function does not solve that structural problem. The model must either define different states, restrict its intended use to one branch, or acknowledge the irreducible discrepancy for the combined dataset.

The anchor studies report incomplete conductivity recovery and sensitivity to cycling [3, 4, 5]. They therefore motivate checking branch dependence before using an empirical stress curve in a changing-pressure simulation. A simple separate-branch model may be adequate for a narrowly specified sequence, but it also needs a rule for switching branches. Test that rule on a withheld excursion rather than assume that fitting the original loading and unloading curves establishes general history prediction.

5. Extrapolation and simulator use

5.1 Derivatives and implementation checks

A curve that predicts conductivity adequately can still provide a poor derivative for a coupled calculation. Numerical solvers may use the rate of conductivity change with stress to update pressure and flow. Inspect the derivative over the intended operating range, especially near branch switches or piecewise joins. A proposed handoff includes the equation, derivative convention, and treatment of points where the derivative is discontinuous or not defined.

For the exponential example without a residual term, the stress derivative is $-aC$. Its magnitude decreases with conductivity when a is positive. With a residual term, the derivative instead depends on the part above the residual. These mathematical distinctions can affect a coupled update even when the two curves appear close over a narrow interval. Report whether the implementation uses an analytical derivative or a finite difference and verify that its units match the simulator's stress unit.

A finite-difference check at several interior stresses can compare the implemented derivative with the change predicted by a small stress perturbation. Reduce the perturbation within a numerically stable range and inspect consistency. This is a verification of the implementation, not validation of the constitutive model. A simulator can implement an unsupported extrapolation perfectly, so code-level agreement should remain separate from evidence that the relationship describes the relevant shale state and loading history.

5.2 Prediction envelopes and model handoff

When several simple relationships fit the measured range comparably, retain their differences where predictions matter. A proposed model envelope evaluates each admissible relationship under the same future stress path and reports the resulting spread. The envelope represents the selected alternatives, not a statistical confidence interval unless a probability model justifies that interpretation. It is especially useful when a decision depends on stresses or exposure states not directly covered by the experiment.

A hypothetical pair of models might agree within 5% over the measured range yet differ by a factor of three at the highest proposed field stress. The laboratory fit alone would not resolve that extrapolation. A new measurement near the diverging predictions could be more informative than many additional points where the curves already coincide. The selected numbers illustrate an experimental-design principle and should not be presented as a measured uncertainty for any cited shale dataset.

The final model package should include calibration observations, held-out evaluations, stress and fluid domains, parameter uncertainty, and explicit behavior outside the accepted range. If extrapolation is disabled, document the stopping condition; if it is allowed, label the assumption and its sensitivity. A simple curve is useful when its scope is clear. Adding parameters is justified only when the available evidence or intended prediction requires the extra behavior and can meaningfully constrain it.

When a fitted relationship is passed to another analyst, include the equation, parameter units, reference state, calibration interval, applicable loading branch, and treatment of values outside the interval. State whether the curve is allowed to approach zero indefinitely or whether a residual term is used. A residual term should not be introduced solely to produce a desired field result; it requires supporting information.

Provide at least one plausible alternative form when extrapolation materially affects the conclusion. The synthetic exponential and power-law comparison in this article supplies an example of structural disagreement despite an exact shared calibration point. The spread between forms is a scenario comparison, not automatically a statistical confidence interval. Keep that distinction explicit in subsequent network or reservoir calculations.

A useful final statement identifies where the model interpolates supported observations and where it becomes a hypothesis. If an operational decision depends on the latter region, additional measurements or independent constraints are needed. The model remains valuable as a transparent approximation, provided its numerical convenience is not mistaken for evidence that shale conductivity follows one universal law.

6. Conclusions

Choose the simplest relation that answers the stated question within a supported interval, and retain model-form uncertainty where alternatives remain plausible. A curve fit is a compact description of specified observations, not proof of a universal shale law. Clear reference conditions, branch identity, validation by independent specimens, and explicit extrapolation limits are more informative than a high fit statistic alone.

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