

Phase-Intersected TPMS (PI-TPMS): Implicit Strut Networks from the Intersection Curves of Paired Level-Set Fields

Matthew Shomper

Not a Robot Engineering LLC, Xenia, Ohio, USA · matt@notarobot-eng.com · ORCID 0009-0002-5312-2864

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Abstract

Triply periodic minimal surfaces (TPMS) are usually turned into solids in one of two ways: by thickening a single surface into a sheet, or by filling one side of it to form a skeletal network. We introduce phase-intersected TPMS (PI-TPMS), in which the solid is instead a tube around the curve where two level-set fields vanish together. In the basic case the two fields are a TPMS and a phase-shifted copy of it. The construction is a single implicit inequality that can be printed directly from the field. Unlike a skeletal TPMS, whose strut graph is fixed by its parent surface, one surface pair yields a whole family of strut networks, selected by the phase shift. We give three forms of the construction and their cross-sections. We show that the tube volume is exactly $\pi^2 L$ below the overlap scale. The curve length L is measured per shift for TPMS pairs, and follows in closed form from the wave spectrum for stochastic fields, agreeing to 0.99 ± 0.01 over 43 spinodoid and Gaussian-random-field pairs. We prove that the intersection curve of a generic pair never branches. Network junctions therefore arise in only two ways. **Crossing junctions** form at isolated points where the two surfaces are tangent; they occur at symmetric shifts and connect the network at any tube radius. **Contact junctions** form where separate curves pass within a tube diameter of each other, which makes the topology depend on the radius. Across the 36 gyroid and Fischer–Koch S shifts on the eighth-cell grid, every shift without crossing points gives separate strands. Crossing points are needed for connectivity at vanishing radius, and they give a lattice when they join strands running in different directions. Where three-fold symmetry places a tangency at a flat point of the surface, the junction has six branches instead of four. Averaged over shifts, the curve length per cell equals the square of the parent surface's area per cell times the mean sine of the angle between normals, so a surface with more area per cell gives a denser network. Shifting along a body diagonal keeps a three-fold symmetry axis, which makes Young's modulus exactly the same in every direction across that axis; one such lattice is transversely isotropic to within 3%, the elastic symmetry commonly used to model bone. FFT homogenization of 35 cells reproduces every trend in independent homogenization data, with stiffness a median 5% lower, a gap that shrinks with grid resolution. The connected PI-gyroid is bending-dominated ($E \propto \rho^{2.1}$). At a matched thinnest feature it has the direction-averaged stiffness of a skeletal gyroid at 1.7 times the solid fraction, about one forty-fifth that of a sheet gyroid, and a shift-selected 4 : 1 ratio of in-plane to axial stiffness. At a fixed solid fraction the shift alone moves the lattice along a trade-off between mean stiffness and anisotropy; some shifts keep their anisotropy as density changes, several give negative Poisson's ratios, and stiffness becomes highly sensitive to tube diameter only where new tube contacts form. The construction is released openly, and the software used here is freely available for research.

Keywords: triply periodic minimal surfaces; implicit modelling; integral geometry; strut lattices; metamaterials; spinodoid; additive manufacturing; orthopaedic implants

1. Introduction

Triply periodic minimal surfaces have become a standard architecture for additively manufactured lattices, owing to their smooth curvature, full periodicity and compact level-set description [1–3]. In practice a TPMS level set $\phi(\mathbf{x}) = c$ is converted to a solid in one of two ways. A sheet (shell) solid keeps the band $|\phi| \leq t$ around the surface. A skeletal (network) solid keeps one side, $\phi \geq c$, and fills one of the two labyrinths the surface separates [2, 4]. As c approaches the extreme of ϕ , a skeletal solid thins into a network of struts following the labyrinth graph, the srs net in the case of the gyroid [5]. Its solid fraction then scales with the square of the strut diameter, as for any strut lattice. Its graph, however, is fixed by the parent surface. A designer who wants a different strut network, whether a different connectivity, different anisotropy, or different pore shapes, must switch to explicit beam lattices. Those need a graph, node geometry and meshing, and they give up the single-equation description that makes TPMS attractive. Within the single-field description, new architectures have mostly come from blending fields: weighted sums of several TPMS primitives interpolate continuously between their shapes, and have recently been searched experimentally for structures with extreme energy dissipation [21]. Blending yields a new level-set surface, and the solid is still built from that surface. PI-TPMS instead combines two fields by intersection, so the solid follows a curve.

The mechanical case for strut networks is well established. At low density, strut lattices deform by strut bending or strut stretching depending on their connectivity [6, 7], and bending-dominated networks give the low stiffness per unit mass needed where a structure must match a compliant tissue. In orthopaedic implants, porous metal that is much stiffer than the surrounding bone shields that bone from load and drives resorption [8]. Trabecular bone offers a biological template. Its microstructure spans a continuum from plate-like to rod-like elements, and the rod-like end dominates at low bone volume fraction and with age [9]. Sheet TPMS resemble the plate-like end, and strut networks the rod-like end.

This paper introduces a way to generate strut networks implicitly, with a topology that can be chosen, while keeping a single closed-form, periodic, mesh-free description. The idea is to take two level-set fields and build a tube around the curve where both vanish. For a TPMS and a phase-shifted copy, that curve is a triply periodic space curve. Its connectivity is set by the phase shift and its thickness by the tube radius. The same construction applies to any pair of smooth fields, including the non-periodic random fields used for spinodoid metamaterials [10].

The contributions are:

1. The PI-TPMS construction in three forms (max, Euclidean and gradient-normalized), with their cross-sections, and a definition of the design parameters (Section 2).
2. An exact tube-volume law. The solid fraction is $\rho = \pi A(r/T)^2$ below the overlap scale. A is measured for all 36 gyroid and Fischer–Koch S shifts. Its average over shifts is fixed by the parent surface's area per cell, and for stochastic fields it is predicted in closed form from the spectrum (Section 3).
3. A topological classification. Generic pairs give unbranched curves. Junctions arise either as crossing junctions at tangency points, present at any radius, or as contact junctions between nearby curves, which depend on the radius. All 36 gyroid and Fischer–Koch S shifts are classified. Three-fold symmetric shifts place six-branch junctions at flat points of the surface. Shifts along a body diagonal give lattices whose Young's modulus is exactly isotropic across the shift axis, a route to bone-like transverse isotropy (Section 4).

4. A mechanical demonstration by FFT homogenization, checked against independent data: the scaling of stiffness with density, a comparison with sheet and skeletal gyroids at a matched thinnest feature, the directional stiffness of lattices joined by each junction type, and the design space the shift opens at fixed density (Section 5).
5. Browser-based tools for building and exporting PI-TPMS and stochastic pipe networks, released open source, and the homogenization solver used here, free for research use (Section 6).

2. The construction

2.1 Two fields and their intersection curve

Let $A, B: \mathbb{R}^3 \rightarrow \mathbb{R}$ be smooth fields. Their common zero set

$$C = \{ \mathbf{x} \in \mathbb{R}^3 : A(\mathbf{x}) = 0 \text{ and } B(\mathbf{x}) = 0 \} \quad (1)$$

is, for a generic pair, a smooth space curve (Section 4.1). For a TPMS pair, A is a TPMS level-set function (Appendix A) and $B(\mathbf{x}) = \varphi_B(R\mathbf{x} + \boldsymbol{\delta})$, where φ_B is the same or a different TPMS, $\boldsymbol{\delta}$ is a phase shift and R an optional rotation. Shifts are written as fractions of the cell, $\boldsymbol{\delta} = 2\pi(s_x, s_y, s_z)$ in the normalized coordinates of Appendix A. Rational shifts keep the pair periodic on the original cell. For a stochastic pair, A and B are two draws of a random field. In the spinodoid and Gaussian-random-field (GRF) generators used here (F13LD.grain), each field is a normalized sum of N plane waves with random directions, wavenumbers and phases, and the two fields differ by random seed or by generator settings.

Figure 1 shows the construction for the gyroid paired with itself at shift $(0, \frac{1}{8}, \frac{1}{2})$. The two surfaces (a, b) cross along the curve C (c), and a tube of radius r around C gives the solid (d).

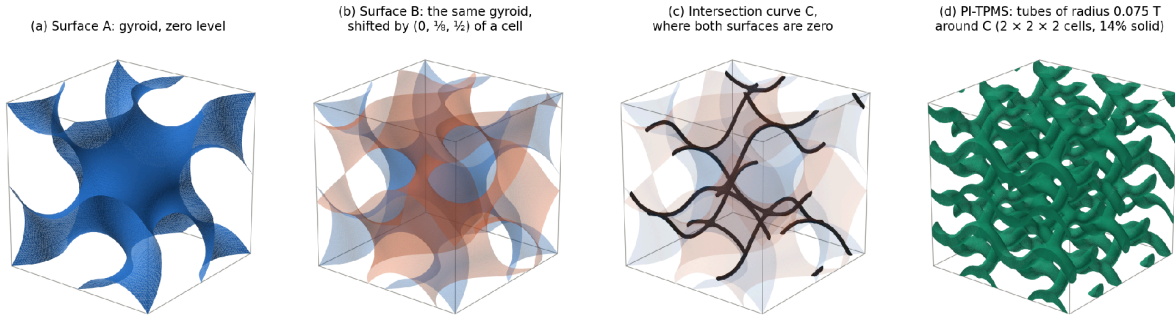


Figure 1. The PI-TPMS construction. (a) Gyroid zero level $A = 0$ in one cell. (b) The same gyroid shifted by $(0, \frac{1}{8}, \frac{1}{2})$ of a cell, $B = 0$. (c) The intersection curve C , where both surfaces vanish; the X-shaped crossings are tangency points of the two surfaces (Section 4.2). (d) The solid: tubes of radius $0.075 T$ around C , gradient-normalized form, $2 \times 2 \times 2$ cells, 14% solid.

2.2 Three tube forms

A tube around C can be written implicitly in several ways. They differ in cross-section, which matters for printability. Near a point of C , linearize $A \approx \nabla A \cdot \mathbf{u}$ and $B \approx \nabla B \cdot \mathbf{u}$, where $\mathbf{u}$ is the offset in the plane normal to C . Let α be the angle between the two surface normals.

Max form. The simplest construction is

$$\Omega_\infty = \{ \mathbf{x} : \max(|A|, |B|) \leq r \} \quad (2)$$

It requires both fields to be small at once, and so isolates C . By contrast, $\min(|A|, |B|) \leq r$ gives the union of two sheets, and $A \cdot B = 0$ gives both surfaces. The cross-section of Ω_∞ is the intersection of two slabs: a rhombus of area $4r^2/(|\nabla A||\nabla B| \sin \alpha)$.

Euclidean form.

$$\Omega_2 = \{ \mathbf{x} : (A^2 + B^2)^{1/2} \leq r \} \quad (3)$$

This form has an elliptical cross-section of area $\pi r^2/(|\nabla A||\nabla B| \sin \alpha)$. It is what a direct implementation in commercial implicit-modelling software typically produces.

Gradient-normalized form. Write $a = A/|\nabla A|$ and $b = B/|\nabla B|$ for the first-order signed distances to the two surfaces, and $c = \cos \alpha$ for the cosine of the angle between their normals. The first-order Euclidean distance to C is then

$$d_C(\mathbf{x})^2 = (a^2 - 2c a b + b^2) / (1 - c^2) \quad (4)$$

and the tube is

$$\Omega_\perp = \{ \mathbf{x} : d_C(\mathbf{x}) \leq r \} \quad (5)$$

Equation (4) is the standard first-order distance to a curve defined by two implicit surfaces [12]. Its cross-section is a circle of area πr^2 , independent of α and of the gradient magnitudes. This makes r a printable tube radius, with tube diameter $= 2r$. It is the form used for all results in this paper unless stated otherwise. Because (4) is first-order, the circle is exact only as $r \rightarrow 0$. At finite radius the tube flattens where the surfaces curve. For PI-gyroid (0, $\frac{1}{8}$, $\frac{1}{2}$) the narrowest tube width is 0.94 of the nominal diameter at $r/T = 0.063$ and 0.73 at $r/T = 0.175$ (Section 5.1). Near tangency ($c \rightarrow \pm 1$) the denominator vanishes. Implementations therefore clamp $|c| \leq 0.95$, which lets the tube widen smoothly at near-tangent points, where it acts as a natural fillet at crossing junctions (Section 4.2).

Figure 2 compares the three cross-sections. Forms (2) and (3) widen wherever the surfaces meet at a shallow angle or the field gradients are small. This produces the web-like fins seen when stochastic fields are paired with the Euclidean form. Form (5) keeps a constant diameter, apart from the bounded widening at near-tangency.

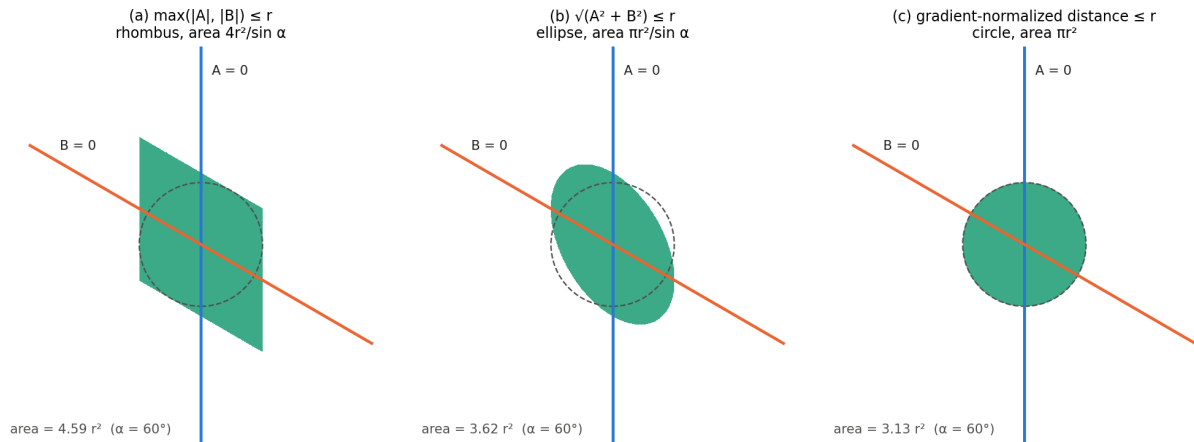


Figure 2. Tube cross-sections in the plane normal to C for unit-gradient fields meeting at $\alpha = 60^\circ$. (a) Max form, equation (2): rhombus, area $4r^2/\sin \alpha$. (b) Euclidean form, equation (3): ellipse, area $\pi r^2/\sin \alpha$. (c) Gradient-normalized form, equation (5): circle, area πr^2 . Dashed: circle of radius r . Areas were measured numerically from the rendered sets.

2.3 Design parameters

A PI-TPMS is specified by the surface pair (ϕ_A, ϕ_B) , the shift δ , the rotation R , the cell size T (and aspect ratios, if the cell is stretched), and the tube radius r . In this paper we write the radius as r/T . An equivalent convention, used in the homogenization data acknowledged in Section 5, is the wall ratio w , the tube diameter over the cell size, so that $r/T = w/2$. The shift, surface pair and rotation set the curve, and with it the network topology at small radius. The radius sets the solid fraction (Section 3). For some shifts the radius also changes the topology, through contact junctions (Section 4.3).

3. Tube volume

3.1 An exact law below the overlap scale

For a smooth curve of length L in $\mathbb{R}^3$, the volume of the set of points within distance r of the curve is exactly $\pi r^2 L$, provided r is below the curve's reach. The reach is set by its curvature radius and by its distance to other parts of itself [13, 14]. This is Weyl's tube formula for a one-dimensional submanifold. The higher-order terms in Weyl's formula involve the intrinsic curvature of the submanifold, and a curve has none. The solid fraction of a PI-TPMS in the gradient-normalized form is therefore

$$\rho = \pi \mathcal{A} (r/T)^2, \quad \mathcal{A} = L_{\text{cell}} / T(6)$$

to first order in the distance approximation (4), for r below the overlap scale. Here L_{cell} is the curve length in one cell. $\mathcal{A}$ is a dimensionless constant of the curve, fixed by the surface pair, shift and rotation. There is no curvature correction at order r^4 for tubes around curves. For the Euclidean and max forms, the area factors of Section 2.2 replace πr^2 pointwise, so ρ still scales with r^2 , but with a prefactor that is a length-weighted average of $1/(|\nabla A| |\nabla B| \sin \alpha)$.

We computed $\mathcal{A}$ for all 18 gyroid–gyroid shifts on the eighth-cell grid by exact curve extraction (Appendix D). The values range from 5.17 at $(0, 0, 1/8)$ to 12.14 at $(0, 1/4, 1/2)$ for the gyroid, and from 15.9 to 45.6 for Fischer–Koch S (Table 1). Figure 3a compares equation (6) with the solid fraction measured on a 240^3 grid for four shifts. Measurement and prediction agree to within 2% at $r/T = 0.02$ (ratio 0.98–1.01). As the radius grows, material is lost to overlap: where tubes meet at crossings or pass close to each other, where r approaches the local curvature radius, and through the first-order distance approximation. The measured-to-predicted ratio falls to 0.90–0.95 at $r/T = 0.05$ and to 0.70–0.77 at $r/T = 0.10$ (Figure 3b). Equation (6) is therefore exact as a limit and accurate to within a few percent below about $r/T = 0.04$. At larger radii the measured $\mathcal{A}(r)$ should be used.

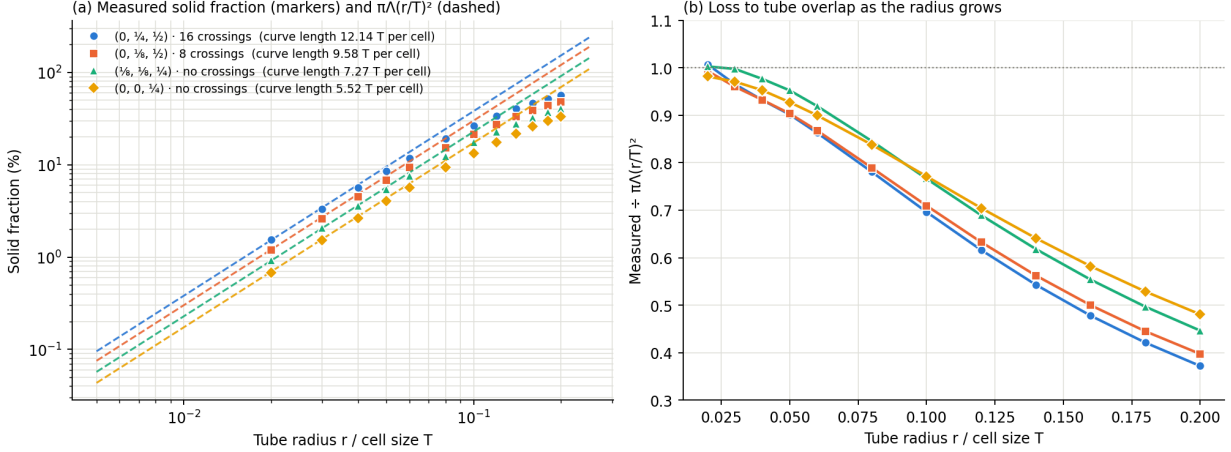


Figure 3. Solid fraction against tube radius for four gyroid–gyroid shifts, gradient-normalized form. (a) Measured solid fraction on a 240^3 grid (markers) and the exact small-radius law $\pi\Lambda(r/T)^2$ (dashed), with Λ from exact curve extraction. (b) Ratio of measured to predicted solid fraction: the loss to tube overlap as the radius grows.

The r^2 law is shared by every strut network, so it is not by itself what distinguishes PI-TPMS. A thin skeletal gyroid follows the srs net, with curve length about $4.24 T$ per cell (twelve edges of length $\sqrt{2} T/4$). That is shorter than any of the 18 PI-gyroid curves. For a thinnest feature of $0.126 T$ ($400 \mu\text{m}$ in a 3.175 mm cell), the skeletal gyroid is 7.0% solid. PI-gyroid $(0, 1/8, 1/2)$ is 10.3% solid at the nominal tube diameter, and 11.6% when the tube is sized so that its measured narrowest width is $0.126 T$ (Table 2, Section 5.1). What PI-TPMS adds is not a new scaling regime but a choice of network: one surface pair gives curves of different length, connectivity and anisotropy, all under the same equation.

3.2 Curve length is set by the surface area

For a TPMS paired with a translated copy of itself, the average of Λ over all shifts follows from translative integral geometry [22, 23]. For two surfaces M and N in $\mathbb{R}^3$, the integral over all translations δ of the length of $M \cap (N + \delta)$ equals $\iint |\sin \theta| dA_M dA_N$, where θ is the angle between the normals at the two surface elements. For a periodic surface and its own copy, dividing by the cell volume gives

$$\langle \Lambda \rangle = a^2 \langle |\sin \theta| \rangle, \quad a = A_{\text{cell}} / T^2 (7)$$

where a is the surface area per cell and $\langle |\sin \theta| \rangle$ is the mean sine of the angle between the normals of two independently chosen, area-weighted surface elements. For surfaces with cubic symmetry the second factor is close to $\pi/4$ (0.782 for the gyroid, 0.785 for Fischer–Koch S). The average curve length is therefore set almost entirely by the square of the surface area per cell. Table 3 tests equation (7) against curve lengths measured at 24 random shifts per surface. They agree within sampling error.

Table 3. Parent-surface properties and the shift-averaged curve length (cubic cell). Genus from the total Gaussian curvature by Gauss–Bonnet, $\int K dA = -4\pi(g - 1)$. Measured $\langle \Lambda \rangle$: mean $\pm$ standard error over 24 random shifts.

Surface	Genus per cubic cell	Area per cell a (T^2)	Total curvature	$\langle \sin \theta \rangle$	Predicted $\langle \Lambda \rangle$	Measured $\langle \Lambda \rangle$	Range over the 18 grid shifts
Gyroid	5	3.092	-16π	0.782	7.48	7.74 ± 0.20	5.17–12.14
Fischer–Koch S	21	5.431	-80π	0.785	23.16	23.22 ± 0.28	15.9–45.6

Genus does not enter equation (7) directly. It fixes the total curvature through Gauss–Bonnet, and it matters for junctions (Section 4.5), but the curve length follows the area. Fischer–Koch S [26] has 1.76 times the gyroid's area per cell and therefore about 3.1 times its average curve length. At equal solid fraction the tube radius scales as $A^{-1/2}$ (equation 6), so a Fischer–Koch S network has tubes about 1.7 times thinner than a gyroid network. It also has about 1.7 times more tube surface per unit volume ($2\rho/r$), and up to 72 crossing junctions per cell against 16. The choice of parent surface therefore sets the density of the curve network, and the shift sets its topology. For a pair of different surfaces, the same argument gives $\langle A \rangle = a_A a_B \langle |\sin \theta| \rangle$; we have not yet tested it.

3.3 Stochastic fields: curve length in closed form

For two independent, stationary random fields, the expected curve length per unit volume follows from the Kac–Rice formula for the zero set of a map $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ [15, 16]:

$$L_V = E[|\nabla A \times \nabla B|] \cdot p_{A,B}(0, 0) \quad (8)$$

where $p_{A,B}$ is the joint density of the two field values. For isotropic Gaussian fields with variance σ^2 and gradient variance λ_2 per component, $p_{A,B}(0, 0) = 1/(2\pi\sigma^2)$ and $E|\nabla A \times \nabla B| = 2\lambda_2$, so

$$L_V = \lambda_2 / (\pi\sigma^2) = \langle k^2 \rangle / (3\pi) \quad (9)$$

where $\langle k^2 \rangle$ is the mean squared wavenumber of the spectrum (Appendix C). The solid fraction of a gradient-normalized stochastic pipe network is then $\rho = \pi r^2 L_V = \langle k^2 \rangle r^2 / 3$. For a narrow-band (spinodoid) field of wavelength λ this gives $\rho \approx (4\pi^2/3)(r/\lambda)^2 \approx 13.2(r/\lambda)^2$.

We tested equation (8) on 43 pairs of F13LD.grain fields: 10 isotropic spinodoid pairs, 10 anisotropic spinodoid pairs, 10 GRF pairs, 10 same-seed pairs differing only in wave count, and 3 isotropic spinodoid pairs in a larger domain. Each pair was evaluated with its own measured spectrum (Appendix D). The extracted curve length matched the prediction to 0.99 ± 0.01 . The 48-wave fields are only approximately Gaussian, but that has no visible effect on the result. For random fields, then, the density of a pipe network can be set from the spectrum before any geometry is built.

4. Topology: crossing and contact junctions

4.1 Generic intersection curves do not branch

Let $F = (A, B): \mathbb{R}^3 \rightarrow \mathbb{R}^2$. If $(0, 0)$ is a regular value of F , meaning ∇A and ∇B are linearly independent at every point of C , then C is a smooth embedded one-dimensional submanifold [17]. It is a disjoint union of smooth curves, each of which has no endpoints and no branch points (Appendix B).

For a periodic pair this has a direct mechanical consequence. On the unit-cell torus, each component of C is a closed loop. Unrolled into $\mathbb{R}^3$, the loop becomes a set of congruent strands, each invariant under translation by a single lattice vector: the loop's winding class. (If that class is zero, the loop closes within a cell and the strands are closed rings.) Thin tubes around a regular curve therefore form **separate strands**. Each carries load along at most one lattice direction and has no shear or transverse stiffness. A load-bearing three-dimensional network needs junctions, and there are only two ways to get them.

4.2 Crossing junctions

Regularity fails where $\nabla A \parallel \nabla B$ on C , that is, where the two surfaces are tangent. Such a point satisfies four scalar conditions ($A = 0$, $B = 0$, and two for parallel gradients) in three unknowns. It is therefore

absent for a generic pair, but it can occur stably when a symmetry of the pair forces it. At a tangency point, write both surfaces as graphs over their common tangent plane. The curve C is then locally the zero set of the difference of their second fundamental forms. If that difference is indefinite, C consists of two branches crossing at the point: a four-valent **crossing junction**. If it is definite, the point is isolated. If it vanishes, the contact is of higher order and more branches can meet (Section 4.5). Crossing junctions exist at zero tube radius, so a network joined by them is connected at any radius. Its topology is fixed by the shift alone.

4.3 Contact junctions

Where two separate branches of C , or two parts of one branch, pass within about one tube diameter ($2r$ in the gradient-normalized form), their tubes fuse into a **contact junction**. Contact junctions depend on the radius. As r grows, more of them form, and a strand system can become connected layers and then a three-dimensional lattice. In a contact-joined network, r sets topology as well as density.

4.4 Classification of the gyroid and Fischer–Koch S shifts

We applied this analysis to all 18 shifts on the eighth-cell grid for the gyroid paired with itself, and for Fischer–Koch S [26] paired with itself. For each shift, the method was as follows (Appendix D):

1. **Crossing points:** found by solving the four tangency conditions from seeds placed where the surfaces are closest to parallel.
2. **Periodic contact graph:** built from the exact curve for tube radii $r/T = 0.025\text{--}0.205$ (gyroid only).
3. **Wrap dimension:** for each connected component, the number of independent lattice directions it wraps through the periodic cell: 1 for strands, 2 for layers, 3 for a lattice.

The results are in Table 1 and Figure 4a.

For the gyroid:

- **Thirteen shifts have no crossing points.** Every one is a set of separate strands at vanishing radius. On these curves the sine of the angle between the surfaces never falls below 0.22, so no near-tangency is hiding. They join only through contacts. The first layers or lattices appear at $r/T = 0.045$, for $(0, \frac{1}{8}, \frac{1}{8})$ and $(\frac{1}{8}, \frac{1}{8}, \frac{1}{4})$. $(0, 0, \frac{1}{8})$ and $(0, 0, \frac{1}{4})$ are still separate strands at $r/T = 0.205$.
- **Five shifts have crossing points:** 4, 8 or 16 per cell. All five are connected at vanishing radius. Four are full three-dimensional lattices: $(0, 0, \frac{1}{2})$, $(0, \frac{1}{8}, \frac{1}{2})$, $(0, \frac{1}{4}, \frac{1}{2})$ and $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, whose junctions have six branches (Section 4.5). One, $(\frac{1}{8}, \frac{1}{4}, \frac{3}{8})$, forms connected layers and becomes a lattice through contacts at $r/T = 0.085$.

For Fischer–Koch S :

- **Eight shifts have no crossing points.** All eight are separate strands at vanishing radius.
- **Nine shifts have resolved crossing points,** 8 to 72 per cell. Seven are three-dimensional lattices at vanishing radius. In the other two, $(0, \frac{3}{8}, \frac{3}{8})$ and $(\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$, the crossings are true four-branch X-junctions lying on the curve, but they join strands that run in the same direction, so the curve remains a set of strands (braided in pairs).
- $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is a single connected lattice in which the two surface copies also touch along whole lines, a non-transverse intersection (Section 4.6). The point search returns many points along those lines (368), so this shift is not counted among the resolved crossings.

Across both surfaces, then: all 21 shifts without crossing points are strands, as Section 4.1 requires. Of the 14 shifts with resolved crossings, 12 are connected at vanishing radius and 2 are not. **Crossing points are necessary for connectivity at vanishing radius, and sufficient when the strands they join run in independent directions.**

For the gyroid, every shift with a half-cell component has crossings and is a lattice at vanishing radius. That gives a simple design rule: **for a gyroid lattice whose topology does not depend on density, choose a shift with a half-cell component, or $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.** The lattice-forming shifts depend on the surface. At $(0, \frac{1}{8}, \frac{1}{8})$, for example, Fischer–Koch S forms a crossing-joined lattice (16 crossings per cell), while the gyroid has no crossings and forms a lattice only through contact from $r/T = 0.045$. For $(0, \frac{1}{8}, \frac{1}{2})$, the connected tubes in a $2 \times 2 \times 2$ block separate into two interpenetrating three-dimensional networks joined at crossing points (Figure 4d).

The classification is purely geometric, but it agrees with independent homogenization data for the same shifts (acknowledged in Section 5). $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is stiff along x, y and z from the smallest wall ratio. $(\frac{1}{8}, \frac{1}{4}, \frac{3}{8})$ has non-zero stiffness along y only until the tubes join its layers. A stack of layers can only carry load along directions that lie inside the layer plane. The full stiffness tensor locates that plane directly: at $w = 0.13$ it is the (101) plane, which contains the y axis but neither x nor z (Section 5, Figure 9h).

Table 1. Gyroid and Fischer–Koch S (FKS), each paired with itself, on the eighth-cell shift grid (gradient-normalized form). A : curve length per cell over T . Crossings: tangency points per cell. Topology at $r \rightarrow 0$: wrap dimension of the bare curve. Lattice from r/T : smallest tube radius at which the gyroid network wraps in three directions (tested at $r/T = 0.025$ – 0.205 in steps of 0.01). *Includes line contacts along three face diagonals (Section 4.6); the count is of points returned along them.

Shift (sx, sy, sz)	Gyroid A	Gyroid crossings	Gyroid at $r \rightarrow 0$	Gyroid lattice from r/T	FKS A	FKS crossings	FKS at $r \rightarrow 0$
$(0, 0, \frac{1}{8})$	5.17	0	strands	> 0.205	15.89	0	strands
$(0, 0, \frac{1}{4})$	5.52	0	strands	> 0.205	20.97	0	strands
$(0, 0, \frac{3}{8})$	6.34	0	strands	0.165	21.86	0	strands
$(0, 0, \frac{1}{2})$	9.24	8	3-D lattice	$\rightarrow 0$	36.11	40	3-D lattice
$(0, \frac{1}{8}, \frac{1}{8})$	7.44	0	strands	0.045	22.82	16	3-D lattice
$(0, \frac{1}{8}, \frac{1}{4})$	6.26	0	strands	0.165	25.24	0	strands
$(0, \frac{1}{8}, \frac{3}{8})$	6.74	0	strands	0.165	22.33	0	strands
$(0, \frac{1}{8}, \frac{1}{2})$	9.58	8	3-D lattice	$\rightarrow 0$	25.73	24	3-D lattice
$(0, \frac{1}{4}, \frac{1}{4})$	7.29	0	strands	0.105	28.38	32	3-D lattice
$(0, \frac{1}{4}, \frac{3}{8})$	8.35	0	strands	0.105	23.14	0	strands
$(0, \frac{1}{4}, \frac{1}{2})$	12.14	16	3-D lattice	$\rightarrow 0$	35.93	72	3-D lattice
$(0, \frac{3}{8}, \frac{3}{8})$	7.93	0	strands	0.125	24.24	8	strands
$(\frac{1}{8}, \frac{1}{8}, \frac{1}{8})$	6.25	0	strands	0.165	23.97	8	3-D lattice
$(\frac{1}{8}, \frac{1}{8}, \frac{1}{4})$	7.27	0	strands	0.175	23.78	0	strands
$(\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$	7.29	0	strands	0.165	22.93	8	strands
$(\frac{1}{8}, \frac{1}{4}, \frac{1}{4})$	7.85	0	strands	0.145	22.10	0	strands
$(\frac{1}{8}, \frac{1}{4}, \frac{3}{8})$	10.56	8	layers	0.085	26.76	24	3-D lattice
$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	11.03	4	3-D lattice	$\rightarrow 0$	45.61	368*	3-D lattice

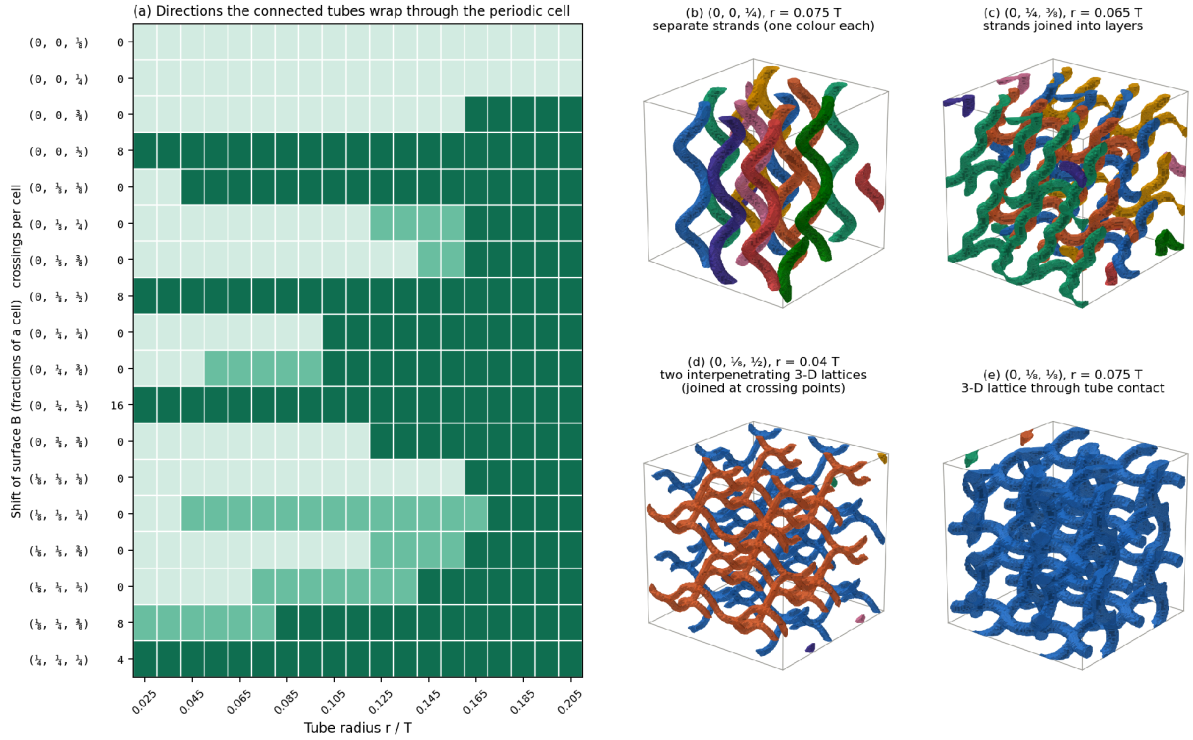


Figure 4. Topology of gyroid-gyroid PI-TPMS. (a) Wrap dimension of the connected tubes against tube radius for all 18 shifts; the number beside each shift is its crossing points per cell. Shifts with crossings are connected at the smallest radius; shifts without them start as strands and connect only through tube contact. (b–e) Examples, $2 \times 2 \times 2$ cells, each connected piece in its own colour: (b) $(0, 0, \frac{1}{4})$, separate strands; (c) $(0, \frac{1}{4}, \frac{3}{4})$, strands joined into layers by contact; (d) $(0, \frac{1}{8}, \frac{1}{2})$ at a small radius, two interpenetrating lattices joined at crossing points; (e) $(0, \frac{1}{8}, \frac{1}{8})$, a lattice formed by tube contact.

4.5 Six-branch junctions at flat points

An ordinary crossing has four branches because the difference of the two second fundamental forms is a quadratic form, whose zero set in the tangent plane is two lines. If that form vanishes, the contact is of third order and the local zero set is that of a cubic. This happens where both surfaces are flat at the tangency point. Symmetry makes it happen stably. Suppose a three-fold rotation axis of the pair passes through the tangency point along the common normal. Any quadratic form on the tangent plane that is invariant under a 120° rotation is a multiple of the identity, so it is either definite, which gives an isolated point, or zero. In the zero case the leading cubic term, invariant under the same rotation, is of the form $\text{Re}(z^3)$: a monkey saddle. C then consists of three curves crossing at the point, a **six-branch junction** (Appendix B).

The gyroid shift $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, taken along a body diagonal, realizes this. Its 4 tangency points per cell lie on three-fold axes at flat points of the gyroid, where both principal curvatures vanish, and the relative curvature form is zero to numerical precision. Six branches leave each point at every sampling radius tested (Figure 5b), against four at the crossings of $(0, \frac{1}{8}, \frac{1}{2})$ (Figure 5a). For an exact minimal surface of genus g per cell, the Gauss map has degree $g - 1$ and, by the Riemann–Hurwitz formula, $4(g - 1)$ branch points counted with multiplicity [24, 25]. These branch points are the flat points. The gyroid has 16 per cubic cell and Fischer–Koch S has 80, so higher genus supplies more sites at which symmetric shifts can place six-branch junctions.

In the gradient-normalized form, the surfaces stay nearly parallel over a broad patch around a flat-point contact. The bounded widening at near-tangency (Section 2.2) therefore produces a hub at the junction (Figure 5b), whose size is set by the clamp on $|\cos \alpha|$. The effect of this lattice on stiffness is taken up in Section 4.6.

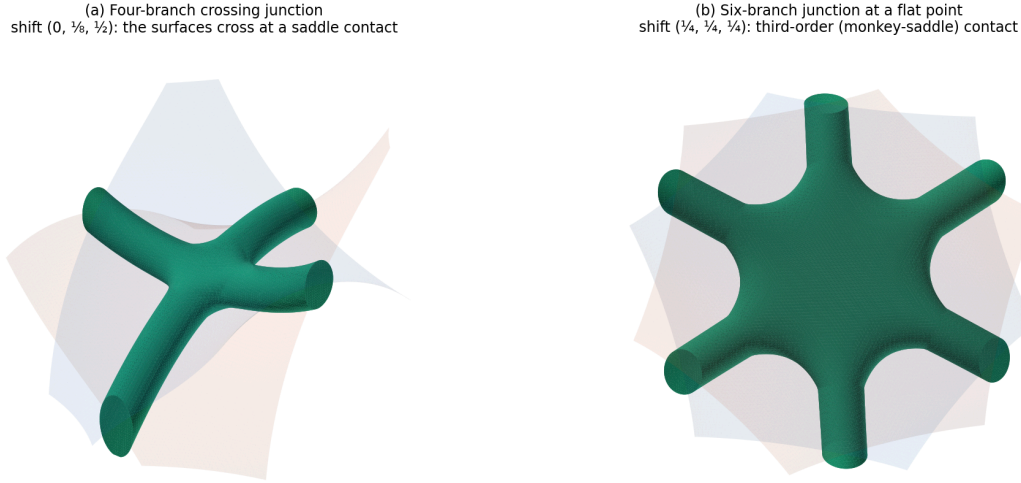


Figure 5. Junction types at tangency points, gradient-normalized form, tube radius 0.022 of the cell (faint: the two surfaces). (a) Four-branch crossing in shift $(0, \frac{1}{8}, \frac{1}{2})$: the relative curvature of the two surfaces is indefinite, and C is two crossing curves. (b) Six-branch junction in shift $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ at a flat point of the gyroid on a three-fold axis: the relative curvature vanishes, the contact is a monkey saddle, and C is three crossing curves. The central hub is the bounded widening of the gradient-normalized form where the surfaces are nearly parallel.

4.6 Shift direction sets the elastic symmetry

A shifted pair keeps only those symmetries of the parent surface that are compatible with the shift, meaning those whose rotational part leaves δ unchanged. The shift direction therefore selects the point symmetry of the lattice, and with it the symmetry of its effective stiffness tensor. For a shift along a body diagonal, the surviving symmetry includes the three-fold rotation about that diagonal. A stiffness tensor with a three-fold axis has the same Young's modulus in every direction perpendicular to the axis [27]. In that plane, $1/E$ is a quartic in the direction cosines, so it contains only the 0th, 2nd and 4th angular harmonics. Invariance under a 120° rotation removes the 2nd and 4th, leaving a constant (Appendix B). A shift along a cube axis leaves at most a four-fold symmetry, which keeps the 4th harmonic, so Young's modulus varies around that axis.

The independent homogenization data confirm this for every connected lattice on the grid, and the F13LD.lab results of Section 5 confirm it again with stiffness tensors computed independently. Across the shift axis, Young's modulus varies by 0.0% in the independent data, and by less than 0.01% in the lab results, for the three body-diagonal shifts: gyroid $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, and Fischer–Koch S $(\frac{1}{8}, \frac{1}{8}, \frac{1}{8})$ and $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. For shifts along a cube axis it varies by 24–700%. The result across the plane is exact. At other tilts from the axis, a three-fold stiffness tensor can still vary with direction around the axis, and the two clean $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ lattices behave very differently (Figure 6d, e; Table 4):

- **Fischer–Koch S $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$** varies by no more than 3% around the axis at any tilt. It is transversely isotropic in practice, which is the elastic symmetry commonly used to model bone. The axis and the

transverse plane are about equally stiff ($E/E_s = 0.044$ and 0.041 at 19% solid in F13LD.lab; 0.046 and 0.042 in the independent data). Between them lies a much softer band centred about 45° from the axis, where E falls to about a quarter of the axial value (0.012).

- **Gyroid** ($\frac{1}{4}, \frac{1}{4}, \frac{1}{4}$) is isotropic only in the plane itself. Along the axis it is about 8 times softer than across it (0.0012 against 0.0094 at 18% solid in F13LD.lab, a ratio of 8.2; 0.0012 against 0.0097 in the independent data). At intermediate tilts it varies up to 5.7-fold around the axis, and it is stiffest toward the other three body diagonals.

For an implant, the diagonal can be aligned with the principal loading axis to give a lattice that is equally stiff in all transverse directions. The surface and the shift then set the ratio of axial to transverse stiffness: 0.12 for the gyroid lattice and 1.07 for the Fischer–Koch S lattice at about 19% solid (1.08 in the independent data).

Table 4. Lattices with a three-fold shift axis ($\nu = 0.403$). F13LD.lab values (Section 5), with the independent homogenization data in brackets. E is normalized by the solid modulus. Across the axis, E is identical in every direction, by symmetry. Variation around the axis: largest ratio of maximum to minimum E over directions at the same tilt, at any tilt. Pieces: composition of the intersection curve.

Lattice	Solid	E along axis	E across axis	Variation around axis	Pieces
Gyroid ($\frac{1}{4}, \frac{1}{4}, \frac{1}{4}$), $w = 0.17$	18.3% (18.7%)	0.0012 (0.0012)	0.0094 (0.0097)	up to $5.7\times$ ($5.6\times$)	two interpenetrating lattices (100%)
Fischer–Koch S ($\frac{1}{4}, \frac{1}{4}, \frac{1}{4}$), $w = 0.09$	19.0% (18.9%)	0.044 (0.046)	0.041 (0.042)	$\leq 3\%$ ($\leq 3\%$)	one lattice (100%)
Fischer–Koch S ($\frac{1}{8}, \frac{1}{8}, \frac{1}{8}$), $w = 0.13$	22.0% (21.9%)	0.034 (0.035)	0.0014 (0.0015)	up to $4.7\times$ ($4.7\times$)	three lattices (65%); strands and rings (35%)

Line contacts. The Fischer–Koch S ($\frac{1}{4}, \frac{1}{4}, \frac{1}{4}$) lattice also shows a third kind of contact. Besides crossing, the two surface copies touch along straight lines. 23.5% of its curve lies where the surfaces are parallel to within 10^{-3} , and almost all of it runs along the three face diagonals perpendicular to the shift, $(1, -1, 0)$, $(1, 0, -1)$ and $(0, 1, -1)$. These lines form the triangular web seen down the axis (Figure 6b). Along a line contact the intersection is doubled and splits into close parallel strands, which the gradient-normalized form merges into thicker rods. All of the curve belongs to one connected lattice.

A caution for homogenization. Fischer–Koch S ($\frac{1}{8}, \frac{1}{8}, \frac{1}{8}$) also has a three-fold axis, and it is 24 times stiffer along that axis than across it (23 times in the independent data). Only 65% of its curve, however, belongs to its three interpenetrating lattices (Figure 6c). The remaining 35% is made up of separate strands, 14.5% of the curve running along the shift axis and 17.6% along a cube axis, plus closed rings (3%). A periodic stiffness calculation counts each axial strand as an infinite load-bearing rod. In a finite part those strands end at the boundary and the rings float free, so they would be removed or left loose. The connectivity of a design should therefore be checked before its homogenized stiffness is used, for example by the wrap dimension of each piece.

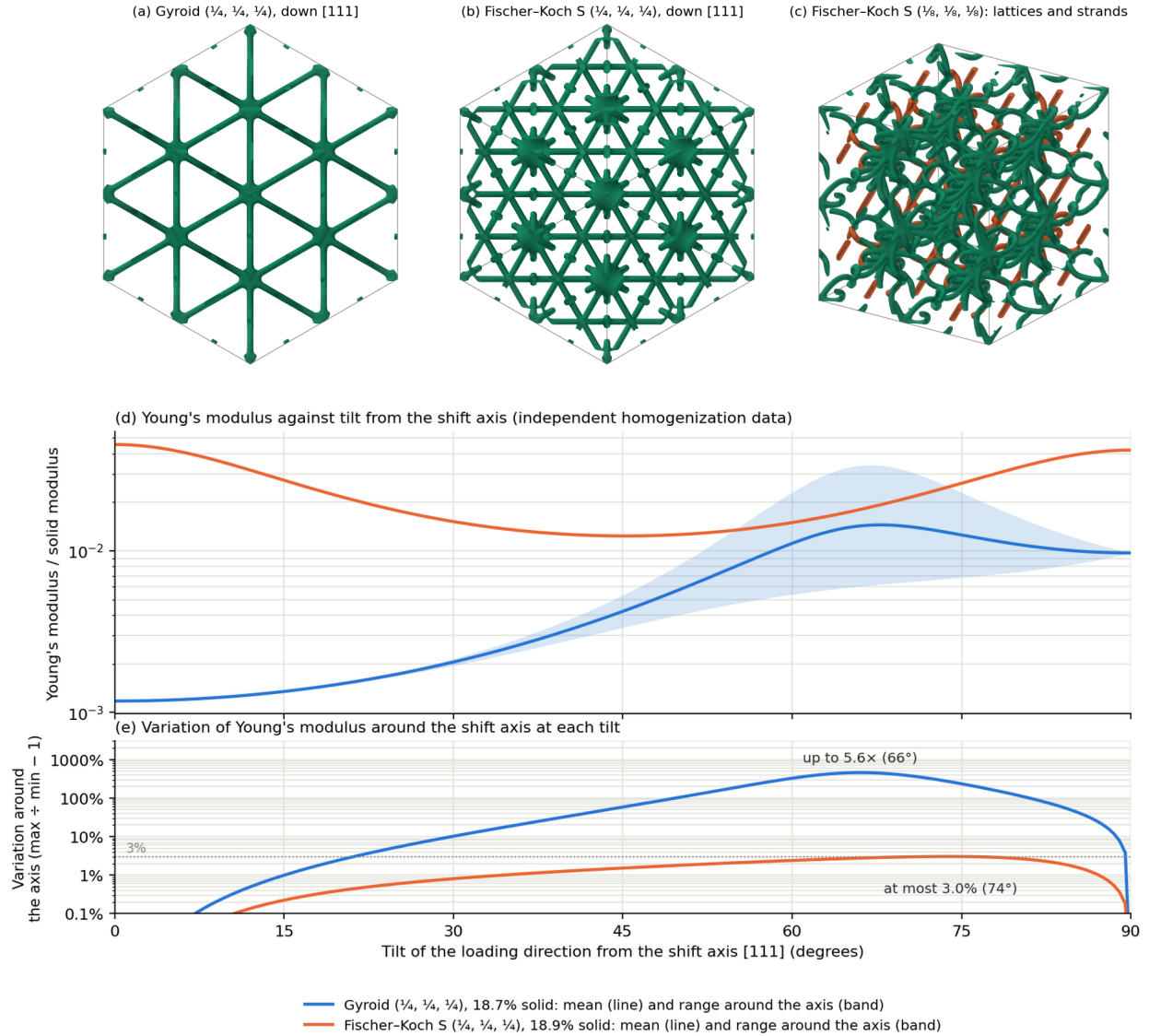


Figure 6. Lattices with a three-fold shift axis. (a) Gyroid ($\frac{1}{4}, \frac{1}{4}, \frac{1}{4}$), viewed down the shift axis [111], $2 \times 2 \times 2$ cells. (b) Fischer-Koch S ($\frac{1}{4}, \frac{1}{4}, \frac{1}{4}$), viewed down [111]: line contacts along the three face diagonals perpendicular to the axis form a triangular web between the hubs. (c) Fischer-Koch S ($\frac{1}{8}, \frac{1}{8}, \frac{1}{8}$): its three lattices (green) and its separate strands and rings (orange), classified from the exact periodic curve. The straight orange rods run along the shift axis. (d) Young's modulus against tilt of the loading direction from the shift axis, from the independent homogenization data at about 19% solid. Line: mean over directions around the axis. Band: range around the axis; for Fischer-Koch S it is narrower than the line. (e) The variation around the axis at each tilt, $\max \div \min - 1$, from the same data. At 90° (across the axis) both are exactly isotropic. Fischer-Koch S stays within 3% at every tilt (largest at 74°); the gyroid varies by up to $5.6\times$ (at 66°).

4.7 Stochastic pairs

A pair of independent random fields has no symmetry to force tangency, so crossing junctions should be absent. In every stochastic pair searched, eight pairs covering isotropic and anisotropic spinodoids, GRFs and same-seed variants, no crossing point was found. The best near-misses left residuals between 7.5×10^{-5} and 1.3×10^{-2} , where the gyroid's exact crossings solve to 10^{-9} or better. The correlated same-seed pair came closest, and in large domains near-tangent points do occur. These behave as contact junctions that form at very small radius. Stochastic pipe networks are therefore joined only through contacts (Figure

7). At small radius they are mostly separate fibres, and they become progressively connected as the tubes thicken. Because the curves are long, contacts are frequent, and connected networks form at low solid fraction. The connection threshold depends on the size of the part, as in random fibre networks [18]. A quantitative study of contact-junction connectivity in stochastic pipe networks, including the share of material on a load path, will be reported separately.

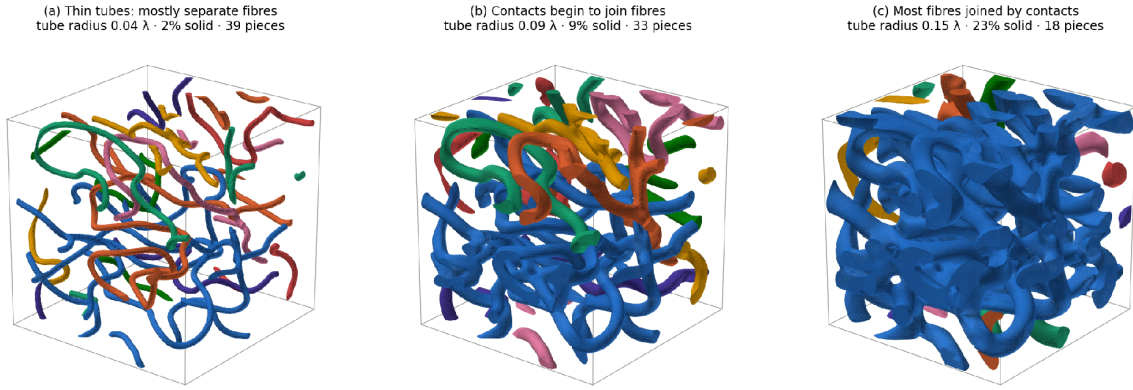


Figure 7. Pipe intersection of two isotropic spinodoid fields (F13LD.grain, 48 waves, seeds 42 and 43), gradient-normalized form, at three tube radii (λ : mean wavelength). Each connected piece is shown in its own colour. There are no crossing junctions; pieces join only where tubes from separate curves fuse. The domain is about 2.8 wavelengths across; larger parts connect at lower solid fraction.

5. Mechanical demonstration

5.1 Method

Effective stiffness tensors were computed in F13LD.lab (v0.12.1), the browser-based GPU solver of the F13LD suite. It performs FFT-based elastic homogenization of a periodic unit cell [19], solving the Lippmann–Schwinger equation by conjugate gradients [30] for six unit-strain load cases, and returns the full 6×6 stiffness tensor. Geometry was voxelized directly from the implicit definitions that F13LD.tpms exports: the gradient-normalized tube (5) for PI-TPMS, $|\phi| \leq c$ for sheets and $\phi \geq c$ for skeletal solids. The void phase was assigned 10^{-6} of the solid modulus. At 10^{-4} it added a stiffness offset that inflated the most compliant results by up to 20% (soft axes and lattices near a contact threshold), while 10^{-6} and 10^{-8} differed by less than 0.6%. Pieces not connected to a spanning network are removed before solving, as they would be in a printed part; this removed no material from any cell reported here. Conjugate gradients ran to a relative residual of 10^{-5} . The solid had $E_s = 1$ and $\nu = 0.403$, matching the independent dataset, with a titanium check at $\nu = 0.34$.

Each case was solved on 64^3 and 128^3 grids (the two coarse grid-study cells on 32^3 and 64^3), and the stiffness tensor was extrapolated term by term to zero voxel size, $C_{\text{ext}} = C_{128} + (C_{128} - C_{64})/(2^p - 1)$, with $p = 2$. A three-grid study (32^3 , 64^3 and 128^3) of PI-gyroid (0, $1/8$, $1/2$) at $w = 0.13$ gave observed orders of 1.8 and 2.0 for the in-plane moduli and 1.4 for E_z . The two-grid extrapolation agreed with the three-grid value to within 0.5% for E_x and E_y and 2% for E_z . Shear moduli converge more slowly (orders 0.5–1.5), so their extrapolated values are 2–13% low, worst for G_{xy} . Before extrapolation, the 128^3 result lies about 2–3% below the extrapolated value for PI lattices, 1.4% for skeletal solids and 0.6% for sheets. The thinnest case ($w = 0.05$, 1.8% solid) has only about 3 voxels across a tube on the 64^3 grid, and its extrapolation adds 12% to the 128^3 value; it is a lower-confidence point. All stiffness values below are extrapolated and

normalized by E_s . Young's modulus in a direction $\mathbf{n}$ was computed from the compliance tensor $S = C^{-1}$ (Appendix B). Its mean over all directions is reported alongside the axis values, because several of these lattices are stiffest away from the cell axes.

The baselines are sheet and skeletal gyroids. Sheets are set by their thinnest wall and skeletal solids by solid fraction, plus one skeletal case matched to a thinnest neck of 0.126 T (Table 2). Thinnest walls and necks were measured as twice the smallest inscribed-sphere radius on the medial surface or skeleton, from a periodic distance transform (Appendix D). For the matched-feature comparison, the PI tube was sized by its measured narrowest width rather than its nominal diameter. Because the distance (4) is first-order, PI-gyroid (0, $\frac{1}{8}$, $\frac{1}{2}$) tubes flatten as they thicken: measured on the continuous field, their narrowest width is 1.00, 0.94, 0.87 and 0.73 of the nominal diameter at $w = 0.05, 0.126, 0.19$ and 0.35 . A narrowest width of 0.126 T therefore needs $w = 0.1364$ (11.6% solid) rather than $w = 0.126$ (10.3% solid). Measured with a single voxel method at 256^3 , the thinnest features of the three matched cells are 0.122 T (PI), 0.124 T (sheet) and 0.130 T (skeletal), all within 4% of the target. The voxel method reads the flattened PI tubes 3–4% thin, which is why the PI case was sized on the continuous field.

Table 2. Baseline level sets for the gyroid $\phi = \sin x \cos y + \sin y \cos z + \sin z \cos x$ (range -1.5 to 1.5). Sheet: solid where $|\phi| \leq c$. Skeletal: solid where $\phi \geq c$.

Case	c	Thinnest feature (T)	Mean or median feature (T)	Solid fraction
Sheet, thinnest wall 0.050 T	0.2424	0.050	0.053	15.7%
Sheet, thinnest wall 0.080 T	0.3926	0.080	0.084	25.4%
Sheet, thinnest wall 0.100 T	0.5006	0.100	0.108	32.5%
Sheet, thinnest wall 0.126 T	0.6453	0.126	0.140	42.0%
Sheet, thinnest wall 0.150 T	0.7499	0.150	0.164	48.9%
Sheet, thinnest wall 0.200 T	0.9737	0.200	0.222	64.1%
Skeletal, 5% solid	1.3303	0.101	0.116	5.0%
Skeletal, 10% solid	1.1972	0.158	0.168	10.0%
Skeletal, 15% solid	1.0578	0.203	0.212	15.0%
Skeletal, 20% solid	0.9138	0.237	0.251	20.0%
Skeletal, 30% solid	0.6162	0.300	0.318	30.0%
Skeletal, 40% solid	0.3099	0.347	0.377	40.0%
Skeletal, thinnest neck 0.126 T	1.2786	0.126	0.138	7.0%

5.2 Verification against independent data

Twenty-two cells were computed by both F13LD.lab and the independent dataset: seven PI-gyroid (0, $\frac{1}{8}$, $\frac{1}{2}$) wall ratios from 0.05 to 0.35, ten lattices at about 20% solid, two grid-study cells and three Fischer–Koch S (0, $\frac{1}{8}$, $\frac{1}{2}$) cells (Table 5). The voxelized solid fractions agree to within 1.5%, except for gyroid ($\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$) (2.0% low) and the deliberately coarse 32^3 grid-study cell (2.9% low). Over the 62 axis moduli that are not near zero, the lab values are lower by a median of 5% (median ratio 0.945; middle half 0.92–0.96). The gap narrows as the solid fraction rises, from about 7% at 1.8% solid to 3% at 42% solid. Without extrapolation the gap is larger (median ratio 0.92 at 128^3 and 0.85 at 64^3), and the three-grid extrapolation of the grid-study cell leaves it at 3–6% with solid fraction matched. The difference therefore behaves as a resolution effect rather than a difference in geometry or material model. Here the FFT

stiffness rises monotonically as the grid is refined. The independent solver uses displacement-based finite elements, whose stiffness converges from above as the mesh is refined, so a residual gap of this sign is expected between the two methods, with the converged value lying between them.

Every trend in the independent data is reproduced. The scaling exponent of PI-gyroid (0, $\frac{1}{8}$, $\frac{1}{2}$) agrees to within 0.01 (below). The anisotropy ratios of the three body-diagonal lattices agree to within a few percent (Table 4), and the 17-fold difference between gyroid and Fischer–Koch S at (0, $\frac{1}{8}$, $\frac{1}{8}$) is matched (17.0 against 16.7). The largest outlier is Fischer–Koch S (0, $\frac{1}{8}$, $\frac{1}{2}$) at $w = 0.17$ (E_z ratio 0.40). It lies where new tube-to-tube contacts form, between $w = 0.16$ and 0.175. There E_z rises 8-fold between $w = 0.17$ and 0.19, so the result is very sensitive to resolution. These three cells were computed to check a step in the independent data. The number of loops per cell in the voxelized solid, from its Euler characteristic at 128^3 , rises in steps from 25 to 33, 41 and 57 between $w = 0.15$ and 0.21, so the step is a change in topology rather than an artefact of grid resolution. The results are in the archive.

Table 5. F13LD.lab against the independent homogenization data ($\nu = 0.403$). Lab values extrapolated from 64^3 and 128^3 (last row: from 32^3 , 64^3 and 128^3). E in units of $10^{-3} E_s$. FKS: Fischer–Koch S. Near-zero moduli (below $2 \times 10^{-5} E_s$ in both datasets) are shown as ≈ 0 and not compared.

Cell	Solid % (lab / ind.)	E_x	E_y	E_z	Lab $\div$ independent (x / y / z)
PI-gyroid (0, $\frac{1}{8}$, $\frac{1}{2}$), $w = 0.05$	1.8 / 1.8	0.110	0.109	0.027	0.93 / 0.92 / 0.95
$w = 0.07$	3.5 / 3.5	0.405	0.389	0.097	0.92 / 0.88 / 0.91
$w = 0.09$	5.6 / 5.6	1.08	1.11	0.264	0.93 / 0.94 / 0.95
$w = 0.13$	10.8 / 10.8	4.33	4.35	1.05	0.95 / 0.94 / 0.95
$w = 0.19$	19.9 / 19.8	15.6	15.7	3.89	0.95 / 0.95 / 0.96
$w = 0.27$	31.7 / 31.8	41.7	42.8	11.1	0.97 / 0.97 / 0.97
$w = 0.35$	42.3 / 42.2	74.8	78.1	21.2	0.97 / 0.97 / 0.97
Gyroid (0, 0, $\frac{1}{2}$), $w = 0.19$	19.0 / 19.1	18.1	18.1	4.00	0.96 / 0.96 / 0.96
Gyroid (0, $\frac{1}{4}$, $\frac{1}{2}$), $w = 0.17$	21.0 / 20.9	48.4	16.6	48.4	0.96 / 0.95 / 0.96
Gyroid ($\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$), $w = 0.17$	18.3 / 18.7	4.52	4.52	4.52	0.96 / 0.96 / 0.96
Gyroid (0, $\frac{1}{8}$, $\frac{1}{8}$), $w = 0.21$	20.3 / 20.0	3.43	19.1	19.1	0.97 / 0.97 / 0.97
Gyroid (0, $\frac{1}{4}$, $\frac{1}{4}$), $w = 0.23$	21.9 / 22.0	2.43	9.19	9.19	0.91 / 0.90 / 0.90
Gyroid ($\frac{1}{8}$, $\frac{1}{4}$, $\frac{3}{8}$), $w = 0.13$	12.1 / 12.0	≈ 0	0.908	≈ 0	— / 0.97 / —
Gyroid (0, 0, $\frac{1}{4}$), $w = 0.19$	12.3 / 12.3	≈ 0	≈ 0	4.13	— / — / 0.94
FKS (0, $\frac{1}{8}$, $\frac{1}{8}$), $w = 0.13$	21.6 / 21.4	0.179	1.13	1.13	0.96 / 0.95 / 0.95
FKS ($\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$), $w = 0.09$	19.0 / 18.9	12.8	12.8	12.8	0.95 / 0.95 / 0.95
FKS ($\frac{1}{8}$, $\frac{1}{8}$, $\frac{1}{8}$), $w = 0.13$	22.0 / 21.9	4.73	4.73	4.73	0.91 / 0.91 / 0.91
PI-gyroid (0, $\frac{1}{8}$, $\frac{1}{2}$), $w = 0.13$, three grids	10.8 / 10.8	4.36	4.35	1.07	0.95 / 0.94 / 0.97

5.3 Stiffness against density and at a matched feature

PI-gyroid (0, $\frac{1}{8}$, $\frac{1}{2}$) follows $E \propto \rho^n$ with $n = 2.08$ for E_x and 2.12 for E_z over 1.8–42% solid (Figure 8a); fitting the independent data at the same cells gives 2.07 and 2.11. The local exponent between neighbouring cells stays between 2.0 and 2.1 for E_x across the whole range, and between 2.0 and 2.25 for E_z . The lattice is therefore bending-dominated at every density tested, as expected for a network whose

nodes join four or six struts, far fewer than the twelve needed for stretching-dominated behaviour in three dimensions [6]. The skeletal gyroid follows $n = 2.20$ and the sheet gyroid $n = 1.42$. The ratio E_z/E_x of PI-gyroid (0, $\frac{1}{8}$, $\frac{1}{2}$) is 0.24 at low density and rises to 0.28 at 42% solid. At equal solid fraction it is the most compliant of the three: at 11.6% solid the skeletal-gyroid fit is 1.7 times stiffer along x and y , 7 times stiffer along z , and 2.9 times stiffer averaged over all directions.

Printed lattices are limited by their thinnest feature rather than by their solid fraction, so the three structures were also compared at a thinnest feature of $0.126 T$, which is $400 \mu\text{m}$ in a 3.175 mm cell (Table 6, Figure 8b). At that feature the skeletal gyroid is only 6.9% solid, but its stiffness is strongly direction-dependent: it is 2.4 times stiffer along the body diagonals than along the cell axes. PI-gyroid (0, $\frac{1}{8}$, $\frac{1}{2}$), at 11.6% solid, is 1.8 times stiffer than the skeletal gyroid along x and y and 0.43 times as stiff along z . Averaged over all directions the two differ by 5%. The sheet gyroid, at 42% solid, is about 45 times stiffer than either on average. In Ti-6Al-4V ($E_s \approx 110 \text{ GPa}$), both strut networks average about 0.5 GPa, within the range reported for trabecular bone across anatomic sites, while the sheet gyroid, at about 22 GPa, is as stiff as cortical bone [28, 29]. These are ideal linear-elastic values; as-built lattices are typically more compliant. Sizing the PI tube by its measured narrowest width rather than its nominal diameter ($w = 0.1364$ against 0.126) adds 13% solid and 31–34% stiffness on every axis, without changing E_z/E_x . At a printable feature size, then, PI-gyroid (0, $\frac{1}{8}$, $\frac{1}{2}$) does not give lower stiffness per unit of material than the skeletal gyroid. What it adds is a choice of network: here, a 4 : 1 ratio of in-plane to axial stiffness, set by the shift.

Changing the solid's Poisson's ratio from 0.403 to 0.34 changes the direction-averaged stiffness of the three matched cells by +0.2% (PI), -0.1% (sheet) and $+1.1\%$ (skeletal), so none of these comparisons depends on it.

Table 6. Stiffness at a matched thinnest feature of $0.126 T$ ($\nu = 0.403$; E in units of $10^{-3} E_s$). Thinnest feature: on the continuous field, with the 256^3 voxel measurement in brackets. Mean, max and min: Young's modulus over all directions. E_y equals E_x to within 0.2% for the sized PI cell (2.3% for the nominal one) and exactly for the cubic baselines. Last column: direction-averaged modulus in Ti-6Al-4V, taking $E_s = 110 \text{ GPa}$.

Cell	Solid	Thinnest feature (T)	E_x	E_z	Mean E	Max E	Min E	Mean, Ti (GPa)
PI-gyroid (0, $\frac{1}{8}$, $\frac{1}{2}$), $w = 0.1364$	11.6%	0.126 (0.122)	5.13	1.24	4.44	7.63	1.24	0.49
PI-gyroid (0, $\frac{1}{8}$, $\frac{1}{2}$), $w = 0.126$ (nominal)	10.3%	0.118–0.120	3.83	0.942	3.30	5.73	0.942	0.36
Skeletal gyroid, $c = 1.2786$	6.9%	0.126 (0.130)	2.90	2.90	4.68	6.89	2.90	0.51
Sheet gyroid, $c = 0.6453$	42.0%	0.126 (0.124)	185	185	201	212	185	22

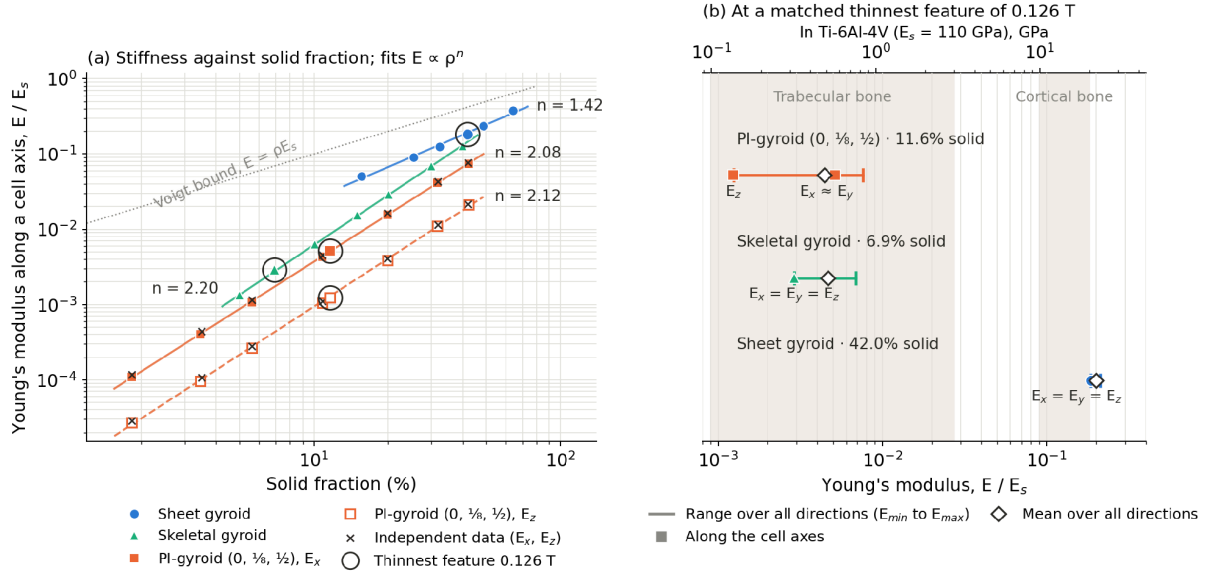


Figure 8. (a) Young's modulus along a cell axis against solid fraction for PI-gyroid (0, $\frac{1}{8}$, $\frac{1}{2}$) (squares; filled E_x , open E_z), sheet gyroid (circles) and skeletal gyroid (triangles), with power-law fits $E \propto \rho^n$. Crosses: independent homogenization data for the same PI cells. Rings: the three cells with a thinnest feature of 0.126 T . Dotted: the Voigt bound. (b) The matched cells: range of Young's modulus over all directions (bar), values along the cell axes (markers) and mean over all directions (diamond). Top axis: in Ti-6Al-4V. Shaded: approximate ranges for trabecular and cortical bone [28, 29].

5.4 Shift, junction type and directional stiffness

Figure 9 shows how Young's modulus varies with direction for lattices of each junction type at about 20% solid, and for the layer and strand cases at 12%. The axis values alone are misleading for most of them.

- **Crossing lattices.** (0, $\frac{1}{8}$, $\frac{1}{2}$) and (0, 0, $\frac{1}{2}$) are four to five times softer along z than along x and y (Figure 9a, b). (0, $\frac{1}{4}$, $\frac{1}{2}$) is softest along y instead ($E_y/E_x = 0.34$). The shift selects the soft axis.
- **Six-branch lattice.** Gyroid ($\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$) has the same modulus, 0.0045, along x , y and z , but that hides strong anisotropy. It is 8.2 times softer along the shift axis (0.0012) than across it (0.0094), and stiffest, at 0.033, toward the other three body diagonals (Figure 9c, Table 4).
- **Contact against crossing junctions at the same shift.** At (0, $\frac{1}{8}$, $\frac{1}{8}$) the gyroid forms a contact-joined lattice and Fischer-Koch S a crossing-joined one. At similar solid fraction (20.3% and 21.6%), the gyroid lattice is 17 times stiffer along y and z and 11 times stiffer averaged over directions. The Fischer-Koch S lattice is stiff only in a narrow cone around the face diagonal [011] (Figure 9e). Its peak, 0.068, exceeds that of the gyroid lattice in any direction (0.052), but only about 1% of directions reach a fifth of that peak. Crossing junctions fix the topology at any radius; they do not by themselves make a lattice stiff. At this density the contact-joined gyroid lattices (0, $\frac{1}{8}$, $\frac{1}{8}$) and (0, $\frac{1}{4}$, $\frac{1}{4}$) carry load in all directions (Figure 9d).
- **Body-diagonal shifts.** All three are isotropic across the shift axis to better than 0.01%, as Section 4.6 requires. Fischer-Koch S ($\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$) varies by at most 2.9% around the axis at any tilt and is transversely isotropic in practice (Figure 9f). Fischer-Koch S ($\frac{1}{8}$, $\frac{1}{8}$, $\frac{1}{8}$) is 24 times stiffer along the axis than across it (Figure 9g); as Section 4.6 explains, part of that axial stiffness is carried by strands that would not survive in a finite part.

- **Layers and strands.** The layered gyroid ($\frac{1}{8}, \frac{1}{4}, \frac{3}{8}$) has only three stiffness modes above $10^{-5} E_s$: the in-plane modes of a stack of sheets. Its Young's modulus is 0.0009–0.0074 within the (101) plane and falls a hundredfold within about 2° of it (Figure 9h). It reads as stiff along y only because y is the only cell axis in that plane. The strand lattice ($0, 0, \frac{1}{4}$) has a single stiffness mode, $E_z = 0.0041$, and is near zero in every other direction (Figure 9i).

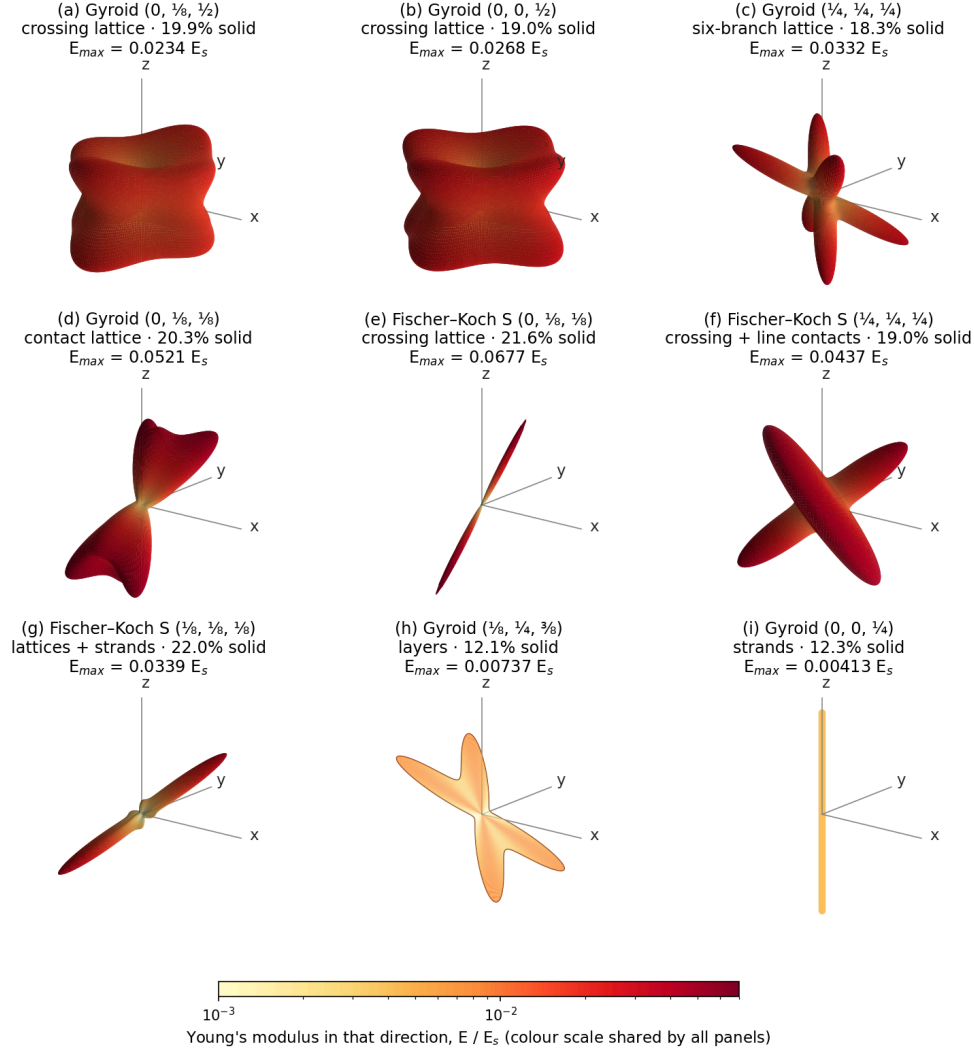


Figure 9. Young's modulus against loading direction, $E(\mathbf{n})$, from the F13LD.lab stiffness tensors. Each surface is drawn at its own scale (radius $E/E_{\max}$); colour gives E/E_s on a shared logarithmic scale. Shift and junction type as labelled; PI-TPMS gradient-normalized form, $\nu = 0.403$. (h) The layered lattice has stiffness only in the (101) plane, drawn as its in-plane polar curve. (i) The strand lattice has stiffness only along z .

5.5 Design space beyond the baselines

A sheet or skeletal gyroid gives one lattice at each solid fraction. A PI-TPMS surface pair gives one for every connected shift. The independent dataset contains full stiffness tensors for all 36 shifts of Table 1 at 20 wall ratios from 0.05 to 0.45, so this design space can be mapped. Figure 10 and Table 7 use it, with the F13LD.lab results of Sections 5.2–5.4 as checks. The independent values run about 5% above the lab values (Section 5.2), which does not affect the comparisons below.

Range at a fixed solid fraction. At about 20% solid (17–23%), the five connected gyroid shifts span a mean modulus of 0.008–0.032 E_s and an anisotropy $E_{\max}/E_{\min}$ of 3–29 (Figure 10a). The eleven connected Fischer–Koch S shifts span 0.0009–0.022 E_s and 4–350. At the same solid fraction the skeletal gyroid is a single point (0.040, 1.8), and so is the sheet gyroid (0.073, 1.1). All of these points fall close to one trend: the more anisotropic the lattice, the lower its mean stiffness. Every PI lattice is less stiff on average than the skeletal gyroid at equal density, and none is close to isotropic. What the shift provides is a choice of position along the compliant, directional part of that trend at a fixed solid fraction. Within the gyroid shifts (0, s , $\frac{1}{2}$), for example, the three grid points are all crossing-joined lattices at about 20% solid with tube diameters of 0.17–0.19 T . Their mean stiffness differs by a factor of 2.1 (0.015 to 0.032), and the soft axis moves from z to y . A shift graded through a part could therefore grade stiffness at nearly constant porosity and feature size, which a sheet or skeletal TPMS can do only by changing density. Shifts between the grid points have not been classified, so a graded design should check that its path stays connected.

Anisotropy as density changes. As a skeletal gyroid thickens, its nodes fill in and its anisotropy falls from 2.5 at 5% solid to 1.4 at 40% (Figure 10b); the sheet gyroid stays at 1.1. Two gyroid crossing lattices hold theirs nearly constant over the same range: (0, $\frac{1}{8}$, $\frac{1}{2}$) at 5.2–6.2 and (0, 0, $\frac{1}{2}$) at 5.6–7.0 in the independent data, with 5.1–6.1 for (0, $\frac{1}{8}$, $\frac{1}{2}$) in F13LD.lab between 1.8% and 42% solid. Others drift as much as the skeletal gyroid or more: (0, $\frac{1}{4}$, $\frac{1}{2}$) falls from 5.5 to 2.2, and ($\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$) from 45 to 16. With the stable shifts, density can be graded through a part without changing its directional character.

Negative Poisson's ratios. The sheet and skeletal gyroids have Poisson's ratios of 0.33–0.35 and 0.32–0.47 along their cell axes. Several PI lattices are auxetic along one or more axis pairs, at every density computed and in both datasets (Table 7). PI-gyroid (0, $\frac{1}{8}$, $\frac{1}{2}$) has ν_{xy} between -0.02 and -0.09 from 1.8% to 42% solid, and (0, 0, $\frac{1}{2}$) reaches -0.13 . The contact-joined gyroid lattice (0, $\frac{1}{8}$, $\frac{1}{8}$) is auxetic for all six axis pairs: at 20% solid, $\nu_{yx} = \nu_{zx} = -0.45$, $\nu_{yz} = -0.17$ and $\nu_{xy} = \nu_{xz} = -0.08$. In the independent data ν_{yx} stays between -0.22 and -0.56 from 7% to 42% solid. The same lattices also have Poisson's ratios above 1 for other axis pairs, which anisotropic solids allow; (0, $\frac{1}{8}$, $\frac{1}{2}$) has $\nu_{xz} \approx 1.1$ and $\nu_{yz} \approx 1.3$, a strong coupling between in-plane and axial strain. The deformation mechanism has not been analysed here. Lateral expansion under tension may matter for how an implant seats against bone, and is left for experimental study.

Sensitivity to tube diameter. A printed tube is rarely exactly its designed size. The relevant measure is the sensitivity $s = d \ln E / d \ln w$: a tube printed 10% thin loses a fraction $1 - 0.9^s$ of its stiffness. For the four gyroid crossing lattices (0, 0, $\frac{1}{2}$), (0, $\frac{1}{8}$, $\frac{1}{2}$), (0, $\frac{1}{4}$, $\frac{1}{2}$) and ($\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$), s stays between 1.5 and 4.1 over the whole range $w = 0.05$ –0.45 (Figure 10c; 2.3–4.1 for the F13LD.lab A set). This is comparable with the skeletal gyroid sized by its thinnest neck (3.3–3.7); the sheet gyroid, sized by its thinnest wall, is 1.3–1.7. Where new tube contacts form, s spikes. Fischer–Koch S (0, $\frac{1}{8}$, $\frac{1}{2}$) reaches 8.6 in mean stiffness, and 19 along z between $w = 0.17$ and 0.19 in F13LD.lab, the contact step of Section 5.2. Fischer–Koch S (0, $\frac{1}{8}$, $\frac{1}{8}$) reaches 20 and (0, $\frac{1}{4}$, $\frac{3}{8}$) 13. At $s = 4$, a 10% undersize costs about a third of the stiffness; at $s = 19$ it costs 86%. Contact thresholds can be located from geometry alone, before any mechanics, by the contact graph of Section 4.4, so a design can place its tube diameter away from them. No contact step appears in the four gyroid crossing lattices over the range computed. Fischer–Koch S, with three times the curve length per cell, forms new contacts even in some of its crossing lattices. The peaks come from the independent data at steps of 0.02 in w ; the one checked with F13LD.lab was confirmed (Section 5.2).

Table 7. Poisson's ratios along the cell axes ($\nu = 0.403$ for the solid). ν_{ij} is the contraction along j per unit extension along i , under uniaxial stress along i . F13LD.lab values at the solid fractions shown; last column: range in the independent data over the solid fractions in brackets, for the lattice once connected.

Lattice	Junctions	Solid %	F13LD.lab	Independent data
PI-gyroid (0, $\frac{1}{8}$, $\frac{1}{2}$)	crossing	1.8–42	ν_{xy} -0.02 to -0.09; ν_{xz} 0.97–1.15; ν_{yz} 1.09–1.38	ν_{xy} -0.04 to -0.10 (2–40%)
PI-gyroid (0, 0, $\frac{1}{2}$)	crossing	19.0	ν_{xy} -0.13; $\nu_{xz} = \nu_{yz}$ 1.41	ν_{xy} -0.06 to -0.13 (2–41%)
PI-gyroid (0, $\frac{1}{8}$, $\frac{1}{8}$)	contact	20.3	$\nu_{xy} = \nu_{xz}$ -0.08; $\nu_{yx} = \nu_{zx}$ -0.45; ν_{yz} -0.17	ν_{yx} -0.22 to -0.56; ν_{yz} -0.14 to -0.17 (7–42%)
PI-gyroid (0, $\frac{1}{4}$, $\frac{1}{4}$)	contact	21.9	ν_{yz} -0.17; ν_{xy} 0.03	ν_{yz} -0.17 to -0.21 (22–40%)
Skeletal gyroid	—	5–40	0.47 to 0.32 (all axis pairs)	—
Sheet gyroid	—	16–64	0.35 to 0.33 (all axis pairs)	—

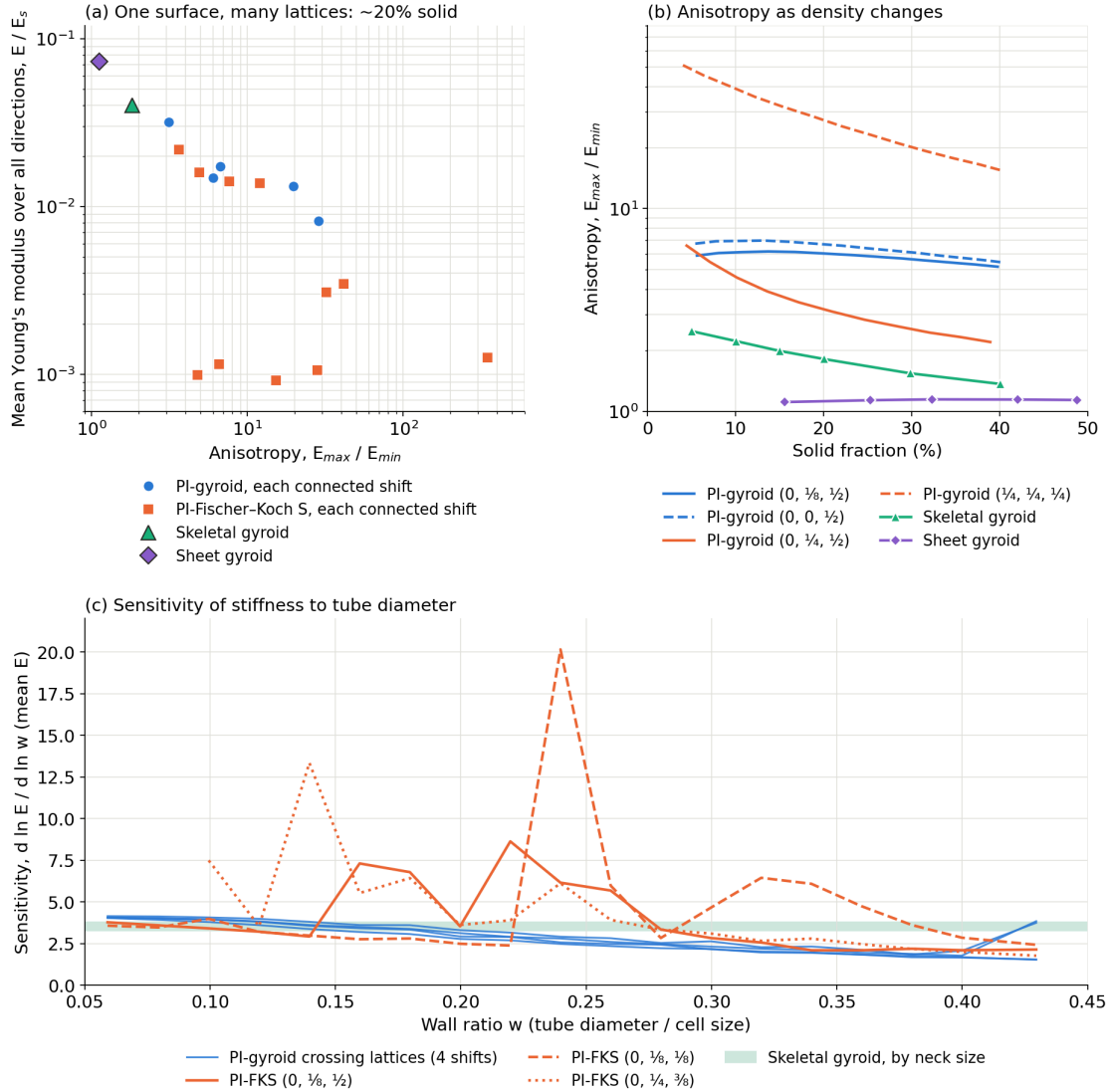


Figure 10. The design space of one surface pair against the sheet and skeletal baselines. (a) Mean Young's modulus over all directions against anisotropy (E_{max}/E_{min}) at about 20% solid: each connected gyroid and Fischer–Koch S shift

(independent data, the wall ratio nearest 20% solid), and the skeletal and sheet gyroids (F13LD.lab). Fischer–Koch S ($\frac{1}{8}$, $\frac{1}{4}$, $\frac{3}{8}$) is included, with the strand caution of Section 4.6. (b) Anisotropy against solid fraction for four gyroid crossing lattices (independent data) and the baselines (F13LD.lab). (c) Sensitivity of mean stiffness to tube diameter, $d \ln E / d \ln w$, from successive wall ratios; shaded, the skeletal gyroid by thinnest neck. Spikes mark the formation of new tube contacts.

6. Implementation and availability

PI-TPMS is implemented in F13LD.tpms, part of the browser-based F13LD suite (f13ld.app). The geometry tools F13LD.tpms and F13LD.grain are open source under the MIT License. The field is evaluated directly on the GPU and rendered by sphere tracing, with no mesh at any stage before export. In the gradient-normalized form, gradients are taken by central differences (step about 0.012 of the cell) with a floor on $|\nabla \phi|$ of 0.08 to guard against critical points, and $|\cos \alpha|$ is clamped to 0.95. Watertight meshes for printing are generated in F13LD.mesh. Stochastic pairs are built in F13LD.grain by pairing two seeds or two generator settings. The surface equations, the mapping between the tube radius and the wall-ratio convention, and all analysis scripts used in Sections 3 and 4 (curve extraction, crossing search, periodic contact graph, wrap dimension and Kac–Rice check) are archived at <https://doi.org/10.5281/zenodo.23090121>. Effective stiffness tensors were computed in F13LD.lab, the suite's WebGPU homogenization solver, which is source-available under the PolyForm Noncommercial License 1.0.0: it is free for research and non-commercial use, and anyone, including commercial organizations, may run it to reproduce or verify published results; the run matrix, the full 6×6 results at both grids and the extrapolated tensors for all 42 runs of Section 5 are archived with the analysis scripts.

Open release. The PI-TPMS construction described here is disclosed openly. The author claims no patent rights in it and intends it to remain freely usable.

7. Discussion

What is new. PI-TPMS belongs to the class of strut networks, and shares its r^2 density scaling with skeletal TPMS and beam lattices. It is distinguished by how the network is generated. One surface pair, varied by shift, produces strands, layers and lattices with different curve lengths and junction counts (Table 1). All of them are defined by one implicit inequality that can be printed without a graph or node model. Crossing junctions give nodes that are smooth by construction. In the gradient-normalized form, the bounded widening at near-tangency adds a natural fillet exactly where the node is. The construction also extends unchanged to non-periodic random fields, where the density follows from the spectrum (equation 9). Two properties of the parent surface enter directly. Its area per cell sets the average curve density (equation 7). Its flat points, whose number is fixed by genus, are where symmetric shifts create six-branch junctions. Mechanically, the shift moves a lattice along a trade-off between mean stiffness and anisotropy at fixed density, so stiffness can be graded through a part at nearly constant porosity and feature size (Section 5.5).

Design rules. Five follow from Sections 3–5:

1. The solid fraction is set by r through equation (6), with A from Table 1 or equation (9), and its average over shifts from equation (7).
2. For a lattice whose topology does not change with density, use a shift whose crossing points already connect it in three directions (Table 1). For the gyroid, that means any shift with a half-cell component, or ($\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$). Crossings alone are not enough: gyroid ($\frac{1}{8}$, $\frac{1}{4}$, $\frac{3}{8}$) has them but forms only layers, and two Fischer–Koch S shifts have them but remain strands.

3. Shifts without crossings, and stochastic pairs, can still give lattices through contact. Their topology then changes with density, and at low density part of the material does not lie on a load path.
4. To set curve density, choose the parent surface by its area per cell (equation 7). For a lattice that is isotropic across a chosen axis, align a body-diagonal shift with that axis, and check that all of the curve belongs to the lattice (Sections 4.6 and 5.4).
5. Choose the tube diameter away from contact thresholds, where stiffness becomes several times more sensitive to print accuracy; the contact graph locates them from geometry alone. To grade density without changing anisotropy, use a shift whose anisotropy is stable with density, such as gyroid $(0, \frac{1}{8}, \frac{1}{2})$ or $(0, 0, \frac{1}{2})$ (Section 5.5).

Relation to bone. Sheet TPMS resemble plate-like trabecular bone, and strut networks resemble its rod-like end [9]. PI-TPMS lets the rod-like end be designed with a chosen network topology and anisotropy. Its low-stiffness, bending-dominated regime ($E \propto \rho^{2.1}$, Section 5.3) [6] is the one relevant to limiting stress shielding [8]. At a 400 μm thinnest feature in titanium, both strut networks fall in the trabecular range, while the sheet gyroid is as stiff as cortical bone (Figure 8b). Body-diagonal shifts add a further match to bone: they give lattices that are isotropic across an axis, and Fischer–Koch S $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is transversely isotropic to within 3%, the symmetry commonly used to model bone.

Limitations.

- Only gyroid and Fischer–Koch S self-pairs on the eighth-cell shift grid are classified here, and contact thresholds are computed for the gyroid only. Other surface pairs, finer shifts and rotations are not classified.
- The mechanics are linear-elastic, homogenized and computational (Section 5). The FFT results lie about 5% below the independent finite-element data, as expected from their opposite directions of convergence, and extrapolated shear moduli are about 6% low. As-built behaviour, including surface roughness on thin tubes, buckling, strength and fatigue, is not addressed.
- Contact junctions may be narrow necks. Their stiffness and strength are not yet characterized.
- The gradient-normalized distance is first-order, so tubes flatten at large r/T : the narrowest width of PI-gyroid $(0, \frac{1}{8}, \frac{1}{2})$ tubes falls to 0.73 of nominal at $r/T = 0.175$. Designs set by a minimum printable feature should be sized on the measured width (Section 5.1).
- For powder-bed processes, the openings between tubes set powder removal. These should be checked for each design.

Future work.

- Classification of other surface pairs and rotations, including mixed pairs, where equation (7) predicts $\langle A \rangle = a_A a_B \langle |\sin \theta| \rangle$.
- Surfaces with four-fold rotation axes, such as Schwarz P and D. There, a fourth-order contact of the form $\text{Re}(z^4)$ would give eight-branch junctions.
- The geometry and mechanics of line contacts, as in Fischer–Koch S $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, and shifts that give a stiffer axis without separate strands.
- Spatially graded shifts $\delta(\mathbf{x})$, which would change topology and stiffness within a part at constant porosity (Section 5.5); the connectivity of shifts between the grid points needs to be checked first.

- The deformation mechanism behind the negative Poisson's ratios of Table 7, and its measurement on printed parts. The strength and fatigue of crossing against contact junctions. Their stiffness at about 20% solid depends more on the direction of the load paths than on the junction type (Section 5.4).
- The connectivity study of stochastic pipe networks.

An intersection of three fields generically gives isolated points rather than curves, so a triple-intersection construction yields separate beads rather than a load-bearing network.

8. Conclusion

Phase-intersected TPMS builds a strut network as a tube around the curve where two level-set fields vanish. The construction is a single implicit inequality. In its gradient-normalized form it has a constant, printable diameter, and its volume is exactly $\pi r^2 L$ below the overlap scale. L is measured per shift for TPMS pairs and predicted from the spectrum for random fields. Generic intersection curves do not branch, so network connectivity comes either from crossing junctions at tangency points, which exist at any radius, or from contact junctions, which depend on the radius. For the 36 gyroid and Fischer–Koch S shifts, every shift without crossings is a set of strands. Crossings joining strands in independent directions give lattices whose topology is independent of density, and at three-fold flat points they form six-branch junctions. The average curve density is set by the square of the parent surface's area per cell. Shifting along a body diagonal makes Young's modulus isotropic across the shift axis, and one such lattice is transversely isotropic to within 3%. Mechanically, the connected PI-gyroid is bending-dominated. At a matched printable feature it is, on average, as stiff as a skeletal gyroid while using more material, so its value lies in the choice of network topology and anisotropy rather than in stiffness per unit mass. The geometry tools are released open source, and the solver is free for research use, so that the construction is easy to use, check and extend.

Acknowledgments

The author thanks Aaron Chow and Vixiv for independent homogenization data on PI-gyroid and PI-Fischer–Koch cells, used here to cross-check the geometric classification and the F13LD.lab results.

AI assistance. The author used Claude (Anthropic), a large language model, to: run analysis code (curve extraction, crossing search, contact graphs and wrap dimension, the stochastic connectivity study and post-processing of the homogenization results), to check derivations, to generate figures, and to generate the LaTeX. The author conceived the PI-TPMS construction, developed the F13LD software including the homogenization solver, ran all homogenization calculations, directed the analyses, reviewed and verified the results, and takes full responsibility for the content. Every AI-assisted analysis can be reproduced from the archived scripts.

Data and code availability

Analysis scripts, curve-length and classification results, figure sources, the Section 5 run matrix and stiffness results, and the script that recomputes every Section 5 number are archived at <https://doi.org/10.5281/zenodo.23090121>. The independent homogenization data were provided by Vixiv and are used with their permission; the archive includes their reference values for the cells compared in Table 5, and requests for the full dataset should be directed to Vixiv. The archive's code is released under

the MIT License, and its data and figures under CC BY 4.0. Software: F13LD.tpms, F13LD.grain, F13LD.mesh and F13LD.lab at f13ld.app. F13LD.tpms and F13LD.grain are licensed under the MIT License; F13LD.lab is licensed under the PolyForm Noncommercial License 1.0.0, with an additional permission for anyone to reproduce or verify published results. The homogenization results were produced with F13LD.lab v0.12.1 (<https://doi.org/10.5281/zenodo.23090621>).

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Appendix A. Level-set equations

Coordinates are normalized so that one cell spans 2π : $x = 2\pi X/T$, and likewise for y and z . All surfaces are taken at their zero level unless noted. The Lidinoid approximation carries a small offset (+0.15) to avoid a self-intersection at the exact zero level of the truncation. This has no effect on the construction, which needs only a smooth periodic field with a well-defined zero set.

Surface	Level-set function $\phi(\mathbf{x}, \mathbf{y}, \mathbf{z})$
Gyroid (G)	$\sin x \cos y + \sin y \cos z + \sin z \cos x$
Schwarz Primitive (P)	$\cos x + \cos y + \cos z$
Schwarz Diamond (D)	$\sin x \sin y \sin z + \sin x \cos y \cos z + \cos x \sin y \cos z + \cos x \cos y \sin z$
Fischer–Koch S	$\cos 2x \sin y \cos z + \cos 2y \sin z \cos x + \cos 2z \sin x \cos y$
Lidinoid (L)	$\frac{1}{2}(\sin 2x \cos y \sin z + \sin 2y \cos z \sin x + \sin 2z \cos x \sin y) - \frac{1}{2}(\cos 2x \cos 2y + \cos 2y \cos 2z + \cos 2z \cos 2x) + 0.15$

For the curve extraction of Appendix D, the gyroid was written as a sum of six plane waves, $\sin a \cos b = \frac{1}{2}[\sin(a + b) + \sin(a - b)]$, with wave vectors $(1, \pm 1, 0)$, $(0, 1, \pm 1)$ and $(\pm 1, 0, 1)$. A shift δ then adds $\mathbf{k} \cdot \delta$ to each wave's phase. Fischer–Koch S was expanded in the same way into twelve plane waves of equal amplitude, using $\cos 2a \sin b \cos c = \frac{1}{4}[\sin(2a + b + c) - \sin(2a - b - c) + \sin(2a + b - c) - \sin(2a - b + c)]$.

Appendix B. Branching, crossings and strands

Regular values. Let $F = (A, B)$ be smooth. Its derivative at $\mathbf{x}$ is the 2×3 matrix with rows $\nabla A(\mathbf{x})$ and $\nabla B(\mathbf{x})$. It has rank 2 unless the two gradients are parallel or one vanishes. If rank 2 holds on all of $C = F^{-1}(0, 0)$, the regular value theorem [17] makes C a smooth embedded submanifold of dimension $3 - 2 = 1$. Every point has a neighbourhood in which C is a single smooth arc, so C has no branch points and no endpoints.

Genericity. A point where both fields vanish and the gradients are parallel satisfies four independent scalar equations in three unknowns. For a pair in general position the solution set is empty. Such points persist only when an additional structure, such as a symmetry of the pair, reduces the number of independent conditions. Rational shifts with small denominators are not generically regular: the half-cell shifts of the gyroid, for example, carry crossing points (Table 1).

Local structure at a tangency. At a tangency point $\mathbf{p}$, rotate coordinates so that the common normal is along z , and write both surfaces as graphs over the tangent plane: $z = \frac{1}{2}\mathbf{u}^T H_A \mathbf{u} + \dots$ and $z = \frac{1}{2}\mathbf{u}^T H_B \mathbf{u} + \dots$, with $\mathbf{u} = (x, y)$. Near $\mathbf{p}$, C is the zero set of $\frac{1}{2}\mathbf{u}^T (H_A - H_B) \mathbf{u} + \dots$. If $H_A - H_B$ is indefinite, this is two transverse branches through $\mathbf{p}$: a four-valent crossing. If it is definite, $\mathbf{p}$ is an isolated point of C .

Strands. For a pair periodic on the lattice Λ , a regular C descends to a compact 1-manifold on the torus $\mathbb{R}^3/\Lambda$: a finite union of closed loops. A loop γ has a homology class $\mathbf{h} \in \Lambda$. Its preimage in $\mathbb{R}^3$ is a union of curves, each invariant under translation by $\mathbf{h}$ alone. A thin tube around such a curve is connected, but it spans only the direction $\mathbf{h}$. Distinct components are disjoint, and at small radius their tubes are disjoint too. The structure therefore has no path joining lattice directions other than $\mathbf{h}$. This is the "strands" topology in Table 1.

Six-branch junctions. Let a rotation S by $2\pi/3$ about the common normal at a tangency point $\mathbf{p}$ map the pair to itself. Then the relative curvature form $Q = H_A - H_B$ on the tangent plane satisfies $S^T Q S = Q$. A symmetric 2×2 matrix commuting with a rotation by $2\pi/3$ is a multiple of the identity, so Q is definite or zero. If $Q = 0$, which holds when both surfaces are flat at $\mathbf{p}$, the leading term of the difference is a cubic form invariant under S . In complex coordinate z on the tangent plane, the invariant real cubics are the multiples of $\text{Re}(e^{i\phi} z^3)$, whose zero set is three lines through $\mathbf{p}$ at 60° spacing. C then has six branches at $\mathbf{p}$. For an exact minimal surface, the flat points are the branch points of the Gauss map. On the quotient surface of genus g , the map has degree $d = g - 1$, and Riemann–Hurwitz gives $2g - 2 = -2d + B$, so the number of branch points is $B = 4(g - 1)$, counted with multiplicity [24, 25].

Isotropy across a three-fold axis. Take a unit vector $\mathbf{n} = \cos t \mathbf{e}_1 + \sin t \mathbf{e}_2$ in the plane perpendicular to the axis. Then $1/E(\mathbf{n}) = S_{ijkl} n_i n_j n_k n_l$ is a homogeneous quartic in $\cos t$ and $\sin t$, and is therefore a trigonometric polynomial with harmonics of order 0, 2 and 4 only. A three-fold axis requires $1/E$ to be unchanged under $t \rightarrow t + 2\pi/3$. A harmonic of order m is unchanged only when m is a multiple of 3. That removes orders 2 and 4 and leaves the constant, so E is the same in every direction across the axis. A four-fold axis requires invariance under $t \rightarrow t + \pi/2$, which keeps order 4, so E can vary around a four-fold axis. For directions tilted out of the plane, the order-3 harmonic survives with a factor $\sin^3 p \cos p$ of the tilt p . This vanishes along the axis and across it, and it is the source of the variation at intermediate tilts in Figure 6e.

Appendix C. Tube volume and curve length of random fields

Tubes around curves. For a compact smooth curve of length L in $\mathbb{R}^3$ and r below its reach, the normal exponential map is a diffeomorphism onto the tube. Weyl's formula [13, 14] expands the tube volume in powers of r , with coefficients that are integrals of intrinsic curvature invariants of the submanifold. A curve has no intrinsic curvature, so every coefficient beyond the first vanishes and $\text{Vol} = \pi r^2 L$ exactly. Deviations at finite radius come only from the tube meeting itself or another tube (reach exceeded) and, in implicit form (4), from the first-order distance approximation.

Translative formula. For compact smooth surfaces $M, N \subset \mathbb{R}^3$, $\int_{\mathbb{R}^3} \text{length}(M \cap (N + \delta)) \, d\delta = \int_M \int_N |\sin \theta(\mathbf{x}, \mathbf{y})| \, dA(\mathbf{x}) \, dA(\mathbf{y})$, where θ is the angle between the normals at $\mathbf{x}$ and $\mathbf{y}$ [22, 23]. For a triply periodic surface paired with its own translate, integrate δ over one cell and divide by its volume T^3 . This gives the shift-averaged curve length per cell, $\bar{L} = A_{\text{cell}}^2 \langle |\sin \theta| \rangle / T^3$, which is equation (7) when divided by T . The mean sine was evaluated from 4×10^5 area-weighted pairs of surface elements from a marching-cubes mesh.

Kac–Rice. For a smooth stationary random map $F = (A, B): \mathbb{R}^3 \rightarrow \mathbb{R}^2$ with independent components, the expected one-dimensional measure of $F^{-1}(0)$ per unit volume is $E[|\nabla A \times \nabla B| \mid F = 0] p_F(0)$ [15]. For independent zero-mean Gaussian fields, the gradients are independent of the values, and $\nabla A, \nabla B \sim N(0, \lambda_2 I)$ for isotropic fields. Write $\nabla A = \sqrt{\lambda_2} \mathbf{z}_1$ and $\nabla B = \sqrt{\lambda_2} \mathbf{z}_2$ with $\mathbf{z}_i$ standard normal. Then $E|\mathbf{z}_1 \times \mathbf{z}_2| = E|\mathbf{z}_1| \cdot E|\mathbf{z}_{2\perp}| = 2\sqrt{(2/\pi)} \cdot \sqrt{(\pi/2)} = 2$, where $\mathbf{z}_{2\perp}$ is the two-dimensional component of $\mathbf{z}_2$ normal to $\mathbf{z}_1$. With $p_F(0) = 1/(2\pi\sigma^2)$, this gives equation (9). For a field that is a normalized sum of plane waves with wave vectors $\mathbf{k}_j$, $\sigma^2 = 1/2$ and the gradient covariance is $(1/2N)\Sigma \mathbf{k}_j \mathbf{k}_j^T$. This reduces to $\lambda_2 = \langle k^2 \rangle / 6$ when isotropic, so $\lambda_2 / \sigma^2 = \langle k^2 \rangle / 3$. For anisotropic spectra we evaluated $E|\nabla A \times \nabla B|$ by Monte Carlo sampling from each field's own gradient covariance.

Appendix D. Numerical methods

Curve extraction. Field A was sampled on a regular grid (160^3 per gyroid cell; 192^3 for the stochastic domains), and its zero surface was extracted by marching cubes [20]. On each triangle whose vertices change sign in B , the two zero crossings of B along its edges define one curve segment, and segments are linked through shared edges. All points were then projected onto C by Gauss–Newton iteration on $(A, B) = (0, 0)$, and segments were resampled to a spacing of 0.01–0.012 (field units). The residuals of the projected points were about 10^{-3} or less. For periodic cells, the sampling window was offset from the cell origin so that curves lying on symmetry planes do not coincide with the window faces. Curve lengths in Tables 1 and 3 were computed on a grid spanning exactly one period ($N + 1$ samples with shared faces), which avoids a thin unsampled slab at the faces. That slab had undercounted length by about $3/N$.

Junction branches. The number of branches at a tangency point was counted as the number of distinct directions, separated by more than 20° , in which curve points leave a spherical shell around the point. Three shells were used, 0.03–0.06, 0.06–0.12 and 0.12–0.20 in field units, and gave the same count. Genus was computed from the total Gaussian curvature of a marching-cubes mesh at 200^3 per cell, using K evaluated from the field's gradient and Hessian. The method recovers genus 3 for Schwarz P and 9 for Schwarz D per cubic cell.

Crossing search. Crossing points were located by least-squares solution of $A = B = 0$ and $\mathbf{n}_A \times \mathbf{n}_B = 0$, seeded from curve points where the sine of the angle between the surfaces fell below 0.15, and de-duplicated modulo the lattice. A residual below 10^{-9} was taken as an exact crossing.

Contact graph and wrap dimension. Pairs of curve points closer than the maximum tube reach were found with a periodic k-d tree. Pairs lying along the same curve, where the connecting vector is within 60° of either tangent, were excluded. For the gradient-normalized form, a pair fuses when its separation is at most the sum of the two local tube reaches along the connecting direction. The reach is r for an unclamped circle, larger where $|\cos \alpha|$ exceeds the 0.95 clamp. The wrap dimension of each connected component was found by a breadth-first search that assigns integer cell offsets to every point. Each cycle that closes with a non-zero offset contributes a wrap vector, and the rank of the collected wrap vectors is the wrap dimension.

Solid fraction and feature size. Solid fractions in Figure 3 were measured on a 240^3 grid per cell. For the stochastic contact model, solid fraction agreed with a 320^3 voxel reference to within 1–5% for both the gradient-normalized and Euclidean forms. Thinnest walls and necks (Table 2) are twice the 1st-percentile inscribed-sphere radius on the mid-surface (sheets) or on the 3-D skeleton (skeletal solids), from a periodic Euclidean distance transform at 160^3 per cell.