

Fraction-valued Piece-wise Linear Linear Programming

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Abstract

We formulate a fraction-valued piece-wise linear linear programming problem as a fraction-valued linear programming programming problem.

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1. Introduction: In this note we are concerned with the maximization of a piece-wise linear function subject to linear inequality constraints where all scalars belong to the ordered field of fractions. We call such problems, fraction-valued piece-wise linear linear programming problems. To be consistent with our terminology, linear programming should be referred to as linear linear programming- which we don't do!!! The definition of piece-wise linear function that we consider can be found in Ovchinnikov (2002, 2018). We assume that the linear inequality constraints define a bounded polyhedron and is thus a convex polytope. (See Frank (2019) and Williamson (2014)). With all scalars being fraction, it is well known that all extreme points of the bounded polyhedron (and hence of the convex polytope) are fraction-valued vectors.

Our main result is that for a fraction-valued piece-wise linear linear programming problem, solutions of a related fraction-valued linear programming problem are solutions of the former problem. Thus, as is well known in linear programming, there exists a solution which is an extreme point of the “constraint set” and, as we have already noted, such a solution must be a fraction-valued vector.

That the problem of finding a *minimizing solution* from the set of solutions of a finite number of linear programming *maximization* problems, can be formulated as a linear programming minimization problem is well known (see page 317 (in section 8.3.3) of Eiselt and Sandblom (2007)). That the problem of finding a *maximizing solution* from the set of solutions of a finite number of linear programming *maximization* problems,

can be formulated as a linear programming maximization problem is proved in Lahiri (2025). A fraction-valued piece-wise linear linear programming problem is about finding *the maximum* from a finite number of *max-min* linear programming problems. Although the objective function of the linear programming problem formulated here is “similar” to the objective function of the linear programming problem formulated in Lahiri (2025), there may be some novelty in our formulation of a fraction-valued piece-wise linear linear programming problem as a single fraction-valued (linear) linear programming problem.

2. Some Notations: Let $\mathbb{Q}$ denote the set of fractions in the real line, $\mathbb{Q}_+$ the set of non-negative fractions and $\mathbb{Q}_{++}$ the set of strictly positive fractions. For any non-empty subset X of $\mathbb{Q}$, and positive integer ‘ r ’, let X^r denote the r -fold Cartesian product of X . Clearly, $X^1 = X$.

For any non-empty subset X of $\mathbb{Q}$, and positive integers ‘ r ’, ‘ s ’, let $X^{r \times s}$ denote the set of all $r \times s$ matrices with entries in X . Clearly, $X^{1 \times 1} = X$.

A point in $X^{1 \times s}$ is referred to as an **s -dimensional (X-valued) row vector**, and a point in $X^{r \times 1}$ is referred to as a **r -dimensional (X-valued) column vector**.

Unless otherwise mentioned a point in X^r will be understood as an r -dimensional (X-valued) column vector.

Given $F \in X^{r \times s}$ and $(h, k) \in \{1, \dots, r\} \times \{1, \dots, s\}$, let $F_h \in X^{1 \times s}$ denote the h^{th} row of the matrix F and $F^k \in X^{r \times 1}$ ($= X^r$) denote the k^{th} column of the matrix F .

3. Fraction-valued Piece-wise Linear Programming Problem: For positive integers m_1, n , consider the pair $(C, d) \in (\mathbb{Q}^{m_1 \times n}, \mathbb{Q}^{m_1})$. Let $\mathcal{H}$ be a non-empty collection of non-empty subsets of $\{1, \dots, m_1\}$.

A **fraction-valued piece-wise linear function** $f: \mathbb{Q}^n \rightarrow \mathbb{Q}$ based on $(C, d, \mathcal{H})$ is defined as follows: for all $x \in \mathbb{Q}^n$, $f(x) = \max_{H \in \mathcal{H}} \min_{h \in H} \{C_h x + d_h\}$.

This definition along with in-depth characterization results are available in Ovchinnikov (2002, 2018).

Example 1: For $m_1 = 3$ and $n = 1$, let $C_1 = 1, C_2 = -1, C_3 = 0, d_1 = 0, d_2 = 2, d_3 = \frac{1}{2}$. Let

$\mathcal{H} = \{\{1, 2\}, \{3\}\}$. For $H = \{1, 2\}$, $\min_{h \in H} \{C_h x + d_h\} = x$ for all $x \leq 1$ and

$\min_{h \in H} \{C_h x + d_h\} = 2 - x$ for all $x \geq 1$.

$f(x) = \max_{H \in \mathcal{H}} \min_{h \in H} \{C_h x + d_h\} = \frac{1}{2}$ for all $x \leq \frac{1}{2}$, $f(x) = x$ for all $x \in [\frac{1}{2}, 1]$, $f(x) = 2 - x$ for all

$x \in [1, \frac{3}{2}]$ and $f(x) = \frac{1}{2}$ for all $x \geq \frac{3}{2}$. ■

Note 1: Two special cases of fraction-valued piece-wise linear function are $\mathcal{H} = \{1, \dots, m\}$ and $\mathcal{H} = \{h | h \in \{1, \dots, m\}\}$. In the first case the function becomes a *concave* piece-wise linear function and in the second case the function becomes a *convex* piece-wise linear function.

For a positive integer m_2 , given $(A, b) \in (\mathbb{Q}^{m_2 \times n}, \mathbb{Q}^{m_2})$ we are interested in the following maximization problem which we refer to as **piece-wise linear linear programming problem (PWL-LP)**: Maximize $f(x)$, subject to $Ax \leq b$.

For what follows we **assume** that $\{x \in \mathbb{Q}^n | Ax \leq b\}$ is a bounded polyhedron. Hence by the Strong Duality Theorem of Linear programming (for instance as in theorem 2 in topic 3 of Lahiri (2023)), we know that every linear programming problem with constraints given by $Ax \leq b$, has a solution which is an extreme point of the bounded polyhedron.

4. Understanding Piece-wise Linear Linear Programming Problem: In this section we provide a proposition that helps in understanding what piece-wise linear linear programming actually is.

Proposition 1: $\hat{x} \in \mathbb{Q}^n$ solves PWL-LP if and only if $A\hat{x} \leq b$ and there exists an array of fractions $\langle \hat{u}^H | H \in \mathcal{H} \rangle$ and an array $\langle \hat{x}^H | H \in \mathcal{H} \rangle$ in $\mathbb{Q}^n$ such that $\hat{x}, \langle \hat{u}^H | H \in \mathcal{H} \rangle, \langle \hat{x}^H | H \in \mathcal{H} \rangle$ satisfy the following:

(i) For each $H \in \mathcal{H}$, let $\hat{x}^H, \hat{u}^H$ solve:

Maximize u^H subject to $C_h x^H + d_h \geq u^H$ for all $h \in H$, $Ax^H \leq b$, $x^H \in \mathbb{Q}^n$, $u^H \in \mathbb{Q}$; and

(ii) $f(\hat{x}) = \hat{u}^{\hat{H}}$ for all $\hat{H} \in \arg\max_{H \in \mathcal{H}} \{\hat{u}^H\}$

Proof: Suppose $\hat{x} \in \mathbb{Q}^n$ solves PWL-LP.

Then, $f(\hat{x}) = \max_{H \in \mathcal{H}} \min_{h \in H} \{C_h \hat{x} + d_h\}$ implies there exists $\hat{H} \in \mathcal{H}$ such that $f(\hat{x}) =$

$\min_{h \in \hat{H}} \{C_h \hat{x} + d_h\}$ and $\min_{h \in \hat{H}} \{C_h \hat{x} + d_h\} \geq \min_{h \in H} \{C_h \hat{x} + d_h\}$ for all $H \in \mathcal{H}$.

$\hat{x} \in \mathbb{Q}^n$ solves PWL-LP implies $A\hat{x} \leq b$ and for all $x \in \mathbb{Q}^n$ satisfying $Ax \leq b$, $f(\hat{x}) =$

$\min_{h \in \hat{H}} \{C_h \hat{x} + d_h\} \geq f(x) \geq \min_{h \in H} \{C_h x + d_h\}$ for all $H \in \mathcal{H}$.

For each $H \in \mathcal{H}$, let $\hat{x}^H, \hat{u}^H$ solve:

Maximize u^H subject to $C_h x^H + d_h \geq u^H$ for all $h \in H$, $Ax^H \leq b$, $x^H \in \mathbb{Q}_+^n$, $u^H \in \mathbb{Q}$.

Since $\{x \in \mathbb{Q}^n | Ax \leq b\}$ is a bounded polyhedron, such a solution always exists.

Thus, it must be the case that $\min_{h \in H} \{C_h \hat{x}^H + d_h\} = \hat{u}^H$.

Clearly, $\min_{h \in \hat{H}} \{C_h \hat{x} + d_h\} \geq f(\hat{x}^{\hat{H}}) \geq \min_{h \in H} \{C_h \hat{x}^H + d_h\} = \hat{u}^H$ for all $H \in \mathcal{H}$.

In particular, $\min_{h \in \hat{H}} \{C_h \hat{x} + d_h\} \geq \min_{h \in \hat{H}} \{C_h \hat{x}^{\hat{H}} + d_h\} = \hat{u}^{\hat{H}}$.

Since $\hat{x}^{\hat{H}}, \hat{u}^{\hat{H}}$ solve:

Maximize $u^{\hat{H}}$ subject to $C_h x^{\hat{H}} + d_h \geq u^{\hat{H}}$ for all $h \in \hat{H}$, $Ax^{\hat{H}} \leq b$, $x^{\hat{H}} \in \mathbb{Q}^n$, $u^{\hat{H}} \in \mathbb{Q}$

and since $\hat{x}, u^{\hat{H}} = \min_{h \in \hat{H}} \{C_h \hat{x} + d_h\}$ satisfies all the constraints of this maximization

problem, it must be the case that $\hat{u}^{\hat{H}} \geq \min_{h \in \hat{H}} \{C_h \hat{x} + d_h\}$.

Combined with $\min_{h \in \hat{H}} \{C_h \hat{x} + d_h\} \geq \hat{u}^{\hat{H}}$ that we obtained earlier, we get $\hat{u}^{\hat{H}} \geq$

$$\min_{h \in \hat{H}} \{C_h \hat{x} + d_h\}.$$

Thus, $f(\hat{x}) = \hat{u}^{\hat{H}}$.

We already know that $f(\hat{x}) = \min_{h \in \hat{H}} \{C_h \hat{x} + d_h\} \geq \min_{h \in H} \{C_h \hat{x}^H + d_h\} = \hat{u}^H$ for all $H \in \mathcal{H}$.

Thus, $\hat{u}^{\hat{H}} \geq \hat{u}^H$ for all $H \in \mathcal{H}$ so that $\hat{H} \in \arg\max_{H \in \mathcal{H}} \{\hat{u}^H\}$.

Thus, $\hat{x}, \langle \hat{u}^H | H \in \mathcal{H} \rangle, \langle \hat{x}^H | H \in \mathcal{H} \rangle$ satisfy (i) and (ii) in the statement of this proposition.

Now suppose, $A\hat{x} \leq b$ and there exists an array of fractions $\langle \hat{u}^H | H \in \mathcal{H} \rangle$ and an array

$\langle \hat{x}^H | H \in \mathcal{H} \rangle$ in $\mathbb{Q}^n$ such that $\hat{x}, \langle \hat{u}^H | H \in \mathcal{H} \rangle, \langle \hat{x}^H | H \in \mathcal{H} \rangle$ satisfy (i) and (ii) in the

statement of this proposition. Let $x \in \mathbb{Q}^n$ satisfy $Ax \leq b$.

Thus, by (i) in the statement of this proposition, for all $H \in \mathcal{H}$, $\min_{h \in H} \{C_h x + d_h\} \leq \hat{u}^H$.

$$\text{Thus, } f(x) = \max_{H \in \mathcal{H}} \min_{h \in H} \{C_h x + d_h\} \leq \max_{H \in \mathcal{H}} \{\hat{u}^H\}.$$

By (ii) in the statement of this proposition, for all $\hat{H} \in \arg\max_{H \in \mathcal{H}} \{\hat{u}^H\}$ it must be the case

that $f(\hat{x}) = \hat{u}^{\hat{H}}$.

Thus, $\hat{x}$ solves PWL-LP. Q.E.D.

5. Representing the piece-wise linear programming problem as a single linear

programming problem: We now provide the main result whereby we characterize a piece-wise linear programming problem as a single linear programming problem.

Proposition 2: (I) $\hat{x} \in \mathbb{Q}^n$ solves PWL-LP if and only if $A\hat{x} \leq b$ and there exists a

fraction $\hat{u}$, an array of fractions $\langle \hat{u}^H | H \in \mathcal{H} \rangle$ and an array $\langle \hat{x}^H | H \in \mathcal{H} \rangle$ in $\mathbb{Q}^n$ such that:

(i) $\hat{u}, \langle \hat{u}^H | H \in \mathcal{H} \rangle, \langle \hat{x}^H | H \in \mathcal{H} \rangle$ solve the following linear programming problem (**LP-**

OPT): Maximize $2\sum_{H \in \mathcal{H}} u^H - u$ subject to $u - u^H \geq 0$, $C_h x^H + d_h \geq u^H$ for all $h \in H$, $H \in \mathcal{H}$,

$Ax^H \leq b$, $x^H \in \mathbb{Q}^n$, $u^H \in \mathbb{Q}$, for all $H \in \mathcal{H}$, $u \in \mathbb{Q}$, and

(ii) $f(\hat{x}) = \hat{u}$.

(II) For all $\hat{H} \in \arg\max_{H \in \mathcal{H}} \{\hat{u}^H\}$, it is the case that $\hat{x}^{\hat{H}}$ solves PWL-LP.

Proof: (I) Suppose $\hat{x} \in \mathbb{Q}^n$ solves PWL-LP.

Then, by proposition 1, there exists an array of fractions $\langle \hat{u}^H | H \in \mathcal{H} \rangle$ and an array $\langle \hat{x}^H | H \in \mathcal{H} \rangle$ in $\mathbb{Q}^n$ such that $\hat{x}$, $\langle \hat{u}^H | H \in \mathcal{H} \rangle$, $\langle \hat{x}^H | H \in \mathcal{H} \rangle$ satisfy the following:

(i) For each $H \in \mathcal{H}$, let $\hat{x}^H$, $\hat{u}^H$ solve:

Maximize u^H subject to $C_h x^H + d_h \geq u^H$ for all $h \in H$, $Ax^H \leq b$, $x^H \in \mathbb{Q}_+^n$, $u^H \in \mathbb{Q}$; and

(ii) $f(\hat{x}) = \hat{u}^{\hat{H}}$ for all $\hat{H} \in \arg\max_{H \in \mathcal{H}} \{\hat{u}^H\}$.

Let $\hat{u} = \max_{H \in \mathcal{H}} \{\hat{u}^H\} = \hat{u}^{\hat{H}}$ where $\hat{H} \in \arg\max_{H \in \mathcal{H}} \{\hat{u}^H\}$.

Thus, $2\sum_{H \in \mathcal{H}} \hat{u}^H - \hat{u} = 2\sum_{H \in \mathcal{H} \setminus \{\hat{H}\}} \hat{u}^H + \hat{u}^{\hat{H}}$

Towards a contradiction suppose that there exists a fraction u^* , an array of fractions $\langle u^{*H} | H \in \mathcal{H} \rangle$ and an array $\langle x^{*H} | H \in \mathcal{H} \rangle$ in $\mathbb{Q}^n$ such that $u^* - u^{*H} \geq 0$, $C_h x^{*H} + d_h \geq u^{*H}$ for all $h \in H$, $H \in \mathcal{H}$, $Ax^{*H} \leq b$, and $2\sum_{H \in \mathcal{H}} u^{*H} - u^* > 2\sum_{H \in \mathcal{H}} \hat{u}^H - \hat{u} = 2\sum_{H \in \mathcal{H} \setminus \{\hat{H}\}} \hat{u}^H + \hat{u}^{\hat{H}}$.

Since, for each $H \in \mathcal{H}$, let $\hat{x}^H$, $\hat{u}^H$ solve the linear programming problem in (i), it must be the case that for each $H \in \mathcal{H}$, $\hat{u}^H \geq u^{*H}$.

$2\sum_{H \in \mathcal{H}} u^{*H} - u^* = 2\sum_{H \in \mathcal{H} \setminus \{\hat{H}\}} u^{*H} + 2u^{*\hat{H}} - u^*$.

Thus, $2\sum_{H \in \mathcal{H} \setminus \{\hat{H}\}} u^{*H} + 2u^{*\hat{H}} - u^* > 2\sum_{H \in \mathcal{H} \setminus \{\hat{H}\}} \hat{u}^H + \hat{u}^{\hat{H}}$ implies

$2u^{*\hat{H}} - \hat{u}^{\hat{H}} - u^* > 2(\sum_{H \in \mathcal{H} \setminus \{\hat{H}\}} \hat{u}^H - \sum_{H \in \mathcal{H} \setminus \{\hat{H}\}} u^{*H}) \geq 0$, i.e., $2u^{*\hat{H}} - \hat{u}^{\hat{H}} - u^* > 0$.

$2u^{*\hat{H}} - \hat{u}^{\hat{H}} - u^* = u^{*\hat{H}} + u^{*\hat{H}} - \hat{u}^{\hat{H}} - u^* \leq u^{*\hat{H}} - u^*$ since $u^{*\hat{H}} - \hat{u}^{\hat{H}} \leq 0$.

Thus, $u^{*\hat{H}} - u^* \geq 2u^{*\hat{H}} - \hat{u}^{\hat{H}} - u^* > 0$, thereby contradicting the requirement $u^* - u^{*H} \geq 0$ for all $h \in H$.

Thus, $\hat{u}$, $\langle \hat{u}^H | H \in \mathcal{H} \rangle$, $\langle \hat{x}^H | H \in \mathcal{H} \rangle$ solve LP-OPT.

Thus, $\hat{u} = \max_{H \in \mathcal{H}} \{\hat{u}^H\}$, since at the maximum it is not possible that all constraints of $u - u^H \geq 0$ for all $H \in \mathcal{H}$, will be satisfied with strict inequality.

Hence, by (ii) in the statement of proposition 1, $f(\hat{x}) = \hat{u} = \max_{H \in \mathcal{H}} \{\hat{u}^H\}$.

Now suppose, $\hat{u}$, $\langle \hat{u}^H | H \in \mathcal{H} \rangle$, $\langle \hat{x}^H | H \in \mathcal{H} \rangle$ solve LP-OPT and $f(\hat{x}) = \hat{u}$.

Since $\hat{u}$, $\langle \hat{u}^H | H \in \mathcal{H} \rangle$, $\langle \hat{x}^H | H \in \mathcal{H} \rangle$ solve LP-OPT it must be the case that $\hat{u} = \max_{H \in \mathcal{H}} \{\hat{u}^H\}$.

Thus, $2\sum_{H \in \mathcal{H}} \hat{u}^H - \hat{u} = 2\sum_{H \in \mathcal{H} \setminus \{\hat{H}\}} \hat{u}^H + \hat{u}^{\hat{H}}$ for all $\hat{H} \in \arg\max_{H \in \mathcal{H}} \{\hat{u}^H\}$.

Towards a contradiction suppose there exists $x \in \mathbb{Q}^n$ satisfying $Ax \leq b$ and $f(x) > f(\hat{x}) = \hat{u}$.

Thus, $f(x) = \max_{H \in \mathcal{H}} \min_{h \in H} \{C_h x + d_h\}$ implies there exists $H^* \in \mathcal{H}$ such that $f(x) =$

$$\min_{h \in H^*} \{C_h x + d_h\} = u^{*H^*} \geq \min_{h \in H} \{C_h x + d_h\} \text{ for all } H \in \mathcal{H}.$$

Further $u^{*H^*} > \hat{u} \geq \hat{u}^{H^*}$

Consider the array $\langle y^H | H \in \mathcal{H} \rangle$ in $\mathbb{Q}^n$ and the array of fractions $\langle v^H | H \in \mathcal{H} \rangle$ where $y^H = x^H$ for all $H \in \mathcal{H} \setminus \{H^*\}$, $y^{H^*} = x$, $v^H = \hat{u}^H$ for all $H \in \mathcal{H} \setminus \{H^*\}$, $v^{H^*} = u^{*H^*}$.

Since $u^{*H^*} > \hat{u} = \max_{H \in \mathcal{H}} \{\hat{u}^H\}$, it must be the case that $v^* = \max_{H \in \mathcal{H}} \{v^H\} = u^{*H^*} > \hat{u}$, so that

$v^* - v^H \geq 0$ for all $H \in \mathcal{H}$.

Thus, $v^*, \langle y^H | H \in \mathcal{H} \rangle, \langle v^H | H \in \mathcal{H} \rangle$ satisfy all the constraints of LP-OPT.

$$\begin{aligned} 2\sum_{H \in \mathcal{H}} v^H - v^* &= 2\sum_{H \in \mathcal{H} \setminus \{H^*\}} \hat{u}^H + 2u^{*H^*} - v^* > 2\sum_{H \in \mathcal{H} \setminus \{H^*\}} \hat{u}^H + 2u^{*H^*} - u^{*H^*} = \\ 2\sum_{H \in \mathcal{H} \setminus \{H^*\}} \hat{u}^H + u^{*H^*} &> 2\sum_{H \in \mathcal{H} \setminus \{H^*\}} \hat{u}^H + \hat{u}^{H^*} \geq 2\sum_{H \in \mathcal{H} \setminus \{H^*\}} \hat{u}^H + 2\hat{u}^{H^*} - \hat{u}, \text{ since } \hat{u} \geq \hat{u}^{H^*}. \end{aligned}$$

Thus, $2\sum_{H \in \mathcal{H}} v^H - v^* > 2\sum_{H \in \mathcal{H} \setminus \{H^*\}} \hat{u}^H + 2\hat{u}^{H^*} - \hat{u} = 2\sum_{H \in \mathcal{H}} \hat{u}^H - \hat{u}$, contradicting our assumption that $\hat{u}, \langle \hat{u}^H | H \in \mathcal{H} \rangle, \langle \hat{x}^H | H \in \mathcal{H} \rangle$ solve LP-OPT.

Thus, $\hat{x}$ solves PWL-LP.

(II) Since $\hat{u} = \max_{H \in \mathcal{H}} \{\hat{u}^H\}$, for all $\hat{H} \in \arg\max_{H \in \mathcal{H}} \{\hat{u}^H\}$, $f(\hat{x}^{\hat{H}}) = \hat{u}^{\hat{H}} = \hat{u}$.

Thus, for all $\hat{H} \in \arg\max_{H \in \mathcal{H}} \{\hat{u}^H\}$, $\hat{x}^{\hat{H}}$ solves PWL-LP. Q.E.D.

References

1. Eiselt, H. A. and Sandblom, C. -L. (2007): Linear Programming and its Applications. Springer-Verlag.
2. Frank, A. (2019): Introduction to Polyhedral Combinatorics. Scribe: Attila Joo. Available at: <https://www.math.uni-hamburg.de/en/forschung/bereiche/dm/personen/ehemalige/joo-attila/wise-2018-polyhedral-combinatorics/pc-introduction.pdf>
3. Lahiri, S. (2023): Essentials of Linear Programming. Available at: https://www.academia.edu/83551516/Essentials_of_Linear_Programming
4. Lahiri, S. (2025): The Max-max Linear Programming Problem. <https://doi.org/10.13140/RG.2.2.30654.96328/1>
5. Ovchinnikov, S. (2002): Max-Min Representations of Piecewise Linear Functions. *Contributions to Algebra and Geometry*, Volume 43, No.1, Pages 297-302.
6. Ovchinnikov, S. (2018): Max-Min Representations of Piecewise Linear Functions. <https://doi.org/10.48550/arXiv.math/0009026>

7. Williamson, D. P. (2014): Lecture 3 of Mathematical Programming I (ORIE 6300, Cornell University). Scribe: Divya Singhvi. Available at:
<https://people.orie.cornell.edu/dpw/orie6300/Lectures/lec03.pdf>