

Numerical Stability Analysis of a Coupled Optical–Terahertz Schrödinger–Boussinesq Model with Quadratic Nonlinearity

Yizhou Wang^{1*} Irina Zakharova^{2†}

¹Lomonosov Moscow State University, Leninskie Gory 1–52, Moscow 119991, Russian Federation.

²Shenzhen MSU–BIT University, 1 International University Park Road, Dayun New Town, Longgang District, Shenzhen, Guangdong Province 518172, P.R. China.

Abstract

We examine a coupled optical–terahertz model of Schrödinger–Boussinesq type with quadratic nonlinearity and the numerical results reported in the source study. The continuous model combines a complex optical envelope equation with a real terahertz wave equation. We present its localized wave profile, normalization, centered finite-difference formulation, and two-stage optical iteration. We then distinguish that formal formulation from the explicit update routines used to generate the numerical figures. In the first experiment, the optical profile remains localized over early propagation layers; the terahertz field subsequently develops large excursions, followed by numerical blow-up in both fields. In a separate experiment, the plotted control factor multiplies fourth-order terahertz dispersion; the larger of two displayed factors produces stronger terahertz excursions. We formulate amplitude, residual, and grid-refinement diagnostics for assessing numerical instability and convergence. The available plots do not establish stability of the continuous soliton or a mesh-independent critical parameter.

Keywords: optical–terahertz soliton; Zakharov–Boussinesq system; quadratic nonlinearity; finite differences; numerical instability; grid convergence.

1 Introduction

Terahertz radiation is used in imaging, spectroscopy, security screening, materials research, and biomedical investigations [11, 13, 12]. Optical rectification of ultrashort laser pulses in media with second-order nonlinearity is an important route to its generation; experiments with tilted pulse fronts have demonstrated energetic terahertz pulses [22]. Because the generated terahertz signal can be broadband and close to a single cycle, a slowly varying envelope description of the terahertz field may be inadequate. Models that retain a wave equation for this field and couple it to an optical envelope therefore provide a useful basis for studying the interaction [2, 1].

The practical interest in this frequency range is accompanied by a difficult modeling problem. The optical pulse and the generated terahertz field occupy widely separated frequency scales, yet they exchange energy through the quadratic response of the medium. The optical component can often be represented by a complex envelope, whereas a broadband terahertz

*Yizhou Wang: Corresponding author. Master’s Student, Department of Automation for Scientific Research, Faculty of Computational Mathematics and Cybernetics, Lomonosov Moscow State University; email: saipeizyug-mail.com.

†Irina Zakharova: Cand. Phys. & Math. Sci., Associate Professor, Faculty of Computational Mathematics and Cybernetics, Shenzhen MSU–BIT University. Associate Professor, Faculty of Physics, Lomonosov Moscow State University.

pulse may contain too few carrier oscillations for an envelope approximation to remain reliable. A model for the coupled system must therefore retain optical group-velocity dispersion, terahertz wave propagation, nonlinear source terms, and the transverse geometry of a tilted pulse front.

Systems of coupled nonlinear Schrödinger equations are useful when both interacting signals contain many carrier periods. The general theory of solitons and ultrashort optical pulses provides the mathematical background for such reductions [16, 14]. For the short terahertz pulses considered here, a more general wave equation is needed for the low-frequency component. Earlier studies of few-cycle and extremely short electromagnetic solitons show why models beyond the conventional envelope approximation are relevant [15, 17, 18]. The Zakharov–Boussinesq-type model used in this work combines a first-order propagation equation for the optical envelope with a second-order wave equation for the terahertz field. The resulting system is sufficiently detailed to contain second- and fourth-order temporal dispersion, transverse diffraction, quadratic terahertz self-action, and optical driving of the terahertz field.

Sazonov and Ustinov developed optical–terahertz soliton solutions with tilted wave fronts for a Zakharov–Boussinesq-type system [2, 23, 1]. Related studies examined waveguide generation, spatial confinement, and numerical stability of optical–terahertz structures [4, 5, 6, 3]. In the reduced Yajima–Oikawa description, optical dispersion and nonlinear coupling already permit a localized wave packet. A Yajima–Oikawa–Kadomtsev–Petviashvili extension adds weak terahertz dispersion and small wave-front tilt while retaining a unidirectional propagation approximation. The more general Zakharov–Boussinesq family retains higher-order terahertz dispersion and transverse diffraction without that approximation. Here we illustrate localized profiles and examine Gaussian fields propagated by related coupled update routines.

We present a staggered finite-difference formulation and analyze two numerical plotting programs from the underlying study. Iterative finite-difference constructions for nonlinear optical propagation and related two-stage solvers motivate this formulation [7, 20]. Numerical work on resonant Boussinesq models, Schrödinger–Boussinesq systems, spectral differentiation, transparent boundaries, and high-resolution finite-difference methods provides a broader computational context [8, 9, 19, 21, 10]. We examine the fields at several propagation labels in the first program and compare two multipliers of fourth-order terahertz dispersion in the second. The resulting plots support a qualitative discussion of numerical blow-up for the displayed calculations.

The objectives are to state the coupled model and localized-profile illustration, present the formal difference equations, identify the updates actually used in the two plotting programs, and evaluate the displayed numerical behavior. The Gaussian numerical initial data and the illustrated localized profile are treated separately. This structure clarifies which conclusions the figures can support.

2 Coupled optical–terahertz model

2.1 From reduced descriptions to the present system

The model belongs to a sequence of descriptions with increasing physical content. In a slowly varying envelope model, both interacting components are represented by first-order propagation equations. This form is convenient but becomes restrictive when the terahertz pulse is broadband or close to a single cycle. A reduced Yajima–Oikawa system retains optical dispersion and resonant coupling between a short-wave envelope and a long-wave component. It explains how a localized coupled structure can arise from a balance between dispersion and nonlinearity.

For small wave-front angles, transverse corrections can be introduced through a Yajima–Oikawa–Kadomtsev–Petviashvili-type extension. That model incorporates weak diffraction

and weak long-wave dispersion while still relying on a preferred propagation direction. The Zakharov–Boussinesq-type system used here removes this restriction for the terahertz component by retaining a second-order wave equation. It includes both second- and fourth-order temporal terms as well as transverse diffraction. This hierarchy motivates the use of the more general system for a numerical stability calculation.

Tilted wave fronts are relevant because they permit matching between the optical group motion and the terahertz phase propagation. In the reduced soliton theory, the tilt angle and the material parameters determine whether the optical and terahertz components can remain localized in the same moving coordinate. In the numerical problem, the same geometry enters through $z' = z \cos \theta + x \sin \theta$ and through the transverse derivatives with respect to x .

Let $\psi(z, x, t)$ denote the complex envelope of the optical field and let $E_T(z, x, t)$ denote the real terahertz electric field. The propagation coordinate is z , x is transverse position, and t is time. We consider the coupled equations

$$\frac{\partial \psi}{\partial z} = i \frac{2}{\beta v_g^2} \psi + i \frac{\beta}{2} \frac{\partial^2 \psi}{\partial t^2} - i \alpha E_T \psi - i \frac{c}{2 n_\omega \omega} \frac{\partial^2 \psi}{\partial x^2}, \quad (1)$$

$$\frac{\partial^2 E_T}{\partial z^2} = \frac{n_T^2}{c^2} \frac{\partial^2 E_T}{\partial t^2} + \frac{\partial^2}{\partial t^2} (\mu E_T^2 + \sigma |\psi|^2) - \gamma \frac{\partial^4 E_T}{\partial t^4} - a \frac{\partial^2 E_T}{\partial x^2}. \quad (2)$$

Here v_g is the optical group velocity, c is the speed of light, n_ω and n_T are the optical and terahertz refractive indices, respectively, and ω is the optical angular frequency. The coefficient α describes the optical–terahertz coupling in Eq. (1); a is the transverse-diffraction coefficient in Eq. (2); β and γ are temporal-dispersion coefficients; and μ and σ describe the terahertz quadratic nonlinearity and optical-intensity driving term, respectively. These meanings are used consistently below.

The original presentation of the model used the letter a in both the optical coupling and terahertz transverse terms, whereas its later difference formulas and Python programs use α for optical coupling and a for the transverse term. We adopt the latter distinction throughout this paper. This resolves a symbol conflict in the underlying material; it does not assert that the two coefficients have the same numerical value.

2.2 Interpretation of the model terms

The linear term $2i\psi/(\beta v_g^2)$ produces a phase contribution in the optical equation. Second-order temporal dispersion is represented by $i\beta\psi_{tt}/2$. Transverse spreading is described by the term containing ψ_{xx} , and $-i\alpha E_T \psi$ provides the local coupling through which the terahertz field changes the optical phase and profile.

In the terahertz equation, $n_T^2 E_{T,tt}/c^2$ is the linear wave contribution. The term $\mu(E_T^2)_{tt}$ describes quadratic self-action of the terahertz field, and $\sigma(|\psi|^2)_{tt}$ is the source generated by the optical intensity. The coefficient γ weights the fourth-order temporal-dispersion term, and a controls transverse diffraction of the terahertz component. The optical field is complex, whereas E_T is real; consequently the terahertz source depends on $|\psi|^2$ rather than on ψ^2 .

Table 1: Notation used throughout the paper.

Symbol	Meaning
z	propagation coordinate
x	transverse coordinate
t	time coordinate
ψ	complex optical-field envelope
E_T	real terahertz electric field
α	optical-terahertz coupling coefficient
a	transverse-diffraction coefficient in the continuous terahertz equation
p	multiplier of fourth-order terahertz dispersion in the second program
β, γ	second- and fourth-order temporal-dispersion coefficients
μ, σ	terahertz self-action and optical driving coefficients
n_ω, n_T	optical and terahertz refractive indices

The computational domain is $0 < z < L_z$, $0 < x < L_x$, and $0 < t < L_t$. The stated propagation data are

$$\psi(0, x, t) = \psi_0(x, t), \quad E_T(0, x, t) = 0, \quad \frac{\partial E_T}{\partial z}(0, x, t) = 0. \quad (3)$$

Homogeneous boundary values for ψ and E_T are imposed at the ends of the finite x and t intervals. More explicitly, the computational boundary conditions are

$$\psi(z, 0, t) = \psi(z, L_x, t) = 0, \quad \psi(z, x, 0) = \psi(z, x, L_t) = 0, \quad (4)$$

$$E_T(z, 0, t) = E_T(z, L_x, t) = 0, \quad E_T(z, x, 0) = E_T(z, x, L_t) = 0. \quad (5)$$

The source difference formulation also sets the temporal curvature of the terahertz field to zero at the two temporal boundaries,

$$E_{T,tt}(z, x, 0) = E_{T,tt}(z, x, L_t) = 0. \quad (6)$$

Together with Eq. (3), these conditions define the finite-domain propagation problem used in the numerical construction.

The localized-profile illustration uses the following fields:

$$\psi = \psi_m \operatorname{sech}^2 \xi, \quad E_T = -E_m \operatorname{sech}^2 \xi, \quad \xi = \frac{t - z'/v}{\tau_p}, \quad z' = z \cos \theta + x \sin \theta. \quad (7)$$

The amplitudes in that appendix are computed from

$$B_\theta = \beta - \frac{c}{n_\omega \omega} \frac{\sin^2 \theta}{v^2}, \quad E_m = \frac{3B_\theta}{4\alpha\tau_p^2}, \quad \psi_m = \frac{3}{4\alpha\tau_p^2} \sqrt{\frac{B_\theta(2\alpha\gamma - \mu B_\theta)}{\sigma}}. \quad (8)$$

These expressions reproduce the plotting code and require a nonnegative radicand for a real optical amplitude. They are used to visualize a localized pair; the appendix does not verify by substitution that the plotted pair solves Eqs. (1)–(2) with the stated finite-domain conditions. The plotted values are $\theta = 30^\circ$, $v = 1$, $\tau_p = 1$, $\alpha = 1$, $\beta = 0.5$, $\mu = 0.2$, $\gamma = 0.1$, and $\sigma = 1$.

The packet center is defined by $\xi = 0$. Hence its trajectory satisfies

$$z \cos \theta + x \sin \theta = vt. \quad (9)$$

Equation (9) shows directly how the tilt angle changes the direction of the localized structure in the (z, x) plane. At fixed x , the center moves in z with time; at fixed t , the transverse

position shifts with the wave-front inclination. The parameter τ_p controls temporal and spatial localization: increasing τ_p broadens the packet, whereas decreasing it produces a narrower profile.

Both plotted components have a $\text{sech}^2 \xi$ dependence and share the same center and width parameter, but have different amplitudes and opposite signs in Eq. (7). In the limiting case $\gamma = 0$, the fourth-order terahertz-dispersion term of the model vanishes; whether the amplitude formula remains real depends on the other parameters through Eq. (8).

2.3 Construction of the localized profiles

The illustration is constructed in a sequence: the angle θ , velocity v , width τ_p , and amplitudes in Eq. (8) are specified; z' is calculated over the (z, x) grid; then $\xi = (t - z'/v)/\tau_p$ is evaluated and the two profiles are plotted. These profiles are separate from the Gaussian initial data of the later numerical programs.

Repeating these operations at several times displays translation of the joint wave packet along the tilted direction. The common value of ξ gives both components the same center trajectory and profile shape, while their amplitude factors set different scales. Figure 1 reproduces this qualitative construction.

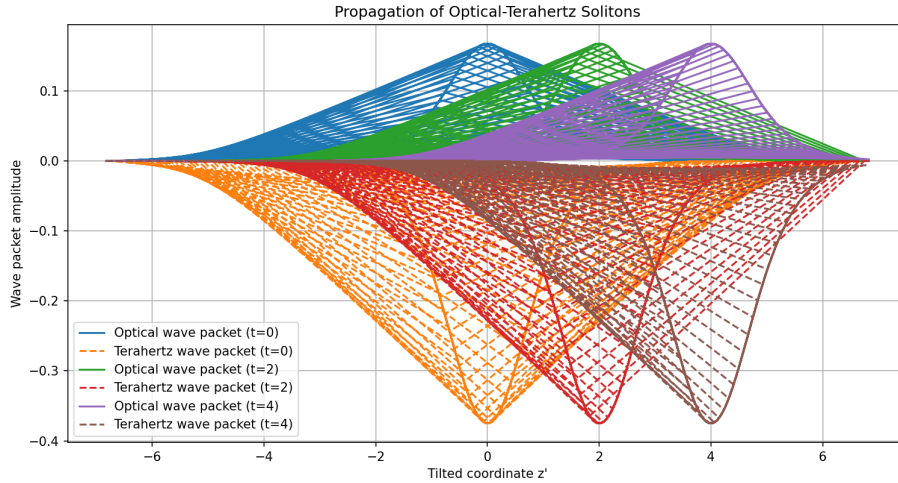


Figure 1: Illustrative optical and terahertz wave packets at different times.

2.4 Dimensionless representation

For a consistent comparison of temporal dispersion, transverse diffraction, and nonlinear coupling, introduce reference scales Z_0 , X_0 , T_0 , Ψ_0 , and E_0 and write

$$z = Z_0 \zeta, \quad x = X_0 \xi_x, \quad t = T_0 \tau_t, \quad \psi = \Psi_0 u, \quad E_T = E_0 e. \quad (10)$$

Substitution into Eqs. (1)–(2) gives

$$u_\zeta = iAu + iBu_{\tau_t \tau_t} - iCeu - iDu_{\xi_x \xi_x}, \quad (11)$$

$$e_{\zeta \zeta} = Pe_{\tau_t \tau_t} + Q(e^2)_{\tau_t \tau_t} + R(|u|^2)_{\tau_t \tau_t} - Ge_{\tau_t \tau_t \tau_t \tau_t} - He_{\xi_x \xi_x}, \quad (12)$$

where the dimensionless coefficient groups are

$$\begin{aligned} A &= \frac{2Z_0}{\beta v_g^2}, & B &= \frac{\beta Z_0}{2T_0^2}, & C &= \alpha E_0 Z_0, & D &= \frac{cZ_0}{2n_\omega \omega X_0^2}, \\ P &= \frac{n_T^2 Z_0^2}{c^2 T_0^2}, & Q &= \frac{\mu E_0 Z_0^2}{T_0^2}, & R &= \frac{\sigma \Psi_0^2 Z_0^2}{E_0 T_0^2}, & G &= \frac{\gamma Z_0^2}{T_0^4}, & H &= \frac{a Z_0^2}{X_0^2}. \end{aligned}$$

The transformation follows from $\partial_z = Z_0^{-1}\partial_\zeta$, $\partial_x = X_0^{-1}\partial_{\xi_x}$, and $\partial_t = T_0^{-1}\partial_{\tau_t}$. It separates the numerical problem from a particular system of physical units and reveals the relative magnitudes of the processes retained in the model. For example, B measures optical temporal dispersion over the reference propagation length, D measures optical transverse spreading, C measures the optical-terahertz coupling, and G measures fourth-order terahertz dispersion. The groups Q and R compare the quadratic terahertz and optical driving contributions with the chosen field scales.

For clarity, the principal derivative transformations are

$$\psi_z = \frac{\Psi_0}{Z_0}u_\zeta, \quad \psi_{tt} = \frac{\Psi_0}{T_0^2}u_{\tau_t\tau_t}, \quad \psi_{xx} = \frac{\Psi_0}{X_0^2}u_{\xi_x\xi_x}, \quad (13)$$

$$E_{T,zz} = \frac{E_0}{Z_0^2}e_{\zeta\zeta}, \quad E_{T,tt} = \frac{E_0}{T_0^2}e_{\tau_t\tau_t}, \quad E_{T,tttt} = \frac{E_0}{T_0^4}e_{\tau_t\tau_t\tau_t\tau_t}, \quad E_{T,xx} = \frac{E_0}{X_0^2}e_{\xi_x\xi_x}. \quad (14)$$

The nonlinear terms acquire factors E_0^2/T_0^2 and Ψ_0^2/T_0^2 , leading to Q and R in Eq. (12).

One may set selected groups to unity by a suitable choice of Z_0 , X_0 , T_0 , Ψ_0 , and E_0 , but this is not necessary for the numerical construction. The calculations below retain the coefficient values used in the source implementation. The scaled equations are included to make the relation among the dimensional coefficients, computational coordinates, and reported parameters explicit.

3 Finite-difference method

3.1 Staggered grid and difference operators

We introduce grid points $z_j = jh_z$, $t_l = lh_t$, and $x_k = kh_x$. The optical field is stored at staggered propagation levels $z_{j+1/2}$, while the terahertz field is stored at integer levels z_j . The optical value entering the nonlinear terms is averaged over neighboring half levels,

$$\bar{\psi}_{l,k}^j = \frac{\psi_{l,k}^{j+1/2} + \psi_{l,k}^{j-1/2}}{2}. \quad (15)$$

Centered approximations to the second derivatives are

$$(D_{tt}u)_{l,k} = \frac{u_{l+1,k} - 2u_{l,k} + u_{l-1,k}}{h_t^2}, \quad (16)$$

$$(D_{xx}u)_{l,k} = \frac{u_{l,k+1} - 2u_{l,k} + u_{l,k-1}}{h_x^2}, \quad (17)$$

The fourth-order temporal derivative is evaluated with the five-point stencil

$$(D_{tttt}u)_{l,k} = \frac{u_{l+2,k} - 4u_{l+1,k} + 6u_{l,k} - 4u_{l-1,k} + u_{l-2,k}}{h_t^4}. \quad (18)$$

This is equivalent to $D_{tttt} = D_{tt}D_{tt}$ at interior points. The terahertz propagation derivative is approximated by

$$(D_{zz}E_T)_{l,k}^j = \frac{E_{T,l,k}^{j+1} - 2E_{T,l,k}^j + E_{T,l,k}^{j-1}}{h_z^2}. \quad (19)$$

These formulas define the centered half-step construction. Equation (3) supplies the propagation data.

For a sufficiently smooth function, Taylor expansion about (t_l, x_k) gives

$$D_{tt}u = u_{tt} + \mathcal{O}(h_t^2), \quad D_{xx}u = u_{xx} + \mathcal{O}(h_x^2), \quad D_{tttt}u = u_{tttt} + \mathcal{O}(h_t^2). \quad (20)$$

The centered propagation derivative in Eq. (19) is likewise second-order accurate in h_z at an interior layer. These local estimates require smooth bounded derivatives and do not alone establish stability of the nonlinear coupled update.

In the formal centered formulation, the prescribed optical profile is placed at the first half layer, $\psi_{l,k}^{1/2} = \psi_0(x_k, t_l)$. The zero initial value and zero propagation derivative of the terahertz field determine its first two propagation layers. The centered stencils in t and x use the prescribed homogeneous boundary values; applying D_{tt} twice gives the fourth-order temporal approximation. The actual plotting programs use different explicit updates, specified below.

At each z_j , the boundary values are set to zero at $t = 0, L_t$ and $x = 0, L_x$. The fourth-order temporal stencil requires two interior neighbors on each side; its boundary-adjacent values are consequently supplied by the homogeneous boundary treatment. This construction localizes the fields inside the finite computational window and prevents values outside the modeled domain from entering the centered differences.

3.2 Two-stage optical iteration and terahertz update

At each propagation layer, the optical equation is advanced by a two-stage iteration. The first stage uses the previous iterate in the nonlinear coupling and transverse-difference terms while evaluating the temporal-difference term at the new iterate. The second stage updates the coupling term with the new optical field and treats the transverse-difference term at the corrected iterate. In both stages, the half-level average in Eq. (15) couples neighboring propagation layers.

To write the iteration explicitly, let s denote the iteration index and define

$$\bar{\psi}_{l,k}^{s,j} = \frac{\psi_{l,k}^{s,j+1/2} + \psi_{l,k}^{j-1/2}}{2}. \quad (21)$$

The first stage is represented by

$$\begin{aligned} \frac{\psi_{l,k}^{s+1,j+1/2} - \psi_{l,k}^{j-1/2}}{h_z} &= i \frac{\beta}{2} D_{tt} \bar{\psi}_{l,k}^{s+1,j} + i \frac{2}{\beta v_g^2} \bar{\psi}_{l,k}^{s,j} - i \alpha E_{T,l,k}^j \bar{\psi}_{l,k}^{s,j} \\ &\quad - i \frac{c}{2n_\omega \omega} D_{xx} \bar{\psi}_{l,k}^{s,j}. \end{aligned} \quad (22)$$

The corrected second stage uses the updated optical value in the phase and coupling terms and evaluates the transverse term at the corrected iterate:

$$\begin{aligned} \frac{\psi_{l,k}^{s+2,j+1/2} - \psi_{l,k}^{j-1/2}}{h_z} &= i \frac{\beta}{2} D_{tt} \bar{\psi}_{l,k}^{s+1,j} + i \frac{2}{\beta v_g^2} \bar{\psi}_{l,k}^{s+1,j} - i \alpha E_{T,l,k}^j \bar{\psi}_{l,k}^{s+1,j} \\ &\quad - i \frac{c}{2n_\omega \omega} D_{xx} \bar{\psi}_{l,k}^{s+2,j}. \end{aligned} \quad (23)$$

Equations (22) and (23) retain the same symbols as the continuous model: α occurs only in the optical coupling term and a occurs only in the transverse term of the terahertz equation.

The corresponding formal terahertz equation uses the centered z difference in Eq. (19), the temporal and transverse differences in Eq. (17), and the computed optical intensity $|\bar{\psi}^j|^2$. The fourth-order temporal term is evaluated by applying D_{tt} twice. This formulation states the numerical approximation to the displayed continuous model; it must not be identified with the explicit update used for the plotted data.

In particular, a centered update consistent with the field equation has the structure

$$E_T^{j+1} = 2E_T^j - E_T^{j-1} + h_z^2 \left[\frac{n_T^2}{c^2} D_{tt} E_T^j + D_{tt} \left(\mu (E_T^j)^2 + \sigma |\bar{\psi}^j|^2 \right) - \gamma D_{tttt} E_T^j - a D_{xx} E_T^j \right]. \quad (24)$$

All products in Eq. (24) are evaluated pointwise. This expression makes explicit how the optical intensity drives the terahertz wave and how the fourth-order dispersion and transverse term enter the propagation step.

For the centered formulation, Eqs. (22), (23), and (24) describe one complete propagation step. The local truncation estimates in Eq. (20) apply to these centered stencils. They do not establish the order, convergence, or stability of the different explicit routines used to generate the figures.

4 Numerical results

4.1 Updates used to generate the figures

The first plotting program does not implement the three-level terahertz step in Eq. (24). Instead, after two explicit optical updates it computes a right-hand side

$$\mathcal{F}_T(E, \psi) = \frac{n_T^2}{c^2} D_{tt} E + \mu D_{tt}(E^2) + \sigma D_{tt} |\psi|^2 - \gamma D_{tttt} E - a D_{xx} E \quad (25)$$

and advances the field by

$$E_T^{j+1} = E_T^j + h_z \mathcal{F}_T(E_T^j, \psi^{j+1}). \quad (26)$$

Thus the plotted terahertz calculation uses one previous propagation layer and an increment proportional to h_z ; the formal second-order propagation equation uses two previous layers and an increment proportional to h_z^2 . Equation (26) is a faithful description of the code, but it is not a consistent discretization of $E_{T,zz} = \mathcal{F}_T$ without an additional velocity variable or a separate reformulation. The figures therefore cannot validate the formal three-level scheme.

The boundary treatment in the executable programs also differs from Eqs. (4)–(6): their difference-operator arrays are set to zero where a full stencil is unavailable, but the optical field values are not generally reset to zero at the edges. The first program additionally sets the first three temporal rows of each updated terahertz layer to zero. The Gaussian in Eq. (31) is uniform in x at $z = 0$ and therefore does not satisfy a homogeneous optical value at $x = 0$ or $x = L_x$. These discrepancies must be removed before a convergence claim about the stated initial-boundary-value problem can be made.

The optical code similarly performs two explicit increments. Its right-hand side is

$$\mathcal{F}_\psi(V, E) = \frac{i\beta}{2} D_{tt} V + \frac{2i}{\beta v_g^2} V - i\alpha E V - \frac{ic}{2n_\omega \omega} D_{xx} V. \quad (27)$$

With $\psi^{(0)} = \psi^j$ copied from the previous layer, the program computes

$$\psi^{(r+1)} = \psi^{(r)} + h_z \mathcal{F}_\psi \left(\frac{\psi^j + \psi^{(r)}}{2}, E_T^j \right), \quad r = 0, 1. \quad (28)$$

The result $\psi^{(2)}$ drives the terahertz update in Eq. (26). These two increments differ from solving the implicit equations (22) and (23) to a stated tolerance.

The second plotting program changes more than the grid. It uses $L_z = 0.1$, $N_z = 20$, and $N_x = 40$, starts from a Gaussian localized in both t and x , and repeats the optical and terahertz updates inside a loop of two passes. Its optical right-hand side is

$$\mathcal{F}_{\psi, \text{sweep}}(V, E) = \frac{i\beta}{2} V + \frac{2i}{\beta} V - i\alpha E V - \frac{ic}{2n_\omega \omega} D_{xx} V. \quad (29)$$

The first term in Eq. (29) contains V where the continuous equation has ψ_{tt} . In its terahertz update, the source-code variable `aaa`, denoted by p here, multiplies the fourth-order coefficient

$\gamma_{\text{sweep}} = 0.05$, while the transverse term is multiplied by zero:

$$\mathcal{F}_{T,p}(E, \psi) = \frac{n_T^2}{c^2} D_{tt} E + \mu D_{tt}(E^2) + \sigma D_{tt} |\psi|^2 - p \gamma_{\text{sweep}} D_{tttt} E. \quad (30)$$

The effective fourth-order coefficients in the two displayed comparisons are therefore $p\gamma_{\text{sweep}} = 0.01$ for $p = 0.2$ and 0.025 for $p = 0.5$. The images label p as **aaa**; it is not the transverse coefficient a in Eq. (2). The separate source-code loops also include $p = 1$ and $p = 0.23$, but no figures for those cases are included in this paper.

4.2 Computational setup

The numerical implementation uses Python arrays for the complex optical field and the real terahertz field. Second-order temporal and transverse differences are evaluated directly from neighboring array elements, while the fourth-order temporal difference uses Eq. (18). Real and imaginary parts of the optical field are differentiated separately and then recombined. The plotted optical quantity is the intensity $|\psi|^2 = (\Re\psi)^2 + (\Im\psi)^2$; the terahertz plots show E_T itself.

The initial optical pulse in the first plotting program is

$$\psi_0(x, t) = \exp\left[-\frac{(t-5)^2}{2}\right], \quad (31)$$

and is initially uniform over the transverse grid. The terahertz field is initially zero. The second plotting program uses a different initial profile,

$$\psi_0(x, t) = \exp\left[-\frac{(t-5)^2}{2}\right] \exp\left[-\frac{(x-1)^2}{2(0.3)^2}\right]. \quad (32)$$

Neither Gaussian equals the illustrated sech^2 profile in Eq. (7). Accordingly, the calculations test propagation of Gaussian inputs in the numerical routines rather than perturbations of that localized profile.

The coefficients and grid values in the two programs are listed in Table 2. The first program sets $a = 0$. The second program also suppresses the transverse terahertz term and varies p in Eq. (30); it does not vary a . In both programs the optical coupling parameter is $\alpha = 0.1$.

Table 2: Parameters recorded in the two plotting programs.

Quantity	First program	Second program
$\alpha, \beta, \mu, \sigma$	0.1, 1.0, 0.2, 0.1	0.1, 1.0, 0.2, 0.1
γ or γ_{sweep}	0.01	0.05 before multiplication by p
a in terahertz update	0	0
p (source label aaa)	not used	0.2, 0.5 in displayed figures
v_g	1.0	absent from second code; phase term uses $2/\beta$
c	3.0×10^8 m/s	3.0×10^8 m/s
n_ω, n_T	1.5, 1.2	1.5, 1.2
ω	$2\pi \times 10^{12}$ rad/s	$2\pi \times 10^{12}$ rad/s
L_t, L_x, L_z	10, 2, 1	10, 2, 0.1
N_t, N_x, N_z	80, 3, 200	80, 40, 20
h_t, h_x, h_z	0.125, 2/3, 0.005	0.125, 0.05, 0.005

The plotting routine labels an optical half layer by

$$z_{\text{plot}} = 10^3 \left(j + \frac{1}{2}\right) h_z. \quad (33)$$

This convention explains why labels such as 27.5, 152.5, 177.5, and 252.5 appear in the figures even though the normalized computational length is $L_z = 1$. They are scaled layer labels from the program rather than dimensional distances. Axis ranges differ across panels, so amplitude comparisons are made from the displayed scales and remain qualitative.

4.3 Propagation of the localized packet

Figure 2 shows a separate localized-profile visualization. Its optical axis is labeled $|\psi|$, and its terahertz axis is nonzero at the displayed $z = 0$. It therefore does not represent the initial data $E_T(0) = 0$ of the numerical code in Eq. (3); it illustrates the shape and opposite signs of a localized optical–terahertz pair. Figures 3–6 are the distinct propagation sequence produced by the first plotting program, whose optical panels display $|\psi|^2$.

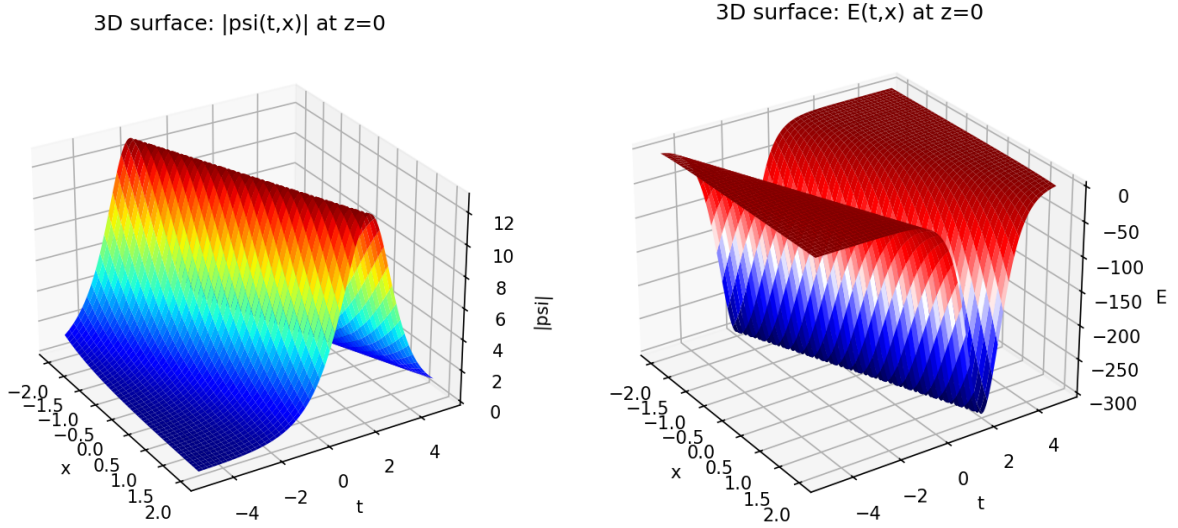


Figure 2: Separate localized-profile illustration at its displayed $z = 0$: optical amplitude $|\psi|$ (left) and terahertz field (right). The nonzero terahertz panel is not the zero-field initial condition of the propagation code.

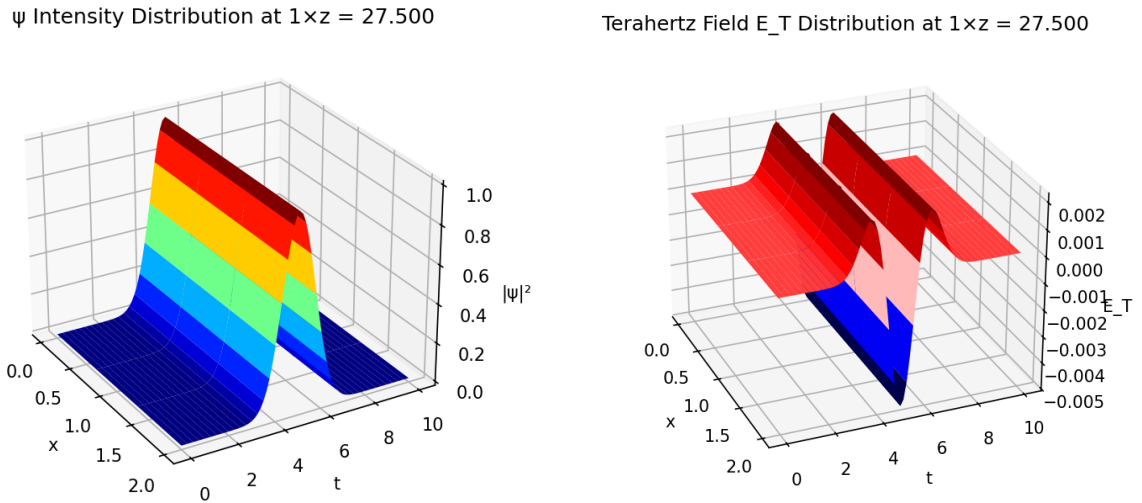
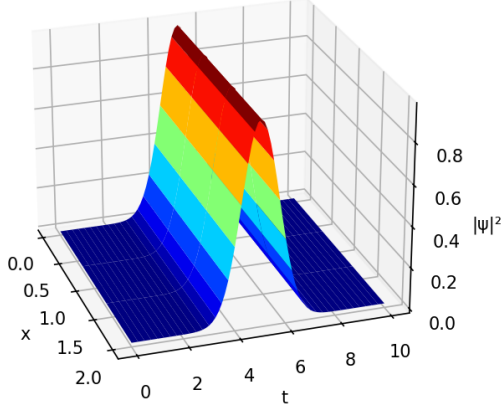


Figure 3: Optical intensity (left) and terahertz field (right) at the plotted propagation label $z = 27.5$.

ψ Intensity Distribution at $1 \times z = 152.500$



Terahertz Field E_T Distribution at $1 \times z = 152.500$

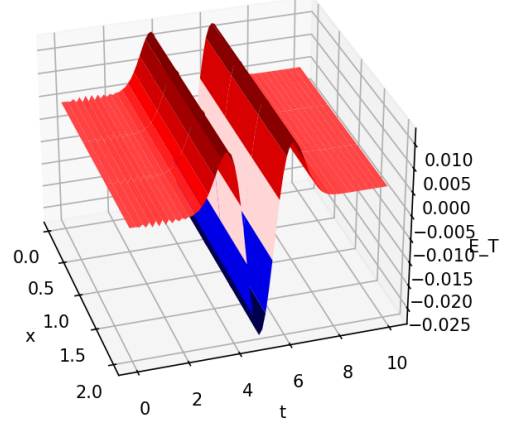


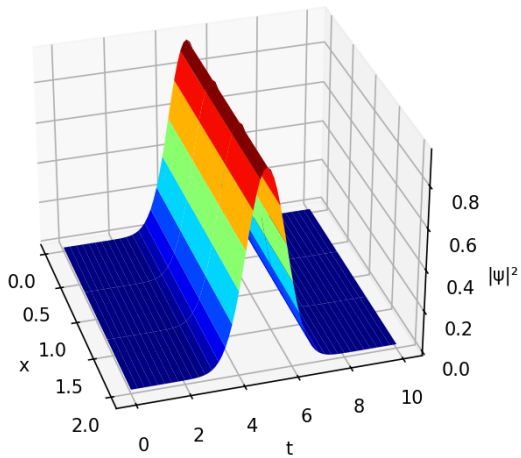
Figure 4: Optical intensity (left) and terahertz field (right) at the plotted propagation label $z = 152.5$.

At $z_{\text{plot}} = 27.5$, the optical intensity remains smooth and localized, and the terahertz response is still confined near the temporal center of the Gaussian input. At $z_{\text{plot}} = 152.5$, the optical peak is retained but the terahertz surface contains more pronounced positive and negative variations. These snapshots indicate that the optical component is less visibly distorted than the terahertz component over this part of the first-program calculation.

At the plotted label $z_{\text{plot}} = 177.5$, the optical intensity still appears localized, but the terahertz surface exhibits stronger, less regular excursions (Fig. 5). The later snapshot at $z_{\text{plot}} = 252.5$ has extremely large plotted amplitudes and sharp spikes in both fields (Fig. 6). This is numerical blow-up in the displayed calculation. It cannot, by itself, distinguish instability of the continuous solution from instability or insufficient resolution of the explicit discretization.

The transition between these figures is important for interpreting stability. Persistence of a localized peak over several layers shows that the program transports the Gaussian input for a finite interval. The rapid amplitude growth at the later layer indicates numerical blow-up. Because this calculation uses $N_x = 3$ transverse intervals and no grid-refinement series is available, the growth cannot be attributed to instability of a continuous exact solution.

ψ Intensity Distribution at $1 \times z = 177.500$



Terahertz Field E_T Distribution at $1 \times z = 177.500$

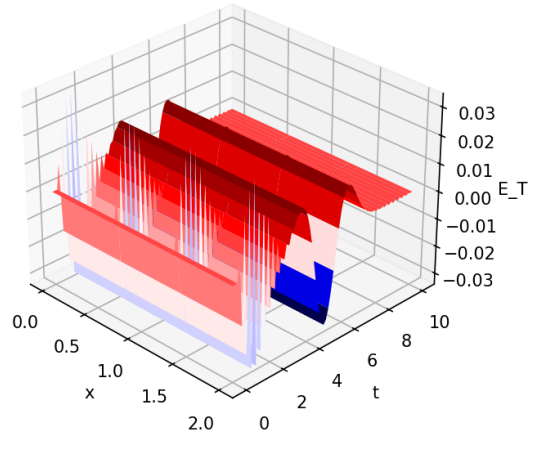
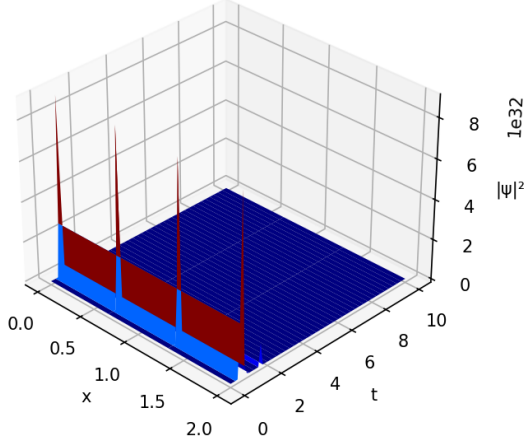


Figure 5: Optical intensity (left) and terahertz field (right) at the plotted propagation label $z = 177.5$. The terahertz field shows pronounced oscillatory excursions.

ψ Intensity Distribution at $1 \times z = 252.500$



Terahertz Field E_T Distribution at $1 \times z = 252.500$

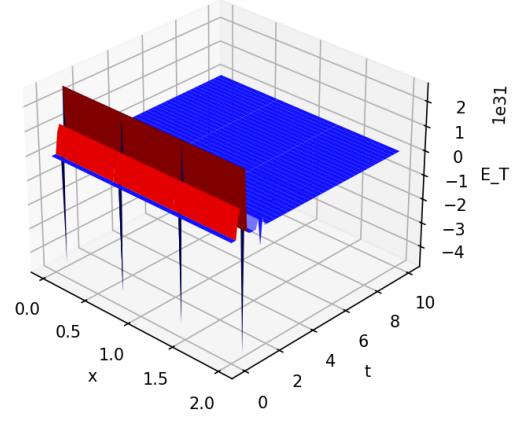


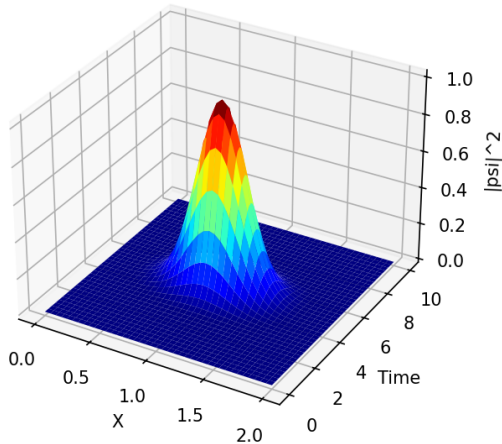
Figure 6: Optical intensity (left) and terahertz field (right) at the plotted propagation label $z = 252.5$. The very large vertical scales indicate breakdown of the reported numerical solution.

4.4 Sensitivity to the fourth-order dispersion multiplier

The second appendix program compares two values of the multiplier p in Eq. (30) while keeping $\alpha = 0.1$. At $p = 0.2$, both fields appear localized in the displayed snapshot (Fig. 7). At $p = 0.5$, the optical peak remains localized, but the terahertz panel shows considerably larger oscillatory excursions (Fig. 8). The source image titles use `aaa` for p ; their value $z = 0.050$ is an unscaled coordinate from the second program.

Since p changes $-p\gamma_{\text{sweep}}D_{tttt}E_T$ while the transverse term is zero in that code, this comparison concerns fourth-order terahertz dispersion, not the transverse coefficient a of the continuous equation. The two displayed cases do not locate a critical value of p . Moreover, the second program differs from the first in its initial pulse, grid, optical right-hand side, and repeated update loop. Its plots must be interpreted as a separate experiment.

Optical Field $|\psi|^2$ at $z=0.050$ (`aaa=0.2`)



Terahertz Field E_T at $z=0.050$ (`aaa=0.2`)

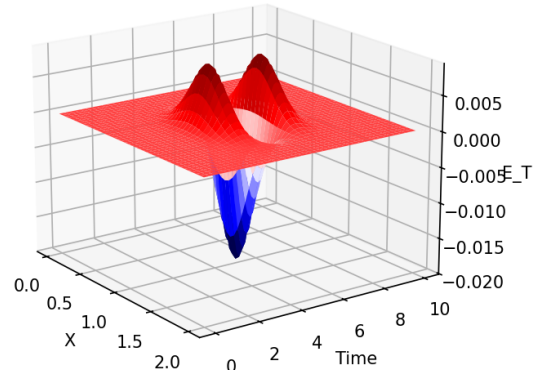


Figure 7: Second-program fields at $z = 0.050$ for dispersion multiplier $p = 0.2$ (`aaa=0.2` in the source image), corresponding to $p\gamma_{\text{sweep}} = 0.01$.

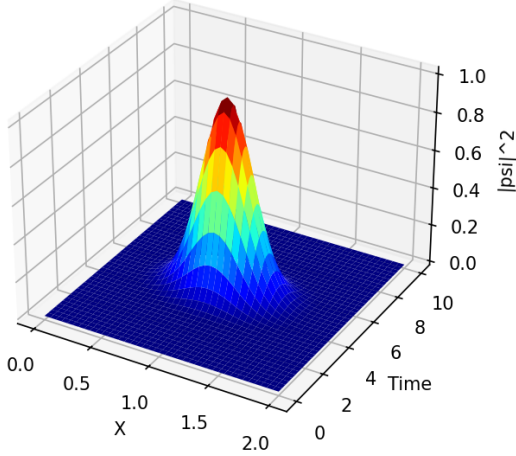
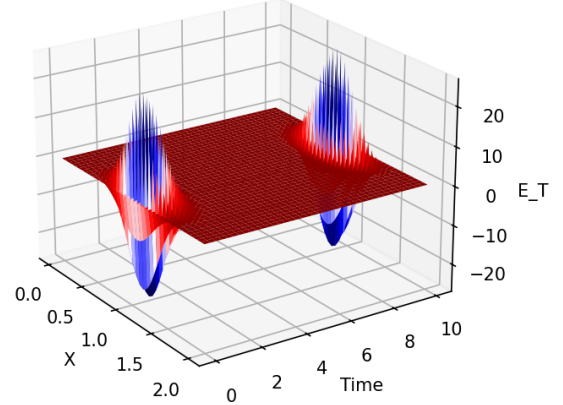
Optical Field $|\psi|^2$ at $z=0.050$ (aaa=0.5)Terahertz Field E_T at $z=0.050$ (aaa=0.5)

Figure 8: Second-program fields at $z = 0.050$ for dispersion multiplier $p = 0.5$ (aaa=0.5 in the source image), corresponding to $p\gamma_{\text{sweep}} = 0.025$. The terahertz surface has a much larger vertical range than in Fig. 7.

4.5 Summary of the displayed evolution

Table 3 collects the observations associated with the figures. It separates the plotted evidence from the interpretation: localization refers to the visible concentration of the surface, whereas numerical instability refers to the rapid growth and spiking seen in the later calculation.

Table 3: Qualitative summary of the numerical surfaces.

Case	Optical field in the source plot	Terahertz field E_T
Separate $z = 0$ illustration	Localized optical amplitude $ \psi $	Nonzero terahertz profile; distinct from the code's initial data
$z_{\text{plot}} = 27.5$	Localized profile retained	Confined structure near the optical pulse
$z_{\text{plot}} = 152.5$	Main peak remains visible	Larger positive and negative variations
$z_{\text{plot}} = 177.5$	Localized peak still present	Stronger and less regular excursions
$z_{\text{plot}} = 252.5$	Sharp growth and loss of a regular profile	Extremely large spikes indicating numerical breakdown
$p = 0.2$ at $z = 0.050$	Localized in the second program	Localized response
$p = 0.5$ at $z = 0.050$	Localized peak remains	Much larger vertical range and stronger excursions

The sequence suggests three stages in the reported calculation. In the first stage, both fields remain localized and the optical pulse changes only weakly. In the second stage, the terahertz field develops stronger oscillatory structure while the optical peak persists. In the final displayed stage, both surfaces contain amplitudes that are incompatible with a bounded numerical soliton. This progression is consistent with the terahertz equation being more sensitive to high-frequency numerical components because it contains both second- and fourth-order temporal derivatives.

5 Criteria for numerical instability and nonconvergence

For a grid field $u_{l,k}$, define $\|u\|_{\infty,h} = \max_{l,k} |u_{l,k}|$ and $\|u\|_{2,h} = (h_t h_x \sum_{l,k} |u_{l,k}|^2)^{1/2}$. At every propagation layer, record

$$M_\psi^j = \max_{l,k} |\psi_{l,k}^j|^2, \quad M_T^j = \max_{l,k} |E_{T,l,k}^j|, \quad (34)$$

as well as $\|\psi^j\|_{2,h}$ and $\|E_T^j\|_{2,h}$. A run fails numerically if either array develops a NaN or infinity. Sustained rapid growth in both the maxima and norms is evidence of numerical blow-up; isolated large points should additionally be checked for grid-scale oscillations. The late first-program plots show intensity and terahertz scales of approximately 10^{32} and 10^{31} , respectively, whereas the initial optical amplitude is of order one. This is direct visual evidence of blow-up in that run.

For a reproducible onset coordinate, choose a finite reference layer $j_{\text{ref}} \geq 1$ with nonzero field norms and fix a reporting factor K before examining the runs. Define

$$B_h^j = \max \left\{ \frac{\|\psi^j\|_{\infty,h}}{\max(\|\psi^{j_{\text{ref}}}\|_{\infty,h}, \varepsilon)}, \frac{\|E_T^j\|_{\infty,h}}{\max(\|E_T^{j_{\text{ref}}}\|_{\infty,h}, \varepsilon)} \right\}, \quad z_B(h; K) = \min\{z_j : B_h^j \geq K\}. \quad (35)$$

For example, $K = 10^6$ marks six orders of magnitude of growth relative to the chosen layer; the chosen K and j_{ref} must be reported. Nonfinite values count as failure even when z_B cannot be evaluated. This diagnostic identifies a numerical onset in a particular run and must be compared across meshes before drawing a continuum conclusion.

For each field, relative changes between consecutive layers can be monitored using

$$\delta_\psi^j = \frac{\|\psi^{j+1} - \psi^j\|_{2,h}}{\max(\|\psi^j\|_{2,h}, \varepsilon)}, \quad \delta_T^j = \frac{\|E_T^{j+1} - E_T^j\|_{2,h}}{\max(\|E_T^j\|_{2,h}, \varepsilon)}, \quad (36)$$

where $\varepsilon > 0$ prevents division by zero at the initial zero terahertz field. Growth of δ_T^j before δ_ψ^j would quantify the ordering suggested by the figures. The available record supplies only plots, so numerical values of these diagnostics are not claimed here.

If the formal optical equations are solved by repeated fixed-point iterations, their algebraic convergence should be checked separately from propagation stability. At each layer define

$$\rho_s^j = \frac{\|\psi^{s+1,j+1/2} - \psi^{s,j+1/2}\|_{2,h}}{\max(\|\psi^{s+1,j+1/2}\|_{2,h}, \varepsilon)}. \quad (37)$$

One should stop only after ρ_s^j falls below a stated tolerance and the discrete optical-equation residual is comparably small. A fixed count of two explicit increments, as in the plotting programs, provides no such convergence evidence.

To test convergence, repeat one and the same program with identical initial and boundary data and with spacings $h = (h_z, h_t, h_x)$, $h/2$, and $h/4$. Compare solutions at the same physical z after restricting fine-grid values to coarse-grid nodes. Let I_h denote that restriction and set

$$e_\psi(h; z) = \frac{\|\psi_h(z) - I_h \psi_{h/2}(z)\|_{2,h}}{\max(\|I_h \psi_{h/2}(z)\|_{2,h}, \varepsilon)}, \quad e_T(h; z) = \frac{\|E_{T,h}(z) - I_h E_{T,h/2}(z)\|_{2,h}}{\max(\|I_h E_{T,h/2}(z)\|_{2,h}, \varepsilon)}. \quad (38)$$

Convergence is supported when both errors decrease under refinement and the onset of rapid growth approaches a stable z value. With three grids, the observed order can be estimated as

$$q_u(z) = \log_2 \left(\frac{e_u(h; z)}{e_u(h/2; z)} \right), \quad u \in \{\psi, E_T\}. \quad (39)$$

This estimate is meaningful only while the errors are nonzero and the solutions remain finite. A nondecreasing error, a strongly grid-dependent blow-up coordinate, or nonfinite values under refinement indicate that a stability threshold inferred from one run is unreliable. No such grid series is available for the reported calculations.

An interior Fourier-mode calculation gives a more specific check on the explicit terahertz update. Isolate its linear fourth-order term, $E_T^{j+1} = E_T^j - h_z \gamma_{\text{eff}} D_{tttt} E_T^j$, with $\gamma_{\text{eff}} = \gamma$ in the first program and $\gamma_{\text{eff}} = p \gamma_{\text{sweep}}$ in the second. For $E_{T,l}^j = G^j \exp(il\kappa)$, the stencil in Eq. (18) has symbol $16 \sin^4(\kappa/2)/h_t^4$. Hence

$$G(\kappa) = 1 - 16 \frac{h_z \gamma_{\text{eff}}}{h_t^4} \sin^4\left(\frac{\kappa}{2}\right), \quad \frac{h_z \gamma_{\text{eff}}}{h_t^4} \leq \frac{1}{8} \quad (40)$$

is the condition $|G(\kappa)| \leq 1$ for every Fourier mode of this isolated explicit term. The ratio is 0.2048 for the first program and for $p = 0.2$ in the second, and 0.512 for $p = 0.5$. All three exceed $1/8$; the respective highest-mode factors are -2.2768 , -2.2768 , and -7.192 . Thus the isolated high-frequency component can grow on the stated meshes, and increasing p makes that mechanism stronger. This is an interior, linearized test of one update term; it does not establish the exact growth rate of the full nonlinear code with its finite-boundary handling, nor instability of the continuous coupled equations. Because the step bound scales as h_t^4 , a refinement test must reduce h_z suitably as h_t is reduced.

For the formal three-level scheme, one can also monitor the discrete terahertz equation residual

$$R_{T,h}^j = D_{zz} E_T^j - \mathcal{F}_T(E_T^j, \bar{\psi}^j). \quad (41)$$

The one-step residual of Eq. (26) is small by construction and cannot validate the different second-order equation. Consistency with the continuous system must be checked with a method that actually discretizes $E_{T,zz}$.

6 Discussion and conclusion

We have presented the optical–terahertz Schrödinger–Boussinesq-type model, its localized profile form, a formal staggered finite-difference scheme, and the available numerical surfaces. The first-program images show a localized optical profile over early layers, increasing irregularity in the terahertz component, and eventual numerical blow-up in both fields. The second-program images show greater terahertz excursions when the multiplier of fourth-order dispersion rises from $p = 0.2$ to $p = 0.5$.

The continuous equations suggest a possible mechanism for stronger terahertz sensitivity: the terahertz equation contains nonlinear self-action, the temporal derivative of optical intensity, and fourth-order temporal dispersion. High-order differences can amplify grid-scale variation in an explicit calculation. The available plots do not isolate this mechanism from step-size effects or from the mismatch between the formal equations and the update routines.

The centered spatial and temporal stencils have second-order local accuracy for smooth fields at interior points. The plotting programs, by contrast, use explicit optical and terahertz updates that differ from the formal centered scheme. In particular, a first-order explicit terahertz increment is used where the continuous model contains $E_{T,zz}$. The second program also uses a different optical right-hand side. The plots therefore document behavior of those programs, not convergence of the stated centered discretization to the continuous model.

The diagnostics in Eqs. (34)–(38) provide a reproducible way to assess numerical blow-up and grid convergence. The late first-program figures demonstrate large amplitude growth, but the available data contain no same-problem grid-refinement comparison.

The isolated Fourier-mode calculation in Eq. (40) identifies a concrete numerical risk: the stated h_z and h_t violate the explicit fourth-order step bound in both programs, more severely for

the larger second-program multiplier. This supports a numerical explanation for the observed sensitivity without assigning a physical stability threshold to the continuous system.

The figures support conclusions about the two numerical programs. They do not establish nonlinear stability of a verified soliton of Eqs. (1)–(2), a critical value of the transverse coefficient a , a critical value of the dispersion multiplier p , or a physical onset distance. Resolving those questions requires consistent equation-to-code implementation, refinement tests in z , t , and x , and perturbation experiments using a profile verified to solve the continuous equations.

Within these limits, the study provides a mathematical model, an explicit description of the two plotting programs, and visual evidence of numerical blow-up in the first program. The second comparison indicates sensitivity to the effective fourth-order coefficient $p\gamma_{\text{sweep}}$ under its own discretization. Separating the formal equations from the executable updates is essential for a reproducible interpretation of the numerical results.

Future calculations should first implement the same differential equations, initial conditions, and boundary conditions in every run. A three-level update or a first-order formulation with an explicit terahertz velocity would be needed for the $E_{T,zz}$ term. Grid refinement and the diagnostics above should precede any claim of a stability boundary. Substituting a candidate localized profile into the continuous equations, followed by perturbations in amplitude, width, and wave-front angle, would permit a direct soliton-stability test.

Acknowledgments

The authors acknowledge the academic support provided by Shenzhen MSU–BIT University and its Faculty of Computational Mathematics and Cybernetics during the original undergraduate study on which this paper is based.

A Layer-by-layer numerical algorithm

For completeness, the first appendix program that produced Figs. 3–6 can be summarized in the following layer-by-layer form. The steps describe the executable routine rather than the formal scheme in Eq. (24).

1. Create the arrays $\psi_{l,k}^j \in \mathbb{C}$ and $E_{T,l,k}^j \in \mathbb{R}$ on $(N_z + 1) \times (N_t + 1) \times (N_x + 1)$ points.
2. Set $\psi_{l,k}^0 = \exp[-(t_l - 5)^2/2]$ and $E_{T,l,k}^0 = 0$ for all transverse indices.
3. At propagation layer j , copy the previous optical field as the initial iterate for the new layer.
4. Evaluate $D_{tt}\psi$, $D_{xx}\psi$, the linear phase term, and $-i\alpha E_T\psi$; then perform the first optical update.
5. Recalculate the half-layer field and apply a second explicit optical increment.
6. Form $|\psi|^2$, $D_{tt}E_T$, $D_{tt}(E_T^2)$, $D_{tt}|\psi|^2$, $D_{ttt}E_T$, and $D_{xx}E_T$.
7. Combine these terms according to Eq. (25), advance E_T with Eq. (26), set the first three temporal rows of the new terahertz array to zero as in the source code, and store the new layer.
8. Save or plot $|\psi|^2$ and E_T at the selected values of z_{plot} from Eq. (33).

The implementation uses NumPy for multidimensional arrays and finite differences, the standard mathematical library for constants and elementary functions, and Matplotlib for three-dimensional surfaces. The plotting coordinates are constructed with a mesh grid in t and x .

The optical and terahertz panels are generated separately because their amplitudes and signs differ.

The organization of the source implementation is summarized in Table 4. This records the numerical content of the program without reproducing a long language-specific listing.

Table 4: Correspondence between implementation steps and mathematical operations.

Implementation component	Mathematical operation
Second temporal difference	Applies D_{tt} to real scalar arrays
Fourth temporal difference	Applies the five-point stencil in Eq. (18)
Second transverse difference	Applies D_{xx} at interior transverse nodes
Half-layer construction	Forms the average in Eq. (21)
First optical update	Applies an explicit increment using the available half-layer value
Second optical update	Applies a second explicit increment using the updated half-layer value
Terahertz layer update	Combines the five terms in Eq. (25) and applies Eq. (26)
Initial optical profile	Evaluates the Gaussian in Eq. (31)
Field visualization	Plots $ \psi ^2$ and E_T over the (t, x) mesh

The complex optical derivative is obtained by applying the same real-valued difference operator to the real and imaginary parts. The optical intensity is then formed as $(\Re\psi)^2 + (\Im\psi)^2$. For the first terahertz update, the program separately evaluates the linear temporal term, quadratic self-action, optical source, fourth-order dispersion, and transverse term before summing them. The explicit propagation step in Eq. (26) differs from the second derivative on the left-hand side of Eq. (2).

The output routine samples selected propagation layers rather than plotting every layer. At layer j , it creates the temporal and transverse coordinates, computes $|\psi^j|^2$, and draws separate three-dimensional surfaces for the optical and terahertz quantities. Equation (33) converts the array index to the label printed in the first-program figures. The second-program figures instead display $z = jh_z$ without the factor of 10^3 .

B Consistency of symbols and indices

The notation distinguishes the coupling coefficient α in the optical equation from the transverse coefficient a in the continuous terahertz equation. The second program’s variable `aaa` is denoted by p because its only active use is to scale the fourth-order term, $-p\gamma_{\text{sweep}}D_{tttt}E_T$. In that program the transverse terahertz coefficient is zero. The displayed $p = 0.2, 0.5$ comparison therefore cannot be described as a sweep of a .

The refractive index at the optical frequency is n_ω and the terahertz refractive index is n_T throughout. The optical quantity used in the terahertz source and in the numerical intensity plots is $|\psi|^2$; the separate localized-profile illustration labels its optical amplitude $|\psi|$. Integer indices j, l, k refer to z, t, x , respectively. Half-integer propagation levels occur in the formal staggered equations, while the appendix programs store fields at integer array indices. Superscript s is used only for the formal optical iteration.

References

- [1] S. V. Sazonov and N. V. Ustinov, “New soliton regime of generation of broadband terahertz radiation by laser pulses with tilted wave fronts,” *JETP Letters* **118**, 408–413 (2023). <https://doi.org/10.1134/S0021364023602531>.

- [2] S. V. Sazonov and N. V. Ustinov, “Soliton mode of terahertz radiation generation using picosecond laser pulses with tilted wave fronts,” *Laser Physics Letters* **19**, 025401 (2022). <https://doi.org/10.1088/1612-202X/ac4914>.
- [3] I. G. Zakharova, A. A. Kalinovich, and S. V. Sazonov, “Finding stability conditions of a soliton-like mode of optical–terahertz radiation,” in *Infrared, Millimeter-Wave, and Terahertz Technologies XI*, Proc. SPIE **13247**, 1324704 (2024). <https://doi.org/10.1117/12.3035600>.
- [4] S. V. Sazonov, I. G. Zakharova, and A. A. Kalinovich, “Generating optic terahertz pulses in waveguides with quadratic nonlinearity,” *Bulletin of the Russian Academy of Sciences: Physics* **84**, 177–179 (2020).
- [5] M. V. Komissarova, S. V. Sazonov, A. A. Kalinovich, and I. G. Zakharova, “The effect of a waveguide on the formation of optical terahertz solitons,” in *Nonlinear Optics and Applications XI*, Proc. SPIE **11026**, 110260G (2019).
- [6] A. A. Kalinovich, M. V. Komissarova, S. V. Sazonov, and I. G. Zakharova, “Trapping of two-component light bullets in a gradient waveguide with normal group dispersion,” *PLOS ONE* **14**, e0220840 (2019). <https://doi.org/10.1371/journal.pone.0220840>.
- [7] I. G. Zakharova, A. A. Kalinovich, M. V. Komissarova, and S. V. Sazonov, “Multi-step iterative algorithm for mathematical modeling of light bullets in anisotropic media,” in *Finite Difference Methods, Theory and Applications*, Lecture Notes in Computer Science **11386**, 651–659 (Springer, 2019). https://doi.org/10.1007/978-3-030-11539-5_79.
- [8] W.-P. Düll and G. Schneider, “Justification of the nonlinear Schrödinger equation for a resonant Boussinesq model,” *Indiana University Mathematics Journal* **55**, 1813–1834 (2006). <https://doi.org/10.1512/iumj.2006.55.2824>.
- [9] G. Karamali, M. Abbaszadeh, and M. Dehghan, “The smoothed particle hydrodynamics method for solving generalized variable coefficient Schrödinger equation and Schrödinger–Boussinesq system,” *Computational Methods for Differential Equations* **6**, 215–237 (2018).
- [10] S. Chu, A. Kurganov, and R. Xin, “New more efficient A-WENO schemes,” *Journal of Scientific Computing* **104**, 53 (2025). <https://doi.org/10.1007/s10915-025-02969-z>.
- [11] B. Ferguson and X.-C. Zhang, “Materials for terahertz science and technology,” *Nature Materials* **1**, 26–33 (2002). <https://doi.org/10.1038/nmat708>.
- [12] O. P. Cherkasova, D. S. Serdyukov, A. S. Ratushnyak, E. F. Nemova, E. N. Kozlov, Y. V. Shidlovskii, K. I. Zaytsev, and V. V. Tuchin, “Effects of terahertz radiation on living cells: A review,” *Optics and Spectroscopy* **128**, 855–866 (2020). <https://doi.org/10.1134/S0030400X20060041>.
- [13] A. Irizawa, S. Lupi, and A. Marcelli, “Terahertz as a frontier area for science and technology,” *Condensed Matter* **6**, 23 (2021). <https://doi.org/10.3390/condmat6030023>.
- [14] P. G. Kryukov, *Femtosecond Pulses* (Fizmatlit, Moscow, 2008).
- [15] S. V. Sazonov and A. F. Sobolevskii, “Zakharov–Benney resonance as a mechanism for generating extremely short pulses in uniaxial crystals,” *JETP Letters* **75**, 621–624 (2002). <https://doi.org/10.1134/1.1503324>.
- [16] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, *Solitons and Nonlinear Wave Equations* (Academic Press, New York, 1982).

- [17] S. V. Sazonov and N. V. Ustinov, “New class of extremely short electromagnetic solitons,” *JETP Letters* **83**, 483–487 (2006). <https://doi.org/10.1134/S0021364006110038>.
- [18] S. V. Sazonov and N. V. Ustinov, “Propagation of few-cycle pulses in a nonlinear medium and an integrable generalization of the sine-Gordon equation,” *Physical Review A* **98**, 063803 (2018). <https://doi.org/10.1103/PhysRevA.98.063803>.
- [19] B. Shizgal, *Spectral Methods in Chemistry and Physics: Applications to Kinetic Theory and Quantum Mechanics* (Springer, Dordrecht, 2015). <https://doi.org/10.1007/978-94-017-9454-1>.
- [20] V. A. Trofimov, M. M. Loginova, and V. A. Egorenkov, “New two-step iteration process for solving the semiconductor plasma generation problem with arbitrary boundary conditions in the two-dimensional case,” *WIT Transactions on Modelling and Simulation* **59**, 85–96 (2015). <https://doi.org/10.2495/CMEM150081>.
- [21] X. Antoine and C. Besse, “Unconditionally stable discretization schemes of non-reflecting boundary conditions for the one-dimensional Schrödinger equation,” *Journal of Computational Physics* **188**, 157–175 (2003). [https://doi.org/10.1016/S0021-9991\(03\)00159-1](https://doi.org/10.1016/S0021-9991(03)00159-1).
- [22] J. A. Fülöp, Z. Ollmann, C. S. Lombosi, C. Skrobel, S. Klingebiel, L. Pálfalvi, F. Krausz, S. Karsch, and J. Hebling, “Efficient generation of THz pulses with 0.4 mJ energy,” *Optics Express* **22**, 20155–20163 (2014). <https://doi.org/10.1364/OE.22.020155>.
- [23] S. V. Sazonov and N. V. Ustinov, “Optical-terahertz solitons with tilted wave fronts of a system of equations of the Zakharov–Boussinesq type,” *Bulletin of the Russian Academy of Sciences: Physics* **87**, 25–29 (2023). <https://doi.org/10.31857/S0367676522700041>.