

Equilibria, Saturation and Basin Selection in Fixed-Gain Sliding-Mode Contact Control

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Abstract

A robot presses on an object whose stiffness it knows only approximately, and its actuator can push only so hard. A fixed-gain sliding-mode controller is usually trusted to cope with such an error as long as its switching gain exceeds the error force. We show that this trust has a sharp limit. With a finite actuator, a second, wrong resting place appears as soon as the stiffness error reaches the ratio of the switching gain to the actuator limit, long before the controller's own margin is used up. Which resting place the robot reaches depends on how fast it enters contact. We give closed-form conditions for when the wrong resting place exists, an exact dividing entry speed for linear contact and closed-form two-sided bounds for nonlinear contact, and the largest hit a robot at rest can absorb. A single fixed controller can serve a whole class of objects if it assumes the stiffness of the softest. A robot that approaches under the same controller always enters safely, so the danger lies in impacts and disturbances. When the exponent of the contact law is also wrong, the direction of the error, not its size, decides failure. The entry-speed bracket covers the sign and boundary-layer laws; the power-rate law is outside it. Simulations of the full saturated loop agree with every formula; convergence to each resting place is shown numerically, not proved. These closed forms were not found reported for this controller and plant.

Keywords: sliding mode control; contact control; model mismatch; actuator saturation; basin of attraction; reaching law; bifurcation

1 Introduction

The problem. A robot that presses on an object, in tasks such as pressing, tapping, assembling or grasping, rarely knows the object's stiffness exactly. Its controller therefore assumes a stiffness $k_t + \Delta k$, where k_t is the true value and Δk the error, and its actuator can push only up to a limit $F_{\max}$. Sliding-mode control is chosen for exactly this situation: its standard promise is that a switching term of size η overcomes the model error [1, 27, 2, 3, 29, 19, 32]. That promise is usually read as a convergence guarantee: if η exceeds the error force at the target, the robot settles at the target.

The objective. For a fixed-gain sliding-mode contact controller with a finite actuator we ask three practical questions. How large a stiffness error can it tolerate? What exactly goes wrong beyond that? And how hard an impact or disturbance can it absorb at a given error? We also ask what changes when the shape of the contact law, its exponent n , is wrong and not only its stiffness.

The main result. With the controller held fixed, a second, wrong resting place of the robot appears as soon as the relative stiffness error reaches $\eta/F_{\max}$. There the actuator is pushing at

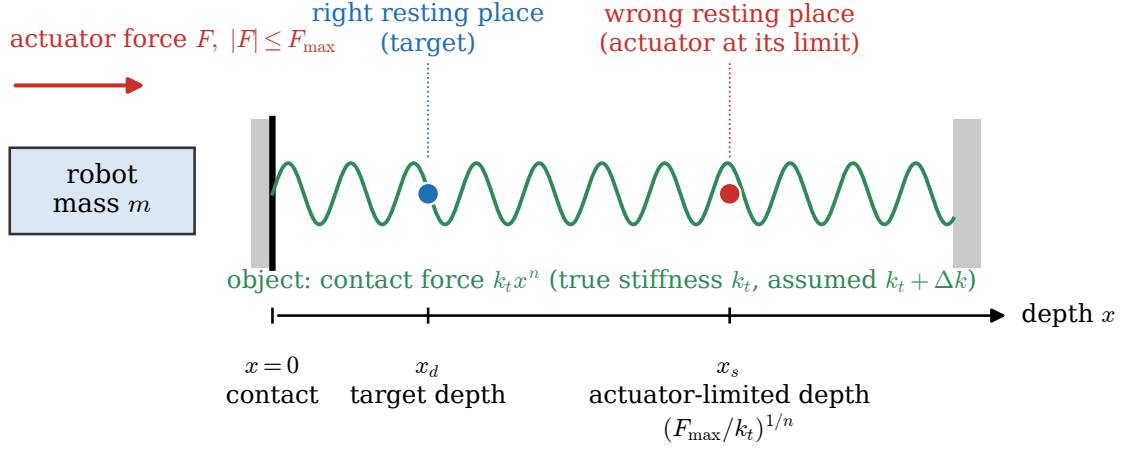


Figure 1: The setting. A robot of mass m is pushed into an object by an actuator that can deliver at most $F_{\max}$. The object pushes back with $k_t x^n$, but the controller assumes $k_t + \Delta k$. The robot can rest at the target depth x_d (blue) or, if the stiffness error is large enough, at the actuator-limited depth x_s (red).

its limit and the robot stays deeper in the object than the target. In the running example of Section 2 this happens at 2.9% error, far below the 8.1% that the controller’s target-point margin suggests. The robot then presses with 700 N instead of the 233 N needed at the target, and sits 15 mm deep instead of 5 mm. Between the two limits the robot has two possible resting places, and the speed at which it enters contact decides which one it reaches. We give the formulas for when this happens, for the speed that separates the outcomes, and for the largest hit the robot can absorb.

Key results at a glance. Boundary-layer or sign controller; symbols are in Table 1: m mass, λ sliding slope, x_d target depth, Φ layer width, η switching gain, $F_{\max}$ actuator limit. Numbers are from the running example of Section 2.

1. **How much stiffness error can it tolerate?** A wrong resting place exists exactly when the relative error reaches $\eta/F_{\max}$:

$$\Delta k \geq \frac{\eta k_t}{F_{\max}} \quad (\text{Corollary 3})$$

Example: $20/700 = 2.9\%$. A gain of $\eta = 210$ N raises this to 30%.

Mechanical stop at depth $s_{\max} < x_s$: replace $F_{\max}$ by $k_t s_{\max}^n$ (Proposition 5).

Rule of thumb: assume the stiffness of the softest object you may meet, plus the margin $\eta/F_{\max}$ (Corollary 4).

2. **Which resting place does the robot reach?** For linear contact it ends at the target if and only if its entry speed is below

$$v_c = \mu(\Delta k) \frac{\eta}{\Delta k}, \quad \mu(q) = \frac{1}{2} \left(\lambda + \sqrt{\lambda^2 + 4q/m} \right) \quad (\text{Theorem 11})$$

Example: $\Delta k = 2500$ N/m gives $v_c = 1.42$ m/s.

For nonlinear contact, two closed-form bounds trap v_c (Theorem 13). The power-rate law is outside this bracket.

3. Which entries are always safe?

- Any entry slower than λx_d (0.75 m/s) is safe (Corollary 15).
- An approach driven by the same controller enters at no more than $\min\{\eta/(m\lambda), \lambda x_d\}$ (0.27 m/s) (Corollary 16).

4. How hard a hit can the robot absorb? A forward velocity kick at rest of up to

$$\mu(\eta/x_u)(x_u - x_+), \quad x_u = (\eta/\Delta k)^{1/n}, \quad x_+ = x_d + \Phi/\lambda \quad (\text{Corollary 17})$$

is absorbed. *Example:* 0.475 m/s certified, 0.4955 m/s exact.

5. What if the contact exponent is also wrong? Only errors that push *away* from the target can trap the robot. An exact stall threshold is given in Section 7.

6. How do I use this? Section 6 gives a step-by-step procedure with a worked example.

Why it matters. A robot that approaches under the same controller always enters contact slowly enough to be safe (Corollary 16). The wrong resting place therefore threatens the situations in which the robot is hit or enters fast: an unexpected impact, a person or a tool bumping the arm, re-contact after a bounce, or an approach phase run by a separate controller. For these the formulas turn the controller’s gain and the stiffness error into a number a designer can use, the impact speed or disturbance the controller is certified to absorb. A simulation of the same model with the parameters of a parallel-jaw gripper of the running-example class (a simulation, not an experiment) shows the effect: a fixed-gain controller tuned for an aluminium object and used on silicone rubber overshoots to 102 mm and ends 73 mm beyond its 5 mm target. This paper explains that failure as the wrong resting place at 78 mm, shows that a local eigenvalue test misreads it (Section 3.6), and shows when it can be reached (Section 6).

What is new. The textbook design rule asks the switching gain to dominate the largest error force the actuator can reach [1, 3], which here reads $\eta \geq \Delta k x_s^n$ with x_s the depth at which the actuator limit is needed. We show that this rule is sharp: it is exactly the condition under which no wrong resting place exists. The local target-point rule, which only asks $\eta > |D(x_d)|$ with D the error force, is weaker; in the example it allows an 8.1% error against 2.9%. What is new is therefore not the rule but what lies beyond it: the exact geometry between the two limits, the same analysis for reaching laws without a dominating constant term (boundary layer, power rate), and the entry-velocity bounds that say which resting place a given impact reaches. With the controller held fixed, a positive stiffness error creates a saturation-locked rest exactly when the relative error reaches $\eta/F_{\max}$. The same branch geometry covers the sign, boundary-layer and power-rate laws. The entry speed that selects the rest is exact for linear contact and bracketed for nonlinear contact. These closed forms were not found reported for this controller and plant.

Why fixed gains. Adaptive sliding-mode laws [20] can in principle remove a stiffness error, but fixed-gain laws remain the common baseline, and an adaptive law saturates too: while the actuator limit is active its estimate can wind up, a problem known from anti-windup design [21]. The map derived here tells a designer when adaptation is needed. Adaptive laws are outside our scope.

Why one paper, and how it is organised. The results answer one question from three sides: where the robot can rest (Section 3), which resting place a given impact reaches (Sections 4

and 5), and what happens when the contact law itself is wrong (Section 7). Section 6 turns them into a design procedure, Section 8 collects the general theory behind the three laws, Section 9 summarises the numerical verification, and Appendix A gives its details. The practical results come first and the general theory last.

Relation to prior work. Reaching-law design under uncertainty is mature (Utkin 1992 [1]; Gao and Hung 1993 [4]; Slotine 1984 [28]). Adaptive-gain error is a different mismatch from a fixed wrong stiffness [30]. Saturated sliding mode already has a domain-of-attraction theory. Fulwani and Bandyopadhyay redesign the surface so the actuator limit is respected [9]. Wang, Gao, Liu and Wu estimate the domain of attraction of a saturated law under magnitude and rate limits by an invariant set (IET Control Theory and Applications, 12(8):1075–1085, 2018, doi:10.1049/iet-cta.2017.1081) [38]. Behera, Chalanga and Bandyopadhyay give an explicit domain for super-twisting control with a bounded actuator (Automatica, 87:437–441, 2018) [11]. Those results rule out any claim that regions of attraction under saturation are new. Tohidi, Yildiz and Kolmanovsky note that actuator limits can create multiple equilibria in a flight controller (Int. J. Robust Nonlinear Control, 2023) [39]. That is a remark, not this contact loop. Contact under an uncertain stiffness is already a design problem: Hogan 1985 [22], Colgate and Hogan 1988 [23], Volpe and Khosla 1993 [24], and set-valued admittance for impact with unknown stiffness and torque saturation (arXiv:2409.19275) [40]. Those works compensate the error. This paper does not propose another compensator. The fold language is from piecewise-smooth bifurcation theory [12, 13, 14]. The contribution is the closed-form classification for this mismatch and this clipping.

2 Setup: robot, controller and symbols

The contact plant is

$$m\ddot{x} + c\dot{x} + k_tx^{n_t} = F, \quad x \geq 0, \quad |F| \leq F_{\max}. \quad (1)$$

Let $x_d > 0$ be the target and

$$s = \dot{x} + \lambda(x - x_d), \quad \lambda > 0. \quad (2)$$

The controller feeds forward the assumed contact force $k_ax^{n_a}$ and adds an odd reaching action $-g(|s|)\operatorname{sgn}(s)$, $g \geq 0$:

$$F_c = c\dot{x} + k_ax^{n_a} - m\lambda\dot{x} - g(|s|)\operatorname{sgn}(s), \quad F = \operatorname{sat}(F_c, F_{\max}). \quad (3)$$

The feedforward uses $\max\{x, 0\}^{n_a}$, so the controller adds no contact force before contact, and the plant has none for $x < 0$. The model residual is $D(x) = k_a \max\{x, 0\}^{n_a} - k_t \max\{x, 0\}^{n_t}$, written $D(x) = k_ax^{n_a} - k_tx^{n_t}$ for $x \geq 0$. Before clipping,

$$m\dot{s} = D(x) - g(|s|)\operatorname{sgn}(s). \quad (4)$$

Two cases matter.

- *Stiffness mismatch* ($n_a = n_t = n$): $D(x) = \Delta k x^n$ with $\Delta k = k_a - k_t$. Sections 3–5 treat this case.
- *Exponent mismatch* ($n_a \neq n_t$): D can change sign. Section 7 treats this case.

Three reaching laws are used: the sign law $g \equiv \eta$, the boundary-layer law $g(r) = \eta \min\{r/\Phi, 1\}$ (that is, $\eta \operatorname{sat}(s/\Phi)$), and the power-rate law $g(r) = Kr^\rho$, $0 < \rho < 1$.

Define the actuator-limited displacement and the layer edge

$$x_s = \left(\frac{F_{\max}}{k_t} \right)^{1/n}, \quad e_s = x_s - x_d, \quad x_+ = x_d + \frac{\Phi}{\lambda}. \quad (5)$$

Every unsaturated equilibrium satisfies $k_t x_e^n \leq F_{\max}$, i.e. $x_e \leq x_s$. We assume $x_s > x_d$. Positive mismatch ($\Delta k > 0$, stiffness overestimated) is the dangerous side.

Running example. A robot of mass 0.5 kg pushes on an object of true stiffness $k_t = 46524$ N/m ($n = 1$) with an actuator limit of $F_{\max} = 700$ N, a target depth of 5 mm, sliding slope $\lambda = 150$ s⁻¹, switching gain $\eta = 20$ N and layer width $\Phi = 0.05$ m/s. The force at the target is $k_t x_d = 233$ N. Then $x_s = 15.05$ mm and $x_+ = 5.33$ mm. The tolerated error is $\Delta k_{\text{sat}} = 1329$ N/m (2.9% of k_t) and the controller's own limit is $\Delta k^* = 3750$ N/m (8.1%). A mismatch of $\Delta k = 2500$ N/m (5.4%) lies between them. A better-tuned gain $\eta = 210$ N ($0.3F_{\max}$) gives 30% and 85%. For $n > 1$ we fix $k_t x_d^n = 233$ N, so the force at the target is the same for every n . Hertzian sphere-on-plane contact has $n = 3/2$ [15, 25]. The dimensionless combination $\varepsilon = \eta/(m\lambda^2 x_d) = 0.356$ will matter in Section 5.

Table 1: Symbols. The last column gives the running example.

Symbol	Meaning	Unit	Running example
m, c	robot mass, viscous damping	kg, N s/m	0.5, 41.7
x	depth into the object	m	
x_d	target depth	m	5 mm
k_t, n	true contact stiffness and exponent, force $k_t x^n$	N/m ⁿ	46524 N/m, 1
$k_a, \Delta k$	assumed stiffness; error $\Delta k = k_a - k_t$	N/m	
$F_{\max}$	actuator force limit	N	700
λ, s	sliding slope; sliding variable $s = \dot{x} + \lambda(x - x_d)$	1/s, m/s	150
η, Φ	switching gain; boundary-layer width	N, m/s	20, 0.05
g	reaching law (sign, boundary layer, power rate)	N	
x_s	depth at which the actuator limit is needed, $(F_{\max}/k_t)^{1/n}$	m	15.05 mm
x_+	layer edge, $x_d + \Phi/\lambda$	m	5.33 mm
Δk_{sat}	smallest error giving a wrong resting place	N/m ⁿ	1329
Δk^*	error at which the controller's target branch ends	N/m ⁿ	3750
x_u	dividing depth between the two resting places, $(\eta/\Delta k)^{1/n}$	m	
v_0, v_c	entry speed; critical (dividing) entry speed	m/s	
$\mu(q)$	decay rate, $\frac{1}{2}(\lambda + \sqrt{\lambda^2 + 4q/m})$	1/s	

3 Where the robot can come to rest

A resting place (an equilibrium) is a state with zero velocity and zero acceleration. Where the robot can rest, and whether a resting place is stable, decides everything that follows. This section finds the resting places of the three reaching laws and says when each is stable.

3.1 Equilibria and a stability test

For $D > 0$ above the target, an unsaturated equilibrium $e = x - x_d > 0$ has $s = \lambda e$ and satisfies $D(x_d + e) = g(\lambda e)$. For stiffness mismatch this is

$$\Delta k = \phi_g(e) := \frac{g(\lambda e)}{(x_d + e)^n}. \quad (6)$$

Define the logarithmic elasticity $E_g(r) = rg'(r)/g(r)$ of the reaching magnitude, and the elasticity $\varepsilon_D(x) = xD'(x)/D(x)$ of the residual ($\varepsilon_D \equiv n$ for stiffness mismatch).

The theorem uses the elasticity of the reaching law: the percentage by which its size grows when the error grows by one percent. It is $E_g = 0$ for a constant (sign) term and $E_g = 1$ for a term proportional to the error, such as the inside of a boundary layer.

Theorem 1 (Branch slope decides stability). *Let g be differentiable and nondecreasing and let $e > 0$ be an unsaturated equilibrium, $D(x_e) = g(\lambda e) > 0$. Then*

$$\text{tr } J = -\lambda - \frac{g'(\lambda e)}{m} < 0, \quad (7)$$

$$\det J = \frac{\lambda g'(\lambda e) - D'(x_e)}{m} = \frac{g(\lambda e)}{m e} \left[E_g(\lambda e) - \varepsilon_D(x_e) \frac{e}{x_e} \right]. \quad (8)$$

So the equilibrium is locally asymptotically stable if $\det J > 0$ and a saddle if $\det J < 0$. For stiffness mismatch, $\text{sign } \det J = \text{sign } \phi'_g(e)$ and

$$\det J = \frac{g(\lambda e)}{m e} \left[E_g(\lambda e) - \frac{n e}{x_d + e} \right]. \quad (9)$$

Proof. In the coordinates (e, v) , $v = \dot{x}$, we have $\dot{e} = v$ and $m\dot{v} = -m\lambda v + D(x_d + e) - g(v + \lambda e)$. Linearising at $v = 0$ gives $\text{tr } J = -\lambda - g'/m$ and $\det J = -\partial_e \dot{v} = [\lambda g'(\lambda e) - D'(x_e)]/m$. Using $\lambda g' = gE_g/e$ and $D' = D\varepsilon_D/x = g\varepsilon_D/x$ gives (8). For stiffness mismatch, $\frac{d}{de} \log \phi_g = [E_g(\lambda e) - ne/(x_d + e)]/e$. $\square$

In plain words: an off-target equilibrium is stable exactly when the reaching action grows faster with error than the mismatch force does.

3.2 Sign law

Here $g \equiv \eta$, $E_g \equiv 0$, and $\phi_g(e) = \eta/(x_d + e)^n$ is strictly decreasing. By Theorem 1 every positive switched equilibrium is a saddle at $x_u = (\eta/\Delta k)^{1/n}$. The target-near sliding equilibrium persists only while Filippov sliding exists at the target, i.e. $|D(x_d)| < \eta$:

$$\Delta k < \Delta k_{\text{sign}}^* = \frac{\eta}{x_d^n}. \quad (10)$$

For $\Delta k < 0$ the switched branch $x_{\text{neg}} = (\eta/|\Delta k|)^{1/n}$ is stable for every $|\Delta k| > \eta/x_d^n$. Ideal sliding at the target therefore exists iff $|\Delta k|x_d^n < \eta$, so the local limits are bilateral, $\Delta k_{\pm}^* = \pm\eta/x_d^n$. On the positive side the switched saddle $x_u = (\eta/\Delta k)^{1/n}$ moves toward the target as Δk grows and reaches it at Δk_+^* , where target sliding is lost. On the negative side the lower branch persists and moves smoothly toward the origin. The controller is therefore asymmetric in consequence: overestimation ends in a saddle collision and, with a finite actuator, a remote stable false equilibrium, while underestimation degrades smoothly. Underestimation is not unconditionally safe, however: it has the same local magnitude limit, and what it avoids is the competing attractor, not the loss of ideal sliding.

3.3 Boundary-layer law

Here $g(r) = \eta \min\{r/\Phi, 1\}$. Inside the layer $E_g = 1$ and outside $E_g = 0$. The in-layer branch is increasing iff $1 > ne/(x_d + e)$, i.e. $e < x_d/(n-1)$. Hence the maximum is a kink at the layer edge

whenever $\Phi/\lambda < x_d/(n-1)$ (always for $n=1$), and otherwise an in-layer interior maximum. At the kink

$$\Delta k^* = \frac{\eta}{x_+^n}, \quad x_+ = x_d + \frac{\Phi}{\lambda}, \quad (11)$$

and on the negative side $\Delta k^* = -\eta/(x_d - \Phi/\lambda)^n$. The kink is a boundary-event mechanism, not a fold: the target-near equilibrium collides with the switched saddle x_u exactly when $x_u = x_+$. As $\Phi \rightarrow 0$ we recover (10).

For $n=1$ the in-layer equilibrium is explicit. Inside the layer $g = \eta\lambda e/\Phi$, so $\Delta k x = (\eta\lambda/\Phi)(x - x_d)$ and

$$x^* = \frac{\eta\lambda x_d}{\eta\lambda - \Delta k \Phi}, \quad (12)$$

valid while the branch stays inside the layer ($x^* \leq x_+$, which is $\Delta k \leq \Delta k^*$) and is actuator-feasible ($x^* \leq x_s$). Theorem 1 gives $\det J = (\eta\lambda/\Phi - \Delta k)/m$ there, so linearising at the origin of the layer changes sign only at $\Delta k = \eta\lambda/\Phi$ (60000 N/m in the running example, against $\Delta k^* = 3750$ N/m). That threshold is not the operative one: the branch is lost at the layer edge or at the actuator limit long before, which is why infinitesimal linearisation at the target is insufficient.

Table 2: Failure mechanisms of the three laws (positive mismatch, $x_s \geq x_+$).

Law	Target-near equilibrium	Upper limit	Mechanism
Sign	exact sliding at x_d	η/x_d^n	saddle collides with the target (existence limit)
Boundary layer	in-layer offset, (12) for $n=1$	η/x_+^n	branch reaches the layer edge, or the actuator limit first if $x_s < x_+$
Power rate	off-surface $e(\Delta k) > 0$	offset Δk_{fold}	saddle-node fold; no exact tracking (Section 8.3)

3.4 Negative mismatch

For $\Delta k = -|\Delta k| < 0$ the lower equilibrium has $r = x_d - x > 0$ and satisfies $|\Delta k| = \psi(r) = g(\lambda r)/(x_d - r)^n$. Since $\frac{d}{dr} \log \psi = \lambda g'(\lambda r)/g(\lambda r) + n/(x_d - r) > 0$ for nondecreasing g , ψ increases from 0 to ∞ , so there is exactly one lower equilibrium. It is stable: $\text{tr } J < 0$ and $\det J = [n|\Delta k|x^{n-1} + \lambda g'(\lambda r)]/m > 0$. Negative mismatch therefore degrades accuracy continuously and never creates a competing attractor. This is the source of the asymmetry between the two signs of Δk for the sign law.

3.5 Three mechanisms compared

The three laws fail in three different ways (Table 2). The sign law keeps exact target sliding until a sharp existence limit. The boundary-layer law never tracks exactly but its offset grows smoothly, (12), until the branch reaches the layer edge or the actuator limit. The power-rate law has no exact target equilibrium for any nonzero mismatch and loses its stable branch in a genuine saddle-node bifurcation, a different phenomenon from a boundary event.

3.6 A common pitfall: reading failure from a local eigenvalue

A frequent argument linearises the closed loop about the target using the equivalent control only, that is, it drops the reaching term. For $n=1$ the characteristic equation is $\mu^2 + \lambda\mu - \Delta k/m = 0$,

with roots $\frac{1}{2}(-\lambda \pm \sqrt{\lambda^2 + 4\Delta k/m})$. For every $\Delta k > 0$ one root is positive, and this is read as “any overestimate of the stiffness, however small, destabilises the loop”. Table 3 shows that this reading is wrong.

The dropped term is the point. Inside the boundary layer the reaching term has slope η/Φ , and Theorem 1 gives the roots of $\mu^2 + (\lambda + \eta/(m\Phi))\mu + (\eta\lambda/\Phi - \Delta k)/m = 0$, which stay in the left half plane until $\Delta k = \eta\lambda/\Phi = 60000$ N/m. The positive root of the first equation is real, but it belongs to a different place: it is the saddle x_u of Theorem 1 with $g' = 0$, which divides the two resting places. For small Δk that saddle lies far outside the working range ($x_u = \eta/\Delta k = 20$ m for $\Delta k = 1$ N/m). The target itself stays stable, and the loop from rest settles at the offset of (12).

Table 3: Eigenvalues (1/s) of the linearisation about the target for the running-example controller (set A, $\eta = 20$ N, $\lambda = 150$ s⁻¹, $\Phi = 0.05$ m/s) on a plant with $k_t = 8969$ N/m and $m = 0.5$ kg, and the position reached from rest in the full clipped loop. “Outer” drops the reaching term; “in-layer” keeps it.

Δk (N/m)	outer eigenvalues	in-layer eigenvalues	final x from rest (mm)
-1000	-135.2, -14.8	-796.9, -153.1	4.918
+1	-150.0, +0.01	-800.0, -150.0	5.000
+1000	-162.3, +12.3	-803.1, -146.9	5.085
+3000	-182.8, +32.8	-809.1, -140.9	5.263
+37555	-359.1, +209.1	-900.1, -49.9	78.047 (peak 102.06)

The overestimate that does fail ($\Delta k = 37555$ N/m, the controller tuned for aluminium used on silicone) fails for a different reason. The error exceeds the controller’s own limit $\Delta k^* = 3750$ N/m, so the in-layer branch has ended at the layer edge, and the robot overshoots to 102 mm (1941%) and rests at $x_s = F_{\max}/k_t = 78.05$ mm with the actuator at its limit, a 73.05 mm error. Three consequences follow. Small overestimates are harmless, since they only shift the resting place by a computable offset. Failure is a statement about the branch structure and the actuator limit, not about an eigenvalue at the target. And a fixed assumed stiffness can be safe over a whole class of objects (Corollary 4).

4 The actuator limit creates a wrong resting place

At $x = x_s$ the true plant needs $F_{\max} = k_t x_s^n$, so the unclipped command is $F_c(x_s) = F_{\max} + \Delta k x_s^n - g(\lambda e_s)$. The saturation-locked equilibrium x_s is consistent iff $\Delta k x_s^n \geq g(\lambda e_s)$, i.e.

$$\Delta k \geq \Delta k_{\text{sat}} = \phi_g(e_s). \quad (13)$$

The clipped linearisation $m\ddot{\xi} + c\dot{\xi} + nk_t x_s^{n-1}\xi = 0$ is asymptotically stable, so this false equilibrium is a stable attractor.

Lemma 2 (Universal saturation ordering). *For every reaching law, $\Delta k_{\text{sat}} = \phi_g(e_s) \leq \sup_{e>0} \phi_g(e)$. If ϕ_g has a unique maximum at e^* , equality holds iff $e_s = e^*$.*

So saturation adds no unrelated threshold: it samples the same branch at the single displacement e_s .

Corollary 3 (Tolerated relative error). *For every reaching law and every n , $\Delta k_{\text{sat}}/k_t = g(\lambda e_s)/F_{\max}$. For the sign and boundary-layer laws with $x_s \geq x_+$ this is $\eta/F_{\max}$. Equivalently, a stable false equilibrium exists exactly when $\eta \leq \Delta k x_s^n$, i.e. exactly when the classical dominance rule is violated at the actuator limit. For $\Delta k < \Delta k_{\text{sat}}$ the only other off-target equilibrium, the switched saddle x_u , lies beyond x_s and is infeasible.*

Proof. $\Delta k_{\text{sat}} = g(\lambda e_s)/x_s^n$ and $k_t x_s^n = F_{\text{max}}$. The locked equilibrium is consistent iff $\Delta k x_s^n \geq g(\lambda e_s)$, and $x_u = (\eta/\Delta k)^{1/n} > x_s$ iff $\Delta k < \eta/x_s^n$. $\square$

In plain words: the stiffness error you can tolerate, as a fraction of the true stiffness, is the switching gain divided by the actuator limit. In the running example that is $20/700 = 2.9\%$. Raising the actuator limit without raising the gain lowers the tolerance.

Corollary 4 (One assumed stiffness for a class of objects). *Let the true stiffness lie in $[k_{\min}, k_{\max}]$ and let the controller assume k_a . If*

$$k_a \leq k_{\min} \left(1 + \frac{\eta}{F_{\text{max}}}\right), \quad (14)$$

then no wrong resting place exists for any object of the class, for every contact exponent n , so no impact or disturbance can lead to one. For $n = 1$ and a stiffer object the price is a smooth offset below the target, $x_d - x^ = x_d |\Delta k| \Phi / (\eta \lambda + |\Delta k| \Phi)$ with x^* from (12), valid while the lower equilibrium stays inside the layer.*

Proof. By Corollary 3, a wrong resting place exists for an object of stiffness k_t^{obj} iff $k_a - k_t \geq \eta k_t / F_{\text{max}}$, i.e. iff $k_a \geq k_t (1 + \eta / F_{\text{max}})$. The right-hand side increases with k_t , so the binding object is the softest and (14) excludes the condition for the whole class. For stiffer objects $\Delta k < 0$, which creates no competing attractor (Section 3); the offset is (12) with $\Delta k < 0$. $\square$

In plain words: assume the stiffness of the softest object you may meet, plus the margin η / F_{max} . Overestimating a soft object is the dangerous side. Underestimating a stiff one costs only a small offset.

Proposition 5 (Mechanical stroke limit). *Let the jaw be stopped by a rigid, inelastic stop at depth s_{max} with $x_+ \leq s_{\text{max}} < x_s$. The robot can rest on the stop iff $\Delta k s_{\text{max}}^n \geq \eta$, i.e. iff the dividing depth satisfies $x_u \leq s_{\text{max}}$. The tolerated relative error is then η / F_{reach} with $F_{\text{reach}} = k_t s_{\text{max}}^n < F_{\text{max}}$, the contact force at the stop, and a velocity kick can drive the robot onto the stop only if $x_u \leq s_{\text{max}}$.*

Proof. On the stop $v = 0$ and the stop supplies any reaction needed, so the robot stays iff the command pushes into the stop. For $s_{\text{max}} \geq x_+$ the outer branch applies and $F_c = k_t s_{\text{max}}^n + \Delta k s_{\text{max}}^n - \eta$, which is at least the contact force $k_t s_{\text{max}}^n$ iff $\Delta k s_{\text{max}}^n \geq \eta$. Clipping at F_{max} does not change this because $k_t s_{\text{max}}^n < F_{\text{max}}$. Since $x_u = (\eta / \Delta k)^{1/n}$, the condition is $x_u \leq s_{\text{max}}$. If $x_u > s_{\text{max}}$, a trajectory that hits the stop has $F_c < k_t s_{\text{max}}^n$ there and leaves it again toward the target. $\square$

In plain words: a mechanical stop makes the robot more tolerant: the wrong resting place on the stop needs a larger stiffness error than the one at the actuator limit, because the contact force at the stop is smaller than F_{max} .

Theorem 6 (Coexistence, capture and monotone branches). *Let g be nondecreasing.*

1. *If ϕ_g has a unique interior maximum e^* (smooth or kink), a coexistence interval $\Delta k_{\text{sat}} < \Delta k < \Delta k_{\text{fold}}$ containing a stable target-near root, a feasible saddle and the stable saturation equilibrium exists iff*

$$E_g(\lambda e_s) < n \left(1 - \frac{x_d}{x_s}\right) \quad (\Leftrightarrow \phi'_g(e_s) < 0). \quad (15)$$

If the inequality is reversed, saturation captures the stable branch before the unconstrained fold.

2. If ϕ_g is strictly decreasing there is no stable positive off-surface branch. If $g(0^+) > 0$, the coexistence window runs from Δk_{sat} to the Filippov reserve $\Delta k_{\text{reserve}} = g(0^+)/x_d^n$.
3. If ϕ_g is strictly increasing, every positive off-surface equilibrium is stable, and capture occurs when the branch reaches x_s . There is no saddle-mediated window.

Proof. Positive off-surface equilibria are the roots of $\phi_g(e) = \Delta k$ with feasibility $e \leq e_s$, and their stability follows from Theorem 1. In the unimodal case, $e_s > e^*$ is equivalent to (15). A horizontal level between $\phi_g(e_s)$ and $\phi_g(e^*)$ meets the feasible branch once on the stable side and once on the saddle side, while x_s is simultaneously consistent by (13). The monotone cases follow from the same root, slope and feasibility arguments. $\square$

In plain words: if the actuator limit sits beyond the peak of the branch, the robot has two possible resting places over a whole range of mismatch, and it can end in the wrong one.

Corollary 7 (Sign and boundary-layer windows). *Let $x_s \geq x_+$. For the boundary-layer law the target-near equilibrium, the saddle x_u and the stable saturation equilibrium coexist for*

$$\Delta k_{\text{sat}} = \frac{\eta}{x_s^n} = \frac{\eta k_t}{F_{\text{max}}} < \Delta k < \Delta k^* = \frac{\eta}{x_+^n}, \quad (16)$$

with normalised width $(\Delta k^ - \Delta k_{\text{sat}})/(\eta/x_d^n) = (1 + \beta)^{-n} - r_F^{-1}$, where $r_F = F_{\text{max}}/(k_t x_d^n)$ and $\beta = \Phi/(\lambda x_d)$. For the sign law the window ends at η/x_d^n . If instead $x_d < x_s < x_+$, the target-near branch becomes infeasible at*

$$\Delta k_{\text{feas}} = \frac{\eta \lambda (x_s - x_d)}{\Phi x_s^n}, \quad (17)$$

which precedes the layer-edge collision for $n = 1$ and, for $n > 1$, whenever $x_+ < n x_d/(n - 1)$.

Proof. Since $E_g = 0$ outside the layer, (15) holds iff $x_s > x_+$. Substituting $\phi_g(e_s) = \eta/x_s^n$ gives Δk_{sat} , and the kink value is Δk^* . The width follows by dividing by η/x_d^n . In the feasibility-limited case the in-layer branch $g_f(x) = \eta \lambda (x - x_d)/(\Phi x^n)$ reaches x_s at $g_f(x_s)$, and $g_f(x_+) = \Delta k^*$. The function g_f is increasing on (x_d, x_+) when $n = 1$, or when $x_+ < n x_d/(n - 1)$. $\square$

A checked feasibility-limited case has $n = 1$, $x_s = 7$ mm ($F_{\text{max}} = 325.7$ N), $\eta = 20$ N and $\Phi = 0.5$ m/s, so $x_+ = 8.333$ mm. Then $\Delta k_{\text{feas}} = 1714$ N/m, below the layer-edge value 2400 N/m, and the loop from rest settles at 6.917 mm at $0.97 \Delta k_{\text{feas}}$ and is captured at $x_s = 7.000$ mm at $1.03 \Delta k_{\text{feas}}$.

In the running example $\Delta k_{\text{sat}} = 1329.26$ N/m, well below the classical layer-edge limit $\Delta k^* = 3750$ N/m. The classical margin is therefore not a convergence guarantee: a competing attractor exists over 65% of the range below it.

Remark 8 (Gain tuning rescales the window). For the boundary-layer law, $\Delta k_{\text{sat}}/\Delta k^* = (x_+/x_s)^n$ depends on geometry only, not on η . Raising the gain therefore rescales the window but does not remove it. With $\eta = 210$ N ($0.3 F_{\text{max}}$) and $n = 1$, the window is $13957 < \Delta k < 39375$ N/m, a 30% tolerance against a local limit of 85%, with the same ratio 0.3545 as the running example.

Remark 9 (A wide layer trades the window for tolerance). If the layer is so wide that $x_+ \geq x_s$ (that is, $\Phi \geq \lambda e_s$), the outer saddle is infeasible and the coexistence window disappears: the law belongs to the monotone class and there is no dependence on the initial state. The price is a lower tolerance. The target branch is lost at $\Delta k_{\text{feas}} = \Delta k_{\text{sat}} \lambda e_s / \Phi \leq \Delta k_{\text{sat}}$, smaller by the factor $\lambda e_s / \Phi$. In the feasibility example above ($x_s = 7$ mm, $\Phi = 0.5$ m/s) the factor is 0.6.

Theorem 10 (Power-rate coexistence and actuator headroom). *For the power-rate law with $n > \rho$, a nonempty interval containing a stable power-rate equilibrium, a saddle and the saturation equilibrium exists iff $e_s > e^*$, i.e.*

$$x_s > \frac{n}{n-\rho} x_d \iff r_F > \left(\frac{n}{n-\rho}\right)^n \iff \rho < n(1 - r_F^{-1/n}), \quad (18)$$

and then $\Delta k_{\text{sat}} < \Delta k < \Delta k_{\text{fold}}$ with $\Delta k_{\text{sat}} = K\lambda^\rho e_s^\rho / x_s^n$. The normalised window fraction is $W = 1 - \Delta k_{\text{sat}} / \Delta k_{\text{fold}}$.

Proof. Apply Theorem 6(1) with $E_g = \rho$: the condition is $\rho < ne_s/x_s$, which rearranges to (18). $\square$

With $n = 1$, $\rho = 0.5$, headroom $r_F = 3.01$ gives $W = 5.8\%$, and raising r_F to 10 expands it to about 40%. When $e_s < e^*$, saturation preempts the fold: for $x_s = 8$ mm one finds $\Delta k_{\text{sat}} = 1677.05 < \Delta k_{\text{fold}} = 1732.05$, yet at $\Delta k = 1704.55$ N/m the trajectory from rest is captured at $x_s = 8.000$ mm.

5 Which resting place? Entry speed, impacts and disturbances

Inside the coexistence window the target-near equilibrium and the saturation equilibrium are separated by the stable manifold of the switched saddle $x_u = (\eta/\Delta k)^{1/n}$. We treat the sign and boundary-layer laws, for which $g \equiv \eta$ outside the layer. The reduced equation is then

$$\ddot{x} + \lambda\dot{x} = f(x), \quad f(x) = \frac{\Delta k x^n - \eta}{m}, \quad f(x_u) = 0. \quad (19)$$

For $q > 0$ let $\mu(q) = \frac{1}{2}(\lambda + \sqrt{\lambda^2 + 4q/m})$, the positive root of $\mu^2 - \lambda\mu = q/m$. It exceeds λ and increases with q , with $\mu'(q) \leq 1/(m\lambda)$.

A robot makes contact at $x = 0$ with entry speed $v_0 = \dot{x}(0) > 0$. The critical entry speed v_c is the intercept $w(0)$ of the stable manifold $v = w(x)$ of $(x_u, 0)$ [37] on the side $x < x_u$.

5.1 Linear contact: the exact separator

Theorem 11 (Exact separator for $n = 1$). *Let $n = 1$ and $\Delta k_{\text{sat}} < \Delta k < \Delta k^*$. The stable manifold of the saddle is the straight line $v = \mu(\Delta k)(x_u - x)$ with $x_u = \eta/\Delta k$. Hence a trajectory entering contact at $x = 0$ with $v_0 > 0$ crosses the saddle iff*

$$v_0 > v_c = \mu(\Delta k) \frac{\eta}{\Delta k}, \quad (20)$$

provided $(0, v_c)$ lies in the outer region $v_c - \lambda x_d > \Phi$ and the command is unsaturated there.

Proof. For $n = 1$, $f(x) = \Delta k(x - x_u)/m$ is linear, so (19) is a linear saddle with eigenvalues $-\mu(\Delta k)$ and $\lambda + \mu(\Delta k)$. The stable eigenvector is the line above, and at $x = 0$ it gives $v = \mu(\Delta k)x_u$. $\square$

At the running-example values $\Delta k = 2000, 2500, 3000$ N/m this gives $v_c = 1.7311, 1.4246, 1.2188$ m/s, and simulations switch the final attractor within $\pm 3\%$ of these speeds.

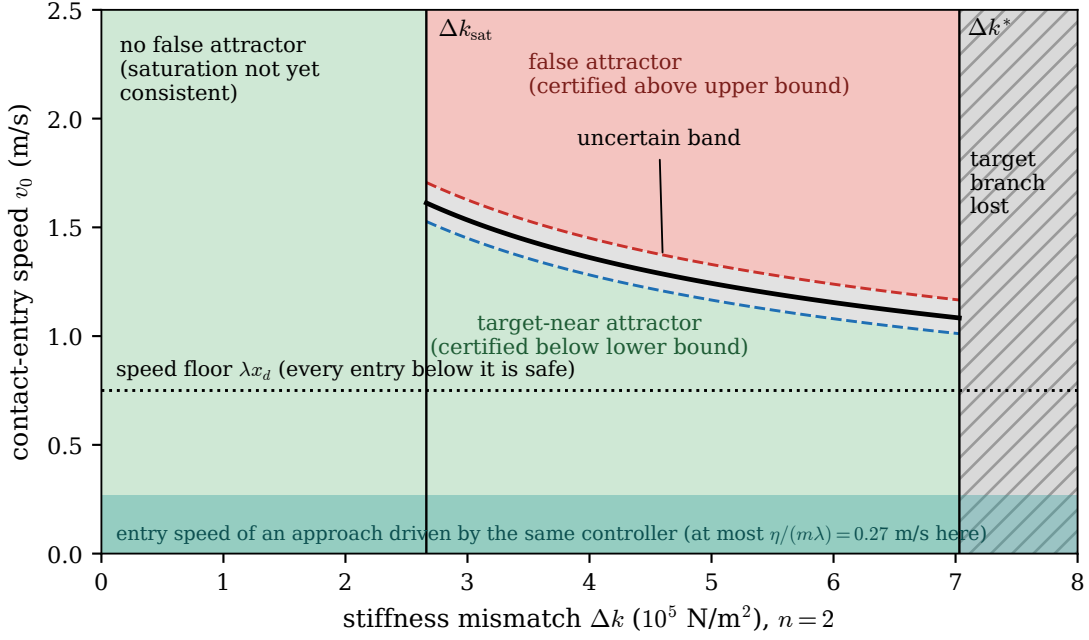


Figure 2: Outcome map for the boundary-layer controller in the plane of stiffness error Δk and contact-entry speed v_0 ($n = 2$, running-example plant). Below Δk_{sat} , the first error at which a wrong resting place exists, the robot can only reach the target. Between Δk_{sat} and Δk^* , where the controller's own target branch ends, the entry speed decides: the target is certified below the lower bound and the wrong resting place above the upper bound, with the exact dividing speed (black) between them. A robot approaching under the same controller enters at or below the speed floor λx_d (Corollaries 15 and 16), so the wrong resting place matters for impacts, disturbances and separately controlled approach phases.

5.2 Nonlinear contact: a two-sided bracket

For $n > 1$ the force $\Delta k x^n$ is convex, so the stable manifold is curved and no line is exact. Convexity pins the force curve between two lines.

Lemma 12 (Comparison lines). *Let $n \geq 1$ and $x_u > 0$. Put the chord rate $q_C = \eta/x_u$ and the tangent rate $q_T = n\eta/x_u$, i.e. q_C/m and q_T/m are the slopes of the chord of f from $(0, f(0))$ to $(x_u, 0)$ and of the tangent of f at x_u . Then*

$$\ell_T(x) \leq f(x) \leq \ell_C(x), \quad 0 \leq x \leq x_u, \quad \ell_{T,C}(x) = \frac{q_{T,C}}{m}(x - x_u),$$

with strict inequalities inside $(0, x_u)$ when $n > 1$. For $q > 0$ put $\sigma_q = v + \mu(q)(x - x_u)$. Along (19),

$$\dot{\sigma}_q = f(x) - \frac{q}{m}(x - x_u) \quad \text{on } \{\sigma_q = 0\}. \quad (21)$$

Hence $\dot{\sigma}_{q_T} \geq 0$ on $\{\sigma_{q_T} = 0\}$ and $\dot{\sigma}_{q_C} \leq 0$ on $\{\sigma_{q_C} = 0\}$.

Proof. $f(0) = -\eta/m$ and $f(x_u) = 0$, so ℓ_C is the chord, and $f'(x_u) = n\Delta k x_u^{n-1}/m = q_T/m$, so ℓ_T is the tangent. A convex function lies between its tangent and its chord. For (21), $\dot{\sigma}_q = (\mu - \lambda)v + f(x)$ and on $\sigma_q = 0$ we have $v = -\mu(x - x_u)$, so $\dot{\sigma}_q = -(\mu^2 - \lambda\mu)(x - x_u) + f(x) = f(x) - \frac{q}{m}(x - x_u)$. $\square$

Theorem 13 (Bracket on the critical entry speed). *Let $n \geq 1$ and $x_+ \leq x_u \leq x_s$ (the coexistence window). The critical entry speed v_c of the reduced system (19) satisfies*

$$\underline{v} := \mu(q_C) x_u \leq v_c \leq \bar{v} := \mu(q_T) x_u, \quad (22)$$

with strict inequalities for $n > 1$, and $\underline{v} = v_c = \bar{v}$ if and only if $n = 1$.

Proof. Two invariant regions. By Lemma 12 and Nagumo's theorem [26] (f is C^1 for $x > 0$), the sets $A = \{\sigma_{q_T} \geq 0, 0 \leq x \leq x_u\}$ and $B = \{\sigma_{q_C} \leq 0, 0 \leq x \leq x_u\}$ are forward invariant until x reaches x_u . In A , $v \geq \mu(q_T)(x_u - x) > 0$ for $x < x_u$, so x increases and cannot converge to a point $x^* < x_u$; every trajectory in A therefore reaches x_u with $v \geq 0$. In B , $v \leq \mu(q_C)(x_u - x)$, which is 0 at $x = x_u$, so no trajectory in B crosses x_u with $v > 0$.

Locating the separator. Let $W = \{v = w(x)\}$ be the stable-manifold branch on $x < x_u$. It separates trajectories that cross x_u from those that turn back. A point of W in the interior of A would have a neighbourhood in A containing points on the turn-back side, contradicting that all trajectories in A reach x_u . So $w(x) \leq \mu(q_T)(x_u - x)$. Likewise W misses the interior of B , so $w(x) \geq \mu(q_C)(x_u - x) > 0$. Evaluating at $x = 0$ gives (22). For $n > 1$ the inequalities in Lemma 12 are strict, so W touches neither line on $x < x_u$. For $n = 1$, $q_C = q_T$. $\square$

In plain words: a line drawn from the contact point to the saddle, and a line drawn tangent to the force curve at the saddle, trap the true dividing curve between them. Their speeds at contact are explicit.

Proposition 14 (From the reduced system to the full closed loop). *Assume, in addition, that on the strip $0 \leq x \leq x_u$, $0 \leq v \leq \bar{v}$ the command is not clipped:*

$$k_t x_u^n + (c - m\lambda)_+ \bar{v} \leq F_{\max} \quad \text{and} \quad \eta + (m\lambda - c)_+ \bar{v} \leq F_{\max}. \quad (23)$$

Then W is also a trajectory of the full closed loop and v_c is the critical speed of the full loop. A trajectory entering with $0 < v_0 < v_c$ stays in $x < x_u$ while in contact, so it never reaches the false attractor. A trajectory entering with $v_0 > v_c$ crosses x_u . In particular entry below $\underline{v}$ is certified safe, and entry above $\bar{v}$ is certified to cross.

Proof. W stays in the outer region. On W we have $v \geq \mu(q_C)(x_u - x)$, so $s = v + \lambda(x - x_d)$ is at least the value $\mu(q_C)(x_u - x) + \lambda(x - x_d)$, which is linear in x . Its endpoint values are $\mu(q_C)x_u - \lambda x_d$ at $x = 0$ and $\lambda(x_u - x_d)$ at $x = x_u$. Since $\mu > \lambda$ and $x_u \geq x_+$, both are at least $\lambda(x_u - x_d) \geq \Phi$. So $s \geq \Phi$ on W without any extra hypothesis.

W is unclipped. On W , $0 \leq v \leq \bar{v}$ and $\Delta k x^n \leq \eta$, so $F_c \leq k_t x^n + (c - m\lambda)v$ and $F_c \geq -\eta - (m\lambda - c)v$. Condition (23) excludes clipping.

Transfer. The full closed loop equals (19) wherever $s > \Phi$ and the command is unclipped, so W is a trajectory of the full loop and its vector field is smooth near W . By uniqueness, no trajectory crosses W . A trajectory above W has $v > w(x) > 0$, so it cannot stall below x_u and it crosses x_u . A trajectory below W lies in $\{0 < x < x_u, v < w(x)\}$, whose boundary is W , the line $x = 0$, and the segment $x = x_u, v < 0$. It cannot leave through W or rightwards through x_u . $\square$

After crossing x_u , all simulations converge to the saturation-locked equilibrium x_s . We do not claim this part as a theorem. Conversely, convergence to the target-near equilibrium for $v_0 < v_c$ is shown by simulation, not by a Lyapunov argument.

Corollary 15 (Speed floor). *Throughout the coexistence window $v_c > \underline{v} > \lambda x_u \geq \lambda x_+ > \lambda x_d$. Hence, if (23) holds, every approach with $v_0 \leq \lambda x_d$ is certified safe, for every n and every mismatch in the window.*

Proof. $\underline{v} = \mu(q_C)x_u > \lambda x_u$ because $\mu > \lambda$, and $x_u \geq x_+ > x_d$. $\square$

In plain words: entering slower than the sliding bandwidth times the target depth is always safe. In the running example that speed is 0.75 m/s.

Corollary 16 (Approach driven by the same controller). *Let the controller (3) (boundary-layer law) also drive the free-space approach from rest at $x(0) < 0$, with the command unclipped there: $\eta + |m\lambda - c|\eta/(m\lambda) \leq F_{\max}$. Then the entry velocity at $x = 0$ satisfies*

$$v_0 \leq \min\left\{\frac{\eta}{m\lambda}, \lambda x_d\right\} \leq \lambda x_d.$$

Hence, if (23) holds, such an approach is certified safe for every n and every mismatch in the window.

Proof. For $x < 0$ there is no contact force and $D = 0$, so $m\dot{s} = -\eta \text{sat}(s/\Phi)$. Starting from $s = \lambda(x(0) - x_d) < 0$: while $s < -\Phi$, $\dot{s} = \eta/m > 0$; inside the layer $\dot{s} = -\eta s/(m\Phi) \geq 0$ for $s \leq 0$ and s cannot cross 0. So $s \leq 0$ throughout, i.e. $v \leq \lambda(x_d - x)$, which at $x = 0$ is λx_d . Separately, $m\dot{v} \leq -m\lambda v + \eta$, so $v \leq \eta/(m\lambda)$. Corollary 15 then applies. $\square$

In plain words: a robot that approaches under its own controller can never enter fast enough to reach the false attractor. The false attractor is a danger for impacts, for disturbances, and for approach phases run by a different controller.

In the running example $\eta/(m\lambda) = 0.267$ m/s and the full loop enters at exactly that speed. With $\eta = 210$ N the robot reaches the sliding surface before contact and enters at $\lambda x_d = 0.750$ m/s, and in both cases the full loop converges to the target even for mismatches deep in the window.

Corollary 17 (Tolerance to velocity kicks). *Assume (23). Any state with $0 < x < x_u$ and $v < \mu(q_C)(x_u - x)$ stays in $x < x_u$ while in contact. In particular, a forward velocity kick $\delta v \leq \mu(q_C)(x_u - x_+)$ applied at the target-near equilibrium (which lies below x_+) is tolerated. The exact tolerance at an equilibrium x_e is $w(x_e)$, which equals $\mu(\Delta k)(x_u - x_e)$ for $n = 1$.*

Proof. By Theorem 13, W lies on or above the chord line, so a state below the chord line is below W , and Proposition 14 applies to any initial state below W . $\square$

In plain words: this converts the basin into a disturbance rating: how hard a hit, in m/s, the robot can take at its resting position without being thrown into the false attractor.

For the running-example plant at $\Delta k = 2500$ N/m the certified kick is 0.475 m/s and the exact value is 0.4955 m/s. Kicks of 0.4856 and 0.5054 m/s lead to 5.217 and 15.046 mm. For $\eta = 210$ N and $\Delta k = 25000$ N/m the certified value is 0.953 m/s and the exact value is 0.9927 m/s, and kicks of 0.9728 and 1.0125 m/s lead to 5.207 and 15.046 mm.

Corollary 18 (Certified speeds and mismatch limits). *Both bounds decrease as Δk increases. In closed form, for $n \geq 1$ and an entry speed v_0 :*

$$v_0 < \underline{v} \iff \Delta k < \Delta k_{\text{cert}}(v_0) := \eta \frac{(\lambda v_0 + \eta/m)^n}{v_0^{2n}}, \quad (24)$$

$$v_0 > \bar{v} \iff \Delta k > \Delta k_{\text{fail}}(v_0) := \eta \frac{(\lambda v_0 + n\eta/m)^n}{v_0^{2n}}. \quad (25)$$

So for a mismatch known only to lie in $[0, \Delta k_{\max}]$ with $\Delta k_{\max}$ inside the window, entry at speed v_0 is certified safe whenever $\Delta k_{\max} < \min\{\Delta k_{\text{cert}}(v_0), \Delta k^\}$. The certified speed for the whole interval is the lower bound evaluated at $\Delta k_{\max}$. For $n = 1$, $\Delta k_{\text{cert}} = \Delta k_{\text{fail}}$ and equals the exact separator.*

Proof. Write $\underline{v}(x_u) = \mu(\eta/x_u)x_u$. Then $\frac{dv}{dx_u} = \mu - q\mu'(q) = \frac{1}{2}\lambda + \frac{1}{4}S + \lambda^2/(4S) > 0$ with $S = \sqrt{\lambda^2 + 4q/m}$ and $q = \eta/x_u$, using $\mu'(q) = 1/(mS)$ and $q/m = (S^2 - \lambda^2)/4$. Since $x_u = (\eta/\Delta k)^{1/n}$ decreases with Δk , $\underline{v}$ decreases with Δk . The same computation holds for $\bar{v}$ with $q = n\eta/x_u$. To invert, solve $\mu(\eta/x)x = v_0$: substituting $\mu = v_0/x$ into $\mu^2 - \lambda\mu = \eta/(mx)$ gives $x = v_0^2/(\lambda v_0 + \eta/m)$, and $\Delta k = \eta/x^n$ gives (24). Replacing η by $n\eta$ in the equation for μ gives (25). $\square$

In plain words: given an entry speed, there is an explicit mismatch below which the target is guaranteed, and an explicit mismatch above which the false attractor is guaranteed.

Corollary 19 (Width of the bracket). $\bar{v} - \underline{v} = x_u [\mu(q_T) - \mu(q_C)] \leq (n-1)\eta/(m\lambda)$, independently of x_u and of the saturation limit. It vanishes at $n = 1$ and grows with the contact exponent.

Proof. $\mu' \leq 1/(m\lambda)$ and $q_T - q_C = (n-1)\eta/x_u$. $\square$

At the running-example values the width is at most $0.267(n-1)$ m/s.

Remark 20 (Dimensionless form). With $\nu = v/(\lambda x_d)$, $\xi = x_u/x_d$ and $\varepsilon = \eta/(m\lambda^2 x_d)$,

$$\underline{\nu} = \frac{1}{2} \left(\xi + \sqrt{\xi^2 + 4\varepsilon\xi} \right), \quad \bar{\nu} = \frac{1}{2} \left(\xi + \sqrt{\xi^2 + 4n\varepsilon\xi} \right).$$

The result depends on three numbers only: $\xi \geq 1 + \beta$, ε and n . It also shows directly that $\underline{\nu} \geq \xi > 1$.

Remark 21 (Scope and extensions). The argument uses only convexity of $\Delta k x^n$. For any convex residual with $D(0) = 0$ and a simple root x_u of $D = \eta$, the same proof gives the bracket with $q_C = \eta/x_u$ and $q_T = D'(x_u)$. For $0 < n < 1$ the force is concave and the roles of chord and tangent swap. The theorem concerns the sign and boundary-layer laws, whose reaching term is constant outside the layer. It does not describe the power-rate law, which lies outside the entry-speed bracket.

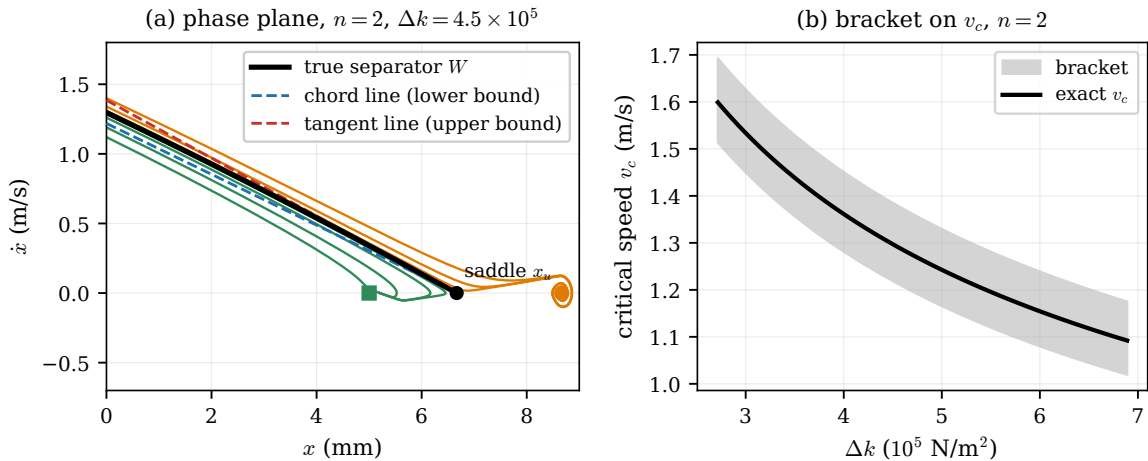


Figure 3: (a) Phase plane for $n = 2$, $\Delta k = 4.5 \times 10^5$. The true separator W (black) lies between the chord line (dashed blue) and the tangent line (dashed red). Green trajectories entering below W converge to the target; orange trajectories entering above it cross the saddle. (b) The bracket (grey) contains the exact v_c (black) across the coexistence window.

Table 4: Bracket versus exact separator and full clipped simulation (running-example plant, $F_{\max} = 700$ N). “True” integrates the stable manifold backwards from the saddle. “Sim” is bisection on the full clipped closed loop from $x(0) = 0$.

n	Δk	x_u (mm)	$\underline{v}$	True v_c	Sim	$\bar{v}$
1	2000	10.000	1.73107	1.73107	1.73107	1.73107
1	3000	6.667	1.21880	1.21880	1.21880	1.21880
1.5	4×10^4	6.300	1.16183	1.20722	1.20722	1.24784
2	3×10^5	8.165	1.44999	1.53376	1.53376	1.62637
2	4.5×10^5	6.667	1.21880	1.29715	1.29715	1.38506
2	6×10^5	5.774	1.07988	1.15436	1.15436	1.23885
3	7×10^7	6.586	1.20634	1.32646	1.32646	1.51102
3	1.2×10^8	5.503	1.03763	1.14974	1.14974	1.32419

Numerical verification. Table 4 compares the bracket with the exact separator and with bisection on the full clipped loop. The bracket holds in every case, the full loop agrees with the stable-manifold value to five digits, and at $n = 1$ all three coincide. A sweep over $n \in \{1.1, 1.2, \dots, 4.0\}$ with seven values of Δk across the window (210 cases) produced no violation of (22). On a finer grid (30 exponents, 25 mismatch values, 750 cases), condition (23) held in every case and the lower bound never fell below $1.35 \lambda x_d$. The closed forms of Corollary 18 were checked at $n = 2$, $v_0 = 1.3$ m/s: $\Delta k_{\text{cert}} = 3.87 \times 10^5$ and $\Delta k_{\text{fail}} = 5.30 \times 10^5$, and the full loop converges to the target at $\Delta k = 3.6 \times 10^5$ and 3.8×10^5 (5.160 and 5.169 mm) and to the false attractor ($x_s = 8.666$ mm) at 5.4×10^5 and 5.8×10^5 . Figure 4(a) shows the cost of nonlinearity at mid-window. The relative error of each bound is below 7% for $n \leq 2$, about 9% and 15% for the lower and upper bounds at $n = 3$, and 11% and 22% at $n = 4$. These are numerical checks of the bracket and of the full loop; they are not proofs of convergence to either resting place.

6 How to use the results: a design procedure

For a fixed-gain sliding-mode contact controller:

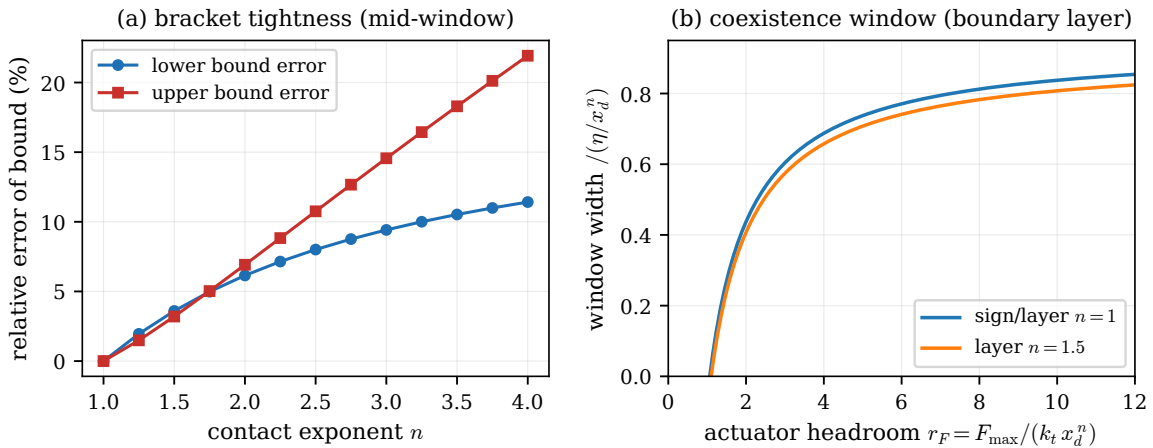


Figure 4: (a) Relative error of the lower and upper bounds against the contact exponent at mid-window; both vanish at $n = 1$. (b) Normalised coexistence-window width for the boundary-layer law as actuator headroom r_F grows.

1. **Classify the reaching law.** Compute E_g and place the law in Table 7. This fixes whether

a fold, a kink, or only a reserve can occur.

2. **Check three margins, not one.** Use the local limit, the saturation onset, and, when $x_s < x_+$, the feasibility threshold. Initial-condition robustness matters whenever coexistence is present.
3. **Find the local limit.** Use η/x_d^n (sign), η/x_+^n (layer) or Δk_{fold} from (28) (power rate).
4. **Find the saturation onset** $\Delta k_{\text{sat}} = \phi_g(e_s)$, equal to $\eta k_t/F_{\text{max}}$ for the sign and layer laws, so the tolerated relative error is η/F_{max} . If it lies below the local limit, the local limit is not a guarantee. Use (15) or (18) to confirm a window, and (16) for its width.
5. **Bound the entry.** Any entry speed $v_0 \leq \lambda x_d$ is safe (Corollary 15), and an approach driven by the same controller enters below that speed (Corollary 16). For impacts and disturbances use the bounds below and the kick tolerance of Corollary 17. For faster entry compute $\Delta k_{\text{cert}}(v_0)$ and $\Delta k_{\text{fail}}(v_0)$ from (24)–(25), after checking (23). With a mismatch interval $[0, \Delta k_{\text{max}}]$, use the lower bound at Δk_{max} .
6. **Check the exponent direction.** Evaluate A . For exponent overestimation use $\hat{\eta}^*$ from (26), and for underestimation use $\hat{\eta} > 1 - \kappa$.
7. **Choose the assumed stiffness for a class of objects.** Use $k_a \leq k_{\text{min}}(1 + \eta/F_{\text{max}})$ (Corollary 4); a stiffer object then costs only a small offset.

Worked example. Running-example plant, boundary-layer law, assumed stiffness error $\Delta k = 2500$ N/m (5.4% of k_t). *Step 1:* the boundary-layer law is in the kink class. *Steps 2–4:* the controller’s own limit is $\eta/x_+ = 3750$ N/m, so the target branch exists; the saturation onset is $\eta k_t/F_{\text{max}} = 1329$ N/m, and since $1329 < 2500 < 3750$ a wrong resting place coexists with the target. The dividing depth is $x_u = \eta/\Delta k = 8$ mm. *Step 5:* any entry slower than $\lambda x_d = 0.75$ m/s is safe, and an approach under the same controller enters at 0.27 m/s. The exact dividing speed is $v_c = 1.42$ m/s, and a forward kick of up to 0.475 m/s at rest is certified (0.4955 m/s exactly). A harder hit can throw the robot into the wrong resting place, where it ends at $x_s = 15.05$ mm with the actuator at its limit. *Step 6:* if the exponent of the contact law is also uncertain, evaluate Section 7.

Table 5: Parallel-jaw gripper simulation, controller tuned for aluminium (46524 N/m). “Final” is the position after settling from rest; “stop” means the loop ends on the 10 mm stroke stop.

	Silicone rubber	ABS
true k_t , error Δk	8969, 37555 N/m (419%)	31752, 14772 N/m (47%)
x_s	78.0 mm	22.0 mm
<i>Fixed-gain set A: $\eta = 20$ N, $\lambda = 150$ s⁻¹, $\Phi = 0.05$ m/s (local limit $\Delta k^* = 3750$ N/m)</i>		
final, no stop	78.05 mm	22.05 mm
final, 10 mm stop	10.00 mm (stop)	10.00 mm (stop)
<i>Optimised gains B: $\eta = 265$ N, $\lambda = 279.1$ s⁻¹, $\Phi = 0.01517$ m/s (tolerated error $\eta/F_{\text{max}} = 37.9\%$, $\Delta k^* = 52430$ N/m)</i>		
Δk_{sat} , stop-lock limit η/s_{max}	3395, 26500 N/m	12020, 26500 N/m
final from rest	5.04 mm	5.02 mm
x_u (dividing depth)	7.06 mm, inside stroke	17.9 mm, outside stroke
kick rating at rest	0.895 m/s certified	unlimited (stop absorbs)

A simulated gripper. The plant of the running example represents a medium-payload parallel-jaw gripper: jaw mass 0.5 kg, damping 41.7 N s/m, actuator limit 700 N (80% of a rated 870 N clamp force), target 5 mm, stroke $s_{\max} = 10$ mm. Three representative object classes, silicone rubber, ABS and aluminium, have contact stiffness 10, 50 and 100 N/mm (the range 10–100 N/mm is also used in a gripper assembly study [35]). In series with the jaw’s own 87000 N/m they give effective stiffness 8969, 31752 and 46524 N/m. The controller assumes the stiffest, 46524 N/m. Table 5 gives the outcome for two gain sets. These are simulations of the same model, from rest, in the full clipped loop; they are not experiments.

With the set-A gains the controller fails on both softer objects: on silicone the loop overshoots to 102 mm (1941%) and ends at 78.05 mm, a 73.05 mm error, against the predicted $x_s = 78.0$ mm. With the optimised set-B gains it reaches the target from rest on both, although both errors exceed the tolerated 37.9% and wrong resting places exist at 78 mm and 22 mm. The kick rating shows what remains. For silicone a forward kick of 0.857 m/s at rest is absorbed (5.04 mm), while 0.947 m/s drives the robot to the wrong resting place (78.05 mm, or the stop at 10.00 mm); the certified rating is 0.895 m/s and the exact value 0.902 m/s. For ABS the dividing depth lies outside the stroke, so the stop absorbs any kick (tested at 1 and 3 m/s) and the robot returns to 5.02 mm (Proposition 5). An approach under the same controller enters at no more than $\min\{\eta/(m\lambda), \lambda x_d\} = 1.40$ m/s, below the dividing speed $v_c = 3.15$ m/s for silicone.

Corollary 4 turns the failure of set A into a design. With the set-B gains the margin is $\eta/F_{\max} = 37.9\%$, so the controller may assume up to $8969 \times 1.379 = 12369$ N/m for all three objects. Table 6 gives the result for $k_a = 12000$ N/m.

Table 6: One assumed stiffness, $k_a = 12000$ N/m, for all three objects (set B), full clipped loop from rest.

	silicone	ABS	aluminium
true k_t (N/m)	8969	31752	46524
error Δk (N/m)	+3031 (+34%)	−19752 (−62%)	−34524 (−74%)
final position (mm)	5.003	4.980	4.965
settling time (2%)	15.4 ms	15.4 ms	15.4 ms

The offsets agree with the corollary (0.003, 0.020 and 0.035 mm). A single fixed value of the assumed stiffness is therefore safe for the whole class, because every object is at or below the margin. The same reasoning explains why a controller tuned for the softest nominal stiffness can serve a wide stiffness range [35].

7 When the contact exponent is also wrong: direction, not size

Now let $n_a \neq n_t$, so the residual $D(x) = k_a x^{n_a} - k_t x^{n_t}$ can change sign along the motion and have interior extrema. A target-point comparison $D(x_d) \geq 0$ then no longer decides the failure direction. Theorem 23 does: only the anti-restoring set A can hold an off-target equilibrium.

Definition 22 (Anti-restoring set). $A = \{x > 0 : D(x)(x - x_d) > 0\}$. At a point of A the residual pushes the state away from the target.

Theorem 23 (Off-target equilibria live in the anti-restoring set). *Let $g(r) > 0$ for $r > 0$ and $x_s > x_d$. Every equilibrium $x_e \neq x_d$, whether unsaturated or saturation-locked at $+F_{\max}$, lies in A . In particular, if $A = \emptyset$, no off-target equilibrium exists: no stall and no false attractor.*

The proof is in Appendix B.

In plain words: a model error can only trap the robot where it pushes away from the target. Where it pushes toward the target, no stall and no false attractor is possible, however large the error is.

This theorem holds for every odd reaching law with $g > 0$ off the origin and for any residual, and it is the organising idea of this section.

Lemma 24 (Sliding existence and switched-branch stability). *Filippov sliding exists at x iff $|D(x)| < \eta$. A switched constant-sign equilibrium $D(x_e) = \eta \operatorname{sgn}(s)$ is locally asymptotically stable if $D'(x_e) < 0$ and a saddle if $D'(x_e) > 0$.*

The proof is in Appendix B. Loss of the sliding condition ($|D| \geq \eta$) is therefore not a failure criterion by itself. For $n_a = 1$, $n_t = 1.5$, $\kappa = 1$ and $\eta = 10$ N the pointwise sliding condition fails over part of the path, yet the residual is restoring there and the state reaches 5.000 mm. Normalise with $F_{\text{ref}} = k_t x_d^{n_t}$: $y = x/x_d$, $\kappa = k_a x_d^{n_a}/F_{\text{ref}}$, $\hat{\eta} = \eta/F_{\text{ref}}$, so $\delta(y) = D/F_{\text{ref}} = \kappa y^{n_a} - y^{n_t}$ (Fig. 5a).

Theorem 25 (Exact one-sided stall threshold). *Approach from below ($0 < y < 1$), in the thin-layer limit $\beta = \Phi/(\lambda x_d) \rightarrow 0$, so that a stall point lies outside the layer. Let $Q(y) = [y^{n_t} - \kappa y^{n_a}]_+$. A switched below-target stall equilibrium, with $y_e^{n_t} \leq F_{\text{max}}/F_{\text{ref}}$ so that it is unsaturated, exists iff $\hat{\eta} \leq \hat{\eta}^* := \max_{0 < y < 1} Q(y)$. When two roots of $Q = \hat{\eta}$ exist, the lower one is a stable stall and the upper one is a saddle, and they merge in a saddle-node at $\hat{\eta} = \hat{\eta}^*$. For $n_a > n_t$ let $y_{\text{stat}} = (n_t/(\kappa n_a))^{1/(n_a - n_t)}$. Then*

$$\hat{\eta}^* = \begin{cases} \frac{n_a - n_t}{n_a} \left(\frac{n_t}{\kappa n_a} \right)^{n_t/(n_a - n_t)}, & y_{\text{stat}} < 1, \\ [1 - \kappa]_+, & y_{\text{stat}} \geq 1. \end{cases} \quad (26)$$

The proof is in Appendix B.

In plain words: when the controller assumes too steep a contact law, there is an exact amount of switching gain needed to avoid stalling short of the target, and below it the robot stalls.

For $\kappa = 1$, $n_a = 2$, $n_t = 1$, $k_t = 8969$ N/m, $x_d = 5$ mm: $\hat{\eta}^* = 1/4$, $F_{\text{ref}} = 44.845$ N and $\eta^* = 11.21$ N. Simulations of the full clipped loop give final positions 1.8703, 2.1568, 5.0000, 5.0000 mm for $\eta = 10.5, 11.0, 11.4, 11.8$ N, so the transition brackets η^* . For the reverse pair $n_a = 1.5$, $n_t = 1$, $\kappa = 1$, $\eta^* = 6.644$ N, with stalls at 1.449 and 1.740 mm for $\eta = 6.0$ and 6.4 N, and target convergence (5.000 mm) at 6.9 and 7.4 N.

Corollary 26 (Exponent-underestimation design rule). *If $n_a \leq n_t$ and $\kappa \leq 1$, the below-target stall-free condition reduces to $\hat{\eta} > 1 - \kappa$, and the positive saturation false attractor cannot occur.*

The proof is in Appendix B. For $n_a = 1$, $n_t = 1.5$, $\kappa = 0.9$ the rule predicts $\eta^* = 23.3$ N, and the full loop gives 4.953 mm at 22 N and 4.9994 mm at 24.5 N. Underestimating the exponent is the benign direction, because it removes the above-target false-attractor mechanism.

Theorem 27 (Exponent-induced false attractor). *Assume $x_s \geq x_+$. A positive saturation-locked equilibrium at x_s is branch-consistent iff $D(x_s) \geq \eta$. If also $D(x_+) < \eta$, continuity gives a root of $D = +\eta$ in (x_+, x_s) , and for a transverse upcrossing ($D' > 0$) it is a saddle and basin separator.*

The proof is in Appendix B. The target can be underestimated at the target point while a remote false attractor still exists. For $n_t = 1$, $n_a = 2$, $\kappa = 0.8$, $F_{\text{max}} = 250$ N and $\eta = 20$ N we have $D(x_d) = -8.97$ N, yet $x_s = 27.87$ mm satisfies $D(x_s) \geq \eta$. The saddle is at about 7.99 mm, trajectories from $x_0 = 0$ and 7 mm converge near the target, and those from 9, 15 and 20 mm converge to 27.87 mm.

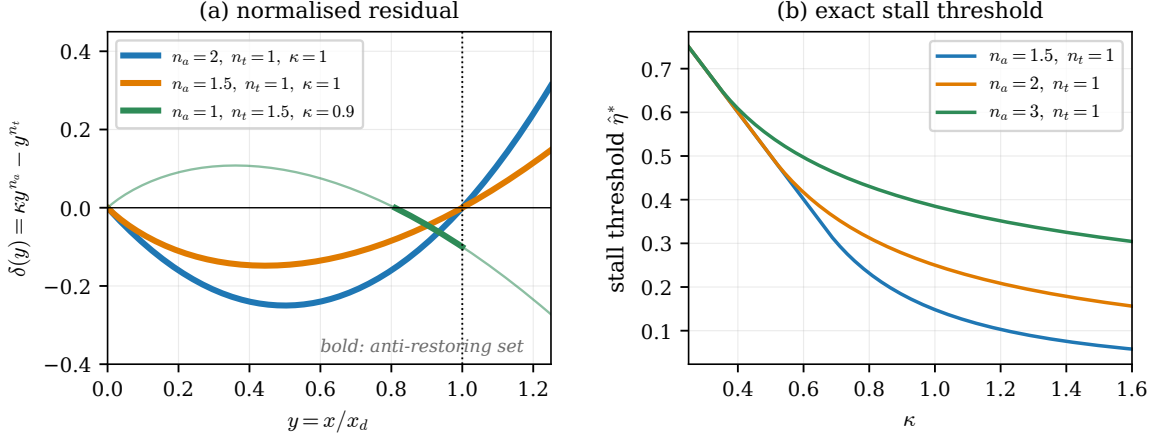


Figure 5: (a) Normalised residual $\delta(y)$ for three exponent pairs. Bold segments, where $\delta(y)(y - 1) > 0$, form the anti-restoring set. (b) The exact stall threshold $\hat{\eta}^*$ against κ for $n_t = 1$.

8 General theory: reaching-law elasticity and branch classes

Sections 3–5 were stated for three specific laws. This section shows why they fall into a few types: the reaching law matters only through its elasticity, and the elasticity decides where a branch folds.

8.1 Fold criterion and unique fold

Corollary 28 (Fold criterion). *Any nondegenerate smooth fold of the positive branch satisfies*

$$E_g(\lambda e^*) = \frac{n e^*}{x_d + e^*}. \quad (27)$$

The left side belongs to the controller and the right side to the plant.

Table 7: Branch classes of reaching laws (positive stiffness mismatch).

Class	Representative laws	Branch geometry	Saturation outcome
Monotone decreasing	sign; ϵ -dominant constant-plus-proportional	no stable positive off-surface branch; finite reserve if $g(0^+) > 0$	coexistence from Δk_{sat} to the reserve
Kink maximum	boundary layer	piecewise branch with a kink maximum	coexistence, or feasibility-limited capture
Smooth maximum	power rate; exponential; tested mixed laws	stable branch, saddle, smooth fold	coexistence iff e_s lies beyond the fold
Monotone increasing	linear; k -dominant constant-plus-proportional	unique stable off-surface branch	capture when the branch reaches x_s

Theorem 29 (A sufficient condition for a unique fold). *If $E_g(r)$ is nonincreasing, ϕ_g has at most one interior stationary point. If in addition $E_g(0^+) > 0$ and $\lim_{e \rightarrow \infty} E_g(\lambda e) < n$, the stationary point exists and is the unique maximum of ϕ_g .*

Proof. Let $H(e) = E_g(\lambda e) - ne/(x_d + e)$. The first term is nonincreasing and the second is strictly increasing, so H is strictly decreasing. Since $\text{sign } \phi'_g = \text{sign } H$, there is at most one change from increasing to decreasing, and the limit conditions force it. $\square$

The condition is sufficient, not necessary. Numerical searches over mixed and two-stage laws found no multi-fold example.

8.2 Four branch classes

Theorems 1 and 29 sort reaching laws into four classes (Table 7, Fig. 6).

The classical constant-plus-proportional law [4], $g(r) = \epsilon_0 + kr$, shows that one standard family can fall into two classes: for $n = 1$ the sign of ϕ'_g is the sign of $k\lambda x_d - \epsilon_0$. For $n > 1$, $\phi'_g(e) \propto k\lambda x_d - n\epsilon_0 + k\lambda(1 - n)e$, with interior maximum $e^* = (k\lambda x_d - n\epsilon_0)/(k\lambda(n - 1))$ whenever $k\lambda x_d > n\epsilon_0$.

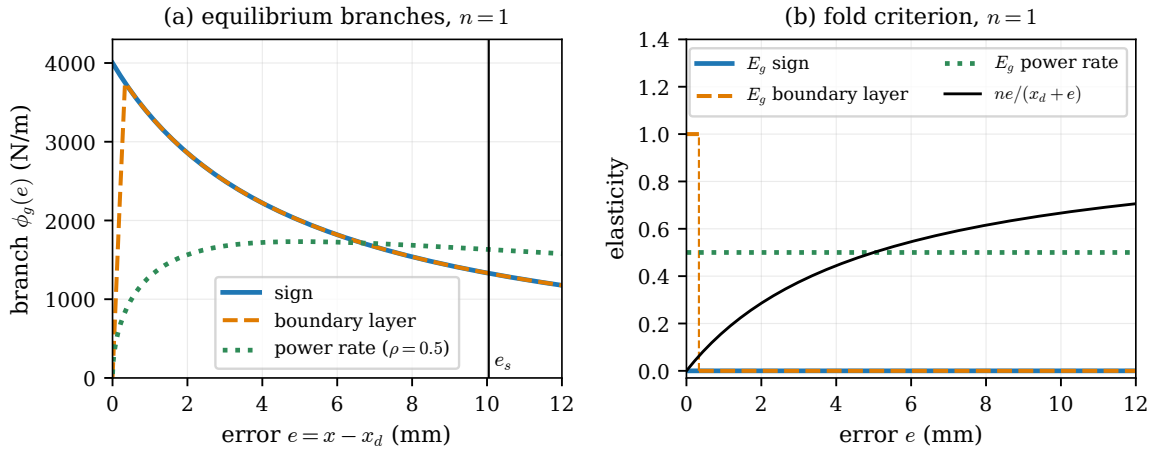


Figure 6: (a) Equilibrium branches $\phi_g(e)$ of the three classical laws for $n = 1$ (running-example plant; power-rate $K = 20$, $\rho = 0.5$). (b) Elasticity of each law against the plant quantity $ne/(x_d + e)$. A smooth fold is where the curves cross.

8.3 The power-rate law

Let $g(r) = Kr^\rho$, $0 < \rho < 1$. Then $E_g \equiv \rho$ is constant and $\phi(e) = K\lambda^\rho e^\rho/(x_d + e)^n$.

Theorem 30 (Power-rate fold under power-law contact). *Let $n > \rho$. Then ϕ has a unique positive maximum at*

$$e^* = \frac{\rho x_d}{n - \rho}, \quad \Delta k_{\text{fold}} = \frac{K\lambda^\rho \rho^\rho (n - \rho)^{n-\rho}}{n^n x_d^{n-\rho}}. \quad (28)$$

The lower branch $e < e^$ is stable and the upper branch $e > e^*$ is a saddle. Under mismatch the controller cannot track the target exactly: $\phi(e) \rightarrow 0$ only as $e \rightarrow 0$.*

Proof. $\phi'/\phi = \rho/e - n/(x_d + e)$, which changes sign from positive to negative at e^* . Substitution gives Δk_{fold} , and stability follows from Theorem 1. $\square$

Proposition 31 (No exact target equilibrium for continuous laws). *Let g be continuous with $g(0) = 0$ (the power-rate law, the linear law, the in-layer part of the boundary-layer law). If $D(x_d) \neq 0$, then $(x_d, 0)$ is not an equilibrium of the unsaturated loop.*

Proof. At $x = x_d$, $s = 0$ and $g(0) = 0$, so (4) gives $m\dot{s} = D(x_d) \neq 0$. $\square$

Unlike the sign law, whose set-valued term on $[-\eta, \eta]$ can absorb a nonzero residual exactly at $s = 0$, a single-valued continuous term has no slack at the origin. Finite-step simulations that appear to converge to x_d therefore do not demonstrate exact continuous-time tracking: the physically relevant objects are the off-surface equilibria. For example, the apparent error of a coarse-step run shrinks as the step is refined, which is compatible with convergence to a small but nonzero excursion.

With $z = e/e^*$ and $\mu = \Delta k/\Delta k_{\text{fold}}$, the whole positive branch collapses to a parameter-free curve,

$$\mu(z) = z^\rho \left[\frac{n}{n - \rho + \rho z} \right]^n, \quad \left. \frac{d\mu}{dz} \right|_{z=1} = 0, \quad (29)$$

independent of K , λ , x_d . For $n = 1$ it reduces to $\mu = z^\rho / [(1 - \rho) + \rho z]$. The stable branch is $z < 1$, the saddle branch $z > 1$, and the fold sits at $(\mu, z) = (1, 1)$ for every parameter set. For $n = 1$, $\rho = 1/2$, $K = 20$ the fold is at $\Delta k_{\text{fold}} = 1732.05$ N/m. Above the fold, trajectories pass a saddle-node bottleneck whose passage time scales as $(\Delta k - \Delta k_{\text{fold}})^{-1/2}$ (Appendix A).

9 Verification

Every threshold, separator and rating in this paper was checked against simulation of the full clipped closed loop (relative tolerance 10^{-10}). The local thresholds agree with from-rest simulations in all 27 sign-law, 63 boundary-layer (linear) and 63 boundary-layer (Hertzian) configurations. The speed bracket held in 210 comparisons with the exact separator and the unclipped condition in 750 cases, with the full loop agreeing with the stable-manifold value to five digits. The power-rate fold was confirmed on 12 random plants to a relative error below 10^{-6} . The certified speeds, the kick rating, the stall thresholds and the gripper examples were each checked in simulation, and the one-assumed-stiffness rule was checked on three object classes. Sampling, contact damping and noise leave the two-attractor picture unchanged; an actuator lag moved the critical speed upward in the one case tested. The tables and the individual runs are in Appendix A. All of these checks are numerical.

10 Conclusion and limitations

The robustness margin of a fixed-gain sliding-mode controller is a local statement. With a finite actuator, a wrong resting place appears once the relative stiffness error reaches η/F_{max} , far below the controller's own target-point margin. Between the two limits the entry velocity decides which resting place the robot reaches: we gave an exact dividing speed for linear contact, closed-form certified bounds for nonlinear contact, and a disturbance rating for the robot at rest. An approach driven by the same controller always enters safely, so the wrong resting place threatens impacts and disturbances, not slow approach. The same elasticity theory covers the sign, boundary-layer and power-rate laws, and under exponent mismatch the direction of the error, not its size, decides the outcome.

Limitations. The model is scalar and restricted to positive contact, with an assumed power-law stiffness. Mass, damping and the sliding parameter are assumed exactly known: only the contact force law is mismatched, and a combined mass, damping and stiffness error is not treated. The sign law is analysed exactly but simulated only through a thin boundary layer. Gains are fixed, so adaptive laws, integral action, persistent disturbance forces and higher-order sliding modes are outside scope. The unique-fold theorem is a sufficient condition. The bracket covers

the sign and boundary-layer laws, requires the unclipped condition (23), and does not describe the power-rate law. Convergence of trajectories below the separator, and convergence to x_s above it, are shown by simulation, not proved. Critical speeds are of order 1 m/s in the running example, so the false attractor is relevant to impacts, disturbances and separately controlled approach phases, not to an approach driven by the same controller (Corollary 16). The bracket is derived for an ideal actuator; an actuator lag raised the critical speed in one example but we make no general claim. The bounds loosen with the contact exponent, to about 22% for the upper bound at $n = 4$, and tighter brackets are possible. The exponent-mismatch stall threshold is one-sided, derived for approach from below with a thin layer, and above-target stable switched roots with $D = +\eta$ and $D' < 0$ are not exhaustively classified. Only one combination of damping, noise, sampling and actuator lag was tested (Appendix A). The sign controller is discontinuous and its simulations are sensitive to time discretisation. The mechanical stop is modelled as rigid and inelastic. All gripper results are simulations of the same model, and the numerical examples verify the theorem structure but do not replace experiments. Multi-contact, viscoelastic and multi-degree-of-freedom extensions are open.

Prior versions

Earlier versions of the sign-law, boundary-layer and power-rate parts of this work were posted by the author on Zenodo in September 2026 (DOIs 10.5281/zenodo.22813154, 10.5281/zenodo.22883933 and 10.5281/zenodo.22884533). This paper contains those results, restated, corrected and extended, and supersedes them.

Data and code availability

No external data are used. The numerical checks reported here can be regenerated from the model equations and parameters given in the paper; the scripts are available from the author on request.

Declaration of generative AI use

Claude (Anthropic) and ChatGPT (OpenAI) assisted with derivation checks, numerical experiments, code, literature search and language editing. The author reviewed the results and is responsible for the content.

Funding and competing interests

This research received no specific grant. The author declares no competing interests.

A Numerical validation

All integrations use LSODA or DOP853 with relative tolerance 10^{-10} and absolute tolerance 10^{-12} , and maximum step 2×10^{-4} s (10^{-4} s for the bracket study). All numbers below can be regenerated from the stated model and parameters. All simulations use the boundary-layer or smooth reaching laws; statements about the discontinuous sign law are analytical.

Table 8: Running-example thresholds for $n = 1$ (N/m).

Law	Δk_{sat}	Upper limit	Upper-limit mechanism
Sign	1329.26	4000	Filippov reserve η/x_d
Boundary layer	1329.26	3750	layer-edge collision η/x_+
Power rate ($K = 20, \rho = 0.5$)	1631.74	1732.05	smooth fold

Coexistence. For the boundary layer at $\Delta k = 1500$ N/m the trajectory from $x_0 = 0$ converges to 5.128205 mm and from $x_0 = 14$ mm to $x_s = 15.045998$ mm. At $\Delta k = 2500$ N/m, $x_0 = 7.9$ mm converges to 5.217391 mm and $x_0 = 8.1$ mm to x_s , so the separator sits at the saddle $x_u = 8.000$ mm. For the power-rate law at the midpoint $\Delta k = 1681.90$ N/m the two unsaturated roots are 8.071645 and 13.138961 mm; the lower is stable, the upper is a saddle, and the trajectory from rest converges to 8.071645 mm while one from 14 mm converges to x_s . For $n = 1.5$ the same construction gives $\Delta k_{\text{fold}} = 18856.2$ and $\Delta k_{\text{sat}} = 16962.5$. For the exponential law $g = 30(1 - e^{-r/0.5})$ N the fold is 2339.2 and $\Delta k_{\text{sat}} = 1896.0$ N/m, and at $\Delta k = 2117.6$ N/m the two initial conditions reach 7.525 mm and x_s . For the linear law $g = 0.4r$, $E_g = 1 > 0.6677 = n(1 - x_d/x_s)$ and both initial conditions are captured at x_s for $\Delta k = 1.03 \Delta k_{\text{sat}}$, after a long settling time because the reaching action is weak.

Beyond the idealised model. At $\Delta k = 2500$ N/m (boundary layer, running-example plant) we added 1 kHz sampling with a zero-order hold, a first-order actuator lag of 2 ms, a contact-force factor $(1 + 0.3\dot{x})$ in the spirit of the Hunt–Crossley model [25] and 20 μm position noise. Sampling, damping and noise, alone or together, preserve both attractors: from $x_0 = 0$ and 14 mm the loop settles at 5.217 and 15.046 mm, and the critical speed of the continuous baseline is 1.427 m/s against the exact 1.4246. With the actuator lag the false attractor still exists (the loop started at 14 mm settles at 15.046 mm), but no entry speed up to 3 m/s reaches it. In this example the lag moved the critical speed from 1.43 m/s to above 3 m/s, so the ideal-actuator bracket was conservative; we do not claim that direction in general, and we did not isolate the effect of sampling and damping on the critical speed.

Threshold sweeps. We verified the local thresholds against from-rest simulations of the full clipped loop, at 0.97 and 1.03 times the predicted value (below: the loop settles inside the boundary layer; above: it is captured at x_s). For the sign law, taken as the thin-layer limit $\Phi = 10^{-3}$ m/s, we swept $\eta \in \{10, 20, 40\}$ N, $x_d \in \{3, 5, 8\}$ mm and $n \in \{1, 1.5, 2\}$: all 27 configurations agree with η/x_d^n . For the boundary-layer law we swept $\lambda \in \{100, 150, 200\}$ s $^{-1}$, $\eta \in \{10, 20, 40\}$ N and seven layer widths $\Phi \in [0.03, 0.3]$ m/s: all 63 rows agree with η/x_+^n for linear contact, and all 63 for Hertzian contact $n = 3/2$. These sweeps test the local threshold only; the coexistence below it is the subject of Section 4. Exploratory runs at $n < 1$ are outside the theorem ($n \geq 1$) and are not counted as verification.

A severe-mismatch plant. With $k_t = 8969$ N/m and an assumed stiffness of 46524 N/m, $\Delta k = 37555$ N/m is 9.4 times the local sign-law limit 4000 N/m. Here $x_s = 78.05$ mm and $\Delta k_{\text{sat}} = 256.3$ N/m, and the loop from rest ends at 78.047 mm. This plant lies far beyond both thresholds and is distinct from the running-example plant used elsewhere, where $k_t = 46524$ N/m.

Power-rate fold on random plants. On 12 random plants ($m \in [0.2, 10]$ kg, $c \in [5, 200]$ N s/m, $k_t \in [10^3, 8 \times 10^4]$ N/m, $\lambda \in [40, 250]$ s $^{-1}$, $x_d \in [2, 20]$ mm, $K \in [5, 50]$, $\rho \in [0.2, 0.75]$, actuator limit 1.6 to 3 times $k_t(x_d + e^*)$), the loop from rest at $0.95 \Delta k_{\text{fold}}$ settles on the predicted stable root with relative error below 1.1×10^{-6} , and at $1.06 \Delta k_{\text{fold}}$ it is captured

at x_s in every case (60 s horizon). On the running-example plant the from-rest transition lies 0.04% above the fold for a 10 s window; shorter windows give larger offsets because of critical slowing down.

Negative mismatch. For $n = 1$ and $\Delta k = -1500$ N/m the power-rate branch gives the lower equilibrium 4.305009 mm, matching direct integration.

Critical slowing down. Just above the power-rate fold, times to reach $x > 12$ mm are 3.39, 2.21, 1.41, 0.88, 0.55, 0.29, 0.155 s for offsets 5, 10, 20, 40, 80, 200, 500 N/m. The $-1/2$ exponent is the asymptotic saddle-node law and should be read close to the fold only, which is why finite-horizon simulations detect the transition slightly above the analytical fold.

B Proofs for Section 7

Proof of Theorem 23. Unsaturated. At an equilibrium $v = 0$ and $s = \lambda(x_e - x_d)$, so (4) gives $D(x_e) = g(\lambda|x_e - x_d|) \operatorname{sgn}(x_e - x_d)$. Multiplying by $x_e - x_d$ gives $D(x_e)(x_e - x_d) = g(\lambda|e|)|e| > 0$.

Saturation-locked. A locked equilibrium needs $k_t x_e^{n_t} = F_{\max}$, so $x_e = x_s > x_d$ (a $-F_{\max}$ lock is impossible since $k_t x^{n_t} > 0$). There $F_c = F_{\max} + D(x_s) - g(\lambda e_s) \geq F_{\max}$, so $D(x_s) \geq g(\lambda e_s) > 0$ and $D(x_s)(x_s - x_d) > 0$. $\square$

Proof of Lemma 24. At $s = 0$ the normal rates are $m\dot{s}_{\pm} = D \mp \eta$, and attraction needs $\dot{s}_+ < 0 < \dot{s}_-$. Linearising $m(\ddot{\xi} + \lambda\dot{\xi}) = D'(x_e)\xi$ gives $mr^2 + m\lambda r - D'(x_e) = 0$. $\square$

Proof of Theorem 25. A stall below the target outside the layer needs $D(x_e) = -\eta$, i.e. $Q(y_e) = \hat{\eta}$ with $y_e < 1$. By the intermediate value theorem a root exists iff $\hat{\eta} \leq \max Q$. At the lower root $Q' > 0$, so $D' < 0$ and the root is stable by Lemma 24; at the upper root $Q' < 0$, so $D' > 0$ and it is a saddle. Where $Q > 0$, $Q' = n_t y^{n_t-1} - \kappa n_a y^{n_a-1} = 0$ gives y_{stat} , and $\kappa y_{\text{stat}}^{n_a-n_t} = n_t/n_a$ gives the closed form. If $y_{\text{stat}} \geq 1$, Q is increasing on $(0, 1)$ with supremum $1 - \kappa$ at the endpoint. $\square$

Proof of Corollary 26. On the anti-restoring interval below the target the largest magnitude is $1 - \kappa$. For $y > 1$, $y^{n_t} \geq y^{n_a} \geq \kappa y^{n_a}$, so $\delta(y) \leq 0$ there and $A \cap \{y > 1\} = \emptyset$. $\square$

Proof of Theorem 27. At x_s , $F_c = F_{\max} + D(x_s) - \eta$, which is $\geq F_{\max}$ iff $D(x_s) \geq \eta$. The root exists by the intermediate value theorem, and its type follows from Lemma 24. $\square$

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