

Aeroelastic Stability Transitions and Divergence Mechanisms of a Z-Fold Wing

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Abstract

The lowest aeroelastic critical speed of a morphing wing can switch between competing flutter modes and static divergence as its configuration changes. This study determines these transitions for a three-segment Z-fold wing and explains the resulting divergence trends. A geometric formulation carries parameter dependence consistently from component kinematics and aerodynamic coupling through wake reduction, including the variation of the reduction bases. Local reduced models incorporate this derivative information and are constructed along continuation paths to resolve the boundaries between competing instabilities. The computed flutter–flutter and flutter–divergence intersection curves partition the configuration plane into regions with distinct governing instabilities. Independent grid-coarsening comparisons demonstrate improved coarse-grid curve accuracy and reduced construction-and-continuation cost when analytic derivative information is used. Spectral comparisons separate interpolation and wake-reduction effects, while neighboring-speed checks recover the corresponding stability transitions in the full-wake model. A two-coordinate torsional model relates the divergence boundary to geometric rotation transfer and intersegment aerodynamic coupling, with a response scale consistent with a classical finite-wing estimate. These results provide mechanism-resolved stability information for selecting structural-axis locations and admissible morphing paths.

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Nomenclature

A_1, B_1	speed-independent wake-state and input operators
C_1, D_1, D_0	aerodynamic force operators multiplying wake, wash, and wash rate
c, L	wing-segment chord and span, m
d, σ	element relative logarithm and reference-anchored section perturbation
$E, \mathcal{L}$	descriptor mass and evolution matrices
$H = [R, r]$	section pose in the special Euclidean group $SE(3)$
M_s, C_s, K_s	assembled structural mass, damping, and stiffness matrices
N_q, N_v	displacement-to-normal-wash and velocity-to-normal-wash maps
$p = (U, \phi, \epsilon_A)$	speed, folding angle, and axis-offset parameters
P, Q	controllability and observability Gramians
q	structural elastic and joint perturbation coordinates
T, W	retained wake trial and test bases, $W^T T = I$
U, V^*	flow speed, m/s, and $U/(\omega_0 c/2)$
x_w, z_w	full and reduced free-wake states
α_g, Γ	geometric incidence perturbation and circulation
ϵ_A	quarter-chord position relative to the structural axis, divided by c
λ, μ	dimensional eigenvalue and λ/ω_0
ρ_f	fluid density, kg/m ³
$\rho(s, \eta)$	section-frame offset of an attached aerodynamic material point, m
ς_i	i th Hankel singular value
ϕ, ω_0	folding angle, rad, and reference angular frequency, rad/s
$\mathcal{C}_{FF}, \mathcal{C}_{FD}$	flutter–flutter and flutter–divergence intersection curves

I. Introduction

Morphing aircraft adapt their aerodynamic and structural characteristics through changes in joint angle, span, camber, and other configuration variables. Such changes redirect motion and load transfer: a folding joint carries a downstream lifting surface, while elastic deformation changes both its velocity and the direction of aerodynamic work. A predictive aeroelastic model must preserve these relationships across the operating configurations. Reviews of morphing-aircraft aeroelasticity identify their interaction with flexibility and actuation as a central modeling and design challenge [1, 2].

Flutter and static divergence impose different stability constraints. Flutter involves an oscillatory mode and motivates feedback designed around its dynamics [3]; divergence reflects loss of static restoring capability. At a fixed configuration, the lower of their critical speeds governs the initial loss of stability. Morphing adds operational

coordinates to this comparison. Even after structural design parameters have been fixed, a configuration change can alter which instability occurs first. Design choices, morphing coordinates, and flow speed therefore define a joint parameter space in which stability margins and controlling mechanisms must be considered together. Its transition boundaries identify where an oscillation-suppression strategy must address a different mode or where a static constraint becomes decisive.

Folding-wing studies show that these changes are physically significant. Component-based analyses and experiments demonstrate the influence of folding angle and hinge stiffness on flutter [4, 5]. Investigations of nonlinear responses and Z-shaped wings further reveal the interaction between configuration and postcritical dynamics [6, 7, 8]. Critical-mode changes have been reported for a modified Z-shaped configuration [9], while divergence and reversal have been studied in a hinged-airfoil morphing concept [10]. Together, these results motivate locating the boundaries between competing instability mechanisms. Near such a boundary, a design change that raises one critical speed must also preserve the margin to its competitor.

Parameterized models make configuration-dependent aeroelastic analysis computationally accessible. Zhan, Zhou, and Huang interpolated structural models in a manifold tangent space and combined them with doublet-lattice aerodynamics, predicting modal and flutter variations with folding angle, including changes in the critical mode [11]. Yue and Zhao developed coherent interpolated aeroservoelastic models for folding-wing control through modal matching and aerodynamic rational-function representations; their analysis also distinguished the neutral boundaries of competing flutter modes in the speed–angle plane [12]. Maraniello and Palacios investigated parameterized reductions of the unsteady vortex-lattice method, including balanced-model and transfer-function interpolation [13]. These studies establish the value of reduced models for configuration-dependent prediction and control and demonstrate that competing modal boundaries are relevant to folding-wing analysis.

Resolving the intersections of these boundaries introduces a further task. In a three-parameter space, each neutral condition defines a surface, and the simultaneous vanishing of two critical growth rates defines an intersection curve. Numerical continuation can follow such a curve from a known point, including through parameter turning points [14]. Its local geometry depends on derivatives of both critical conditions. For a parameterized aeroelastic ROM, these derivatives depend jointly on component motion, aerodynamic interaction, and the changing reduction bases. This motivates a construction that carries parameter derivatives consistently through the coupled model and its reduction,

supplies local surrogates along the continuation path, and preserves a direct relation to the unreduced operators for verification and physical interpretation.

This study develops and evaluates that construction for the three-segment wing investigated in Ref. [8]. Section-pose perturbations and component recursion provide a common geometry for structural inertia, normal wash, and aerodynamic virtual work, building on Ref. [15]. The resulting parameter derivatives are propagated through the continuous-time wake model and balanced reduction, including the motion of the retained trial and test bases. Local Hermite models retain these derivatives, while new parameter cells are constructed as continuation advances. This formulation resolves flutter–flutter and flutter–divergence intersection curves and identifies the mechanisms governing the critical-speed envelope. Independent grid-coarsening experiments compare the derivative-informed construction with value-only interpolation in both curve accuracy and total construction-and-continuation cost. Spectral and neighboring-speed comparisons connect the predicted boundaries to the full-wake system. Finally, a two-coordinate torsional model explains the divergence trends through geometric rotation transfer and induced aerodynamic response.

II. Parameterized Aeroelastic Model

A. Configuration and component kinematics

The wing consists of three equal rectangular segments connected by two chordwise revolute joints. Their reference rotations are ϕ and $-\phi$, so the inner and outer segments remain parallel. The structural axis lies at $x = 0$ in each component frame. With x directed from leading to trailing edge, the aerodynamic quarter-chord is at $x = c\epsilon_A$ and the leading edge is at $x = c(\epsilon_A - 1/4)$. Thus negative ϵ_A places the quarter-chord ahead of the structural axis. The family considered here is parameterized by

$$p = (U, \phi, \epsilon_A)^\top, \quad U > 0 \tag{1}$$

The perturbation coordinates q describe three elastic modes per segment and two joint rotations about each prescribed reference configuration. The reference state is unloaded and has zero circulation; the reference free stream is tangent to all three surfaces.

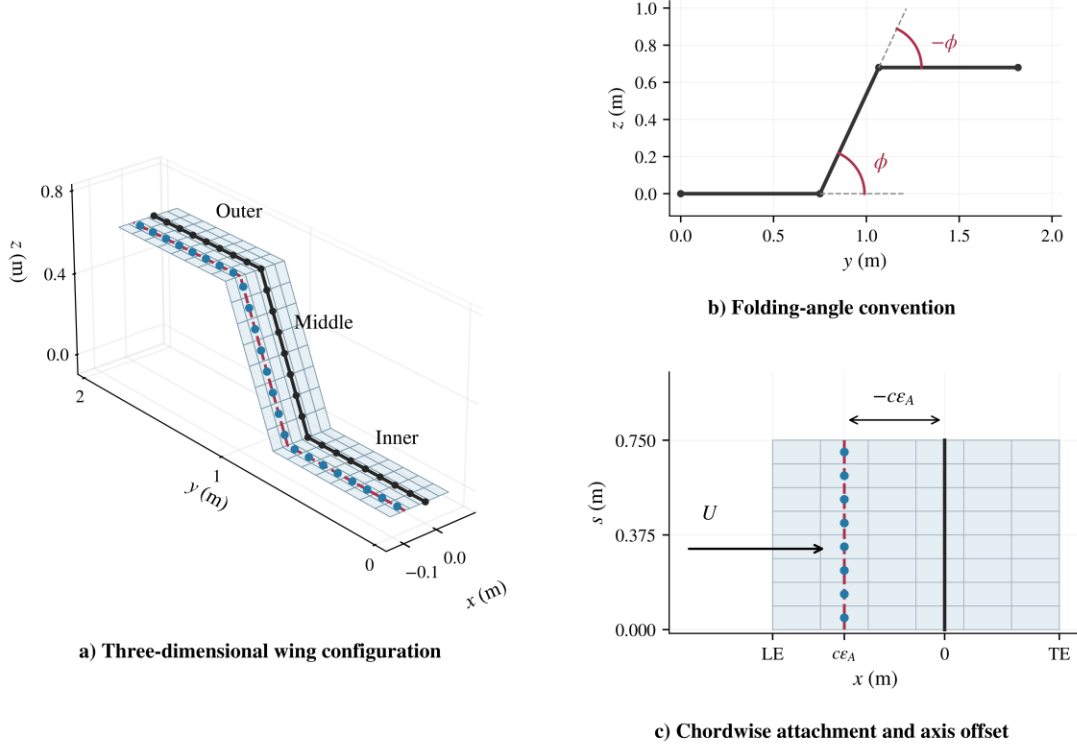


Fig. 1 Z-fold geometry at $\phi = 65$ deg and $\epsilon_A = -0.35$: solid black lines denote structural axes, dashed red lines denote quarter-chords, and blue points denote attached material points.

Figure 1a shows the surface lattice and beam stations at $\phi = 65$ deg and $\epsilon_A = -0.35$. Panel (b) shows the opposite joint rotations in the folding plane; panel (c) locates the structural axis, quarter-chord, and attached material points using the same chordwise offset.

A section pose is written $H_i(s, t) = B_i(t)\bar{H}_i(s, q_i)$, with B_i the component root pose. The identity

$$H_i^{-1}H_{i,s} = \bar{H}_i^{-1}\bar{H}_{i,s} \quad (2)$$

separates component strain from upstream rigid motion. In contrast, the body velocity contains both root motion and local elastic motion. Twists and wrenches are ordered as $V = (v, \omega)$ and $w = (f, m)$, with pairing $w^\top V$. For a joint pose $g(\theta)$ between a parent port and a child root, the common tangent and virtual-displacement maps are

$$V_c = \text{Ad}_{g^{-1}}V_p + J_g\dot{\theta}, \quad \zeta_c = \text{Ad}_{g^{-1}}\zeta_p + J_g\delta\theta, \quad \widehat{J_g\theta} = g^{-1}\dot{g} \quad (3)$$

Virtual work then gives the reverse force maps $\text{Ad}_{g^{-1}}^\top w_c$ and $J_g^\top w_c$. This product structure makes the role of topology explicit: velocities enter through component roots, section dynamics are evaluated locally, and their forces return through transposed kinematic maps. It is consistent with geometrically exact beam and flexible-multibody formulations [16, 17, 18].

B. Section-pose perturbations, inertia, and restoring forces

Let H_a^0, H_b^0 be the reference end-section poses of a beam element of length L_e , and let δ_a, δ_b be their right perturbations. A constant-strain Lie-group element uses

$$\begin{aligned} d_0 &= \log[(H_a^0)^{-1}H_b^0]^\vee, & d &= \log[\exp(-\hat{\delta}_a)\exp(\hat{d}_0)\exp(\hat{\delta}_b)]^\vee, \\ H_h(s, q) &= H_a^0 \exp(\hat{\delta}_a) \exp(\tau \hat{d}) = H_h^0(s) \exp(\hat{\sigma}(s, q)), & \tau &= \frac{s - s_a}{L_e} \end{aligned} \quad (4)$$

The relative logarithm d determines the constant body strain $\kappa = d/L_e$, while σ describes section motion relative to the reference interpolation. Equation (4) makes the two representations compatible with the same endpoint perturbations and constant-strain element. The former is convenient for strain energy; the latter supplies local velocity and surface-geometry expansions. First-order state expansions determine the stress-free linearization used here. The higher-order construction and nonlinear static condensation are described in Ref. [15].

At section level, let $\xi = (H^{-1}\dot{H})^\vee$, $\mu_s = \mathcal{M}_s \xi$, and $n_s = \mathcal{K}_s(\kappa - \kappa_0)$. The Euler–Poincaré balance provides a compact local force equation [19]:

$$-\dot{\mu}_s + \text{ad}_\xi^* \mu_s + \partial_s n_s - \text{ad}_\kappa^* n_s + f_s = 0, \quad \langle \text{ad}_a^* b, c \rangle = \langle b, [a, c] \rangle \quad (5)$$

The convention for ad^* in Eq. (5) is the algebraic transpose under the stated pairing. Linearizing and assembling through Eq. (3) gives

$$M_s(\phi)\ddot{q} + C_s(\phi)\dot{q} + K_s(\phi)q = Q_a \quad (6)$$

For example, if $J_i(s, \phi)$ maps $\dot{q}$ to the section twist, then $M_s = \sum_i \int J_i^\top \mathcal{M}_i J_i ds$. A large upstream rotation does not itself create strain in a downstream rigidly transported segment, although it changes this assembled inertia and the aerodynamic work maps.

The joint springs are specified by target frequency ratios. With joint-coordinate index j , their stiffnesses are

$$k_j(\phi) = M_{s,jj}(\phi)(r_j \omega_0)^2, \quad r_1 = 1.05, \quad r_2 = 1.15 \quad (7)$$

Here $M_{s,jj}$ is the inertia associated with unit joint rate while the other generalized rates vanish. Equation (7) defines the configuration-scheduled spring family studied in this paper. Its derivatives and damping prescription are included in every numerical comparison. Appendix A specifies the structural coefficients and first-order kinematics.

C. Attached aerodynamic geometry and normal wash

The reference lifting surface is first specified continuously and then attached to a material beam section. A locally invertible map $(u, v) \mapsto (s, \eta)$ identifies the section and the position within that section. The deformed material point is

$$X(u, v, q) = r_s(s, q) + R_s(s, q)\rho(s, \eta) \quad (8)$$

The attachment idealizes the role of ribs or equivalent section constraints in carrying the aerodynamic surface. The surface has no independent deformation dynamics in this model. The beam-element interpolation in Eq. (4) supplies r_s, R_s ; aerodynamic discretization subsequently samples Eq. (8). Thus beam strain and surface geometry use compatible, explicitly related representations.

The surface tangents and normal are

$$\begin{aligned} X_s &= r_{s,s} + R_{s,s}\rho + R_s\rho_s, & X_\eta &= R_s\rho_\eta, \\ n &= \frac{\nu}{\|\nu\|}, & \nu &= \kappa_o(X_s \times X_\eta), & \kappa_o &\in \{-1, 1\}, \\ \delta n &= \frac{I - nn^\top}{\|\nu\|} \kappa_o(\delta X_s \times X_\eta + X_s \times \delta X_\eta) \end{aligned} \quad (9)$$

The fixed orientation κ_o reconciles the section coordinates with the aerodynamic panel convention. For a regular material patch its sign is chosen once. In particular, $R_{s,s}\rho$ remains present even when $\rho_s = 0$; a chordwise arm can affect the perturbed normal through the spatial variation of section rotation.

With the boundary-condition input defined as $r = n^\top(\dot{X} - Ue_x)$, its linearization about the stationary, zero-incidence reference is

$$\delta r = UN_q q + N_v \dot{q}, \quad (N_q)_j = -e_x^\top n_{q,j}, \quad (N_v)_j = n^\top X_{q,j} \quad (10)$$

The pressure-to-generalized-force map follows from the same geometry:

$$\delta W_a = \int_s \Delta p n^\top X_{q,j} \delta q dA = \delta q^\top O \Delta p, \quad Q_a = O \Delta p \quad (11)$$

Panel integration and load-site choices are included in O . This common source for wash and loading is essential when differentiating with respect to the structural-axis offset.

D. Continuous-time wake realization and coupled descriptor

The unsteady vortex-lattice method discretizes potential-flow circulation on the three lifting surfaces and their wakes [20]. Three relations determine the linear aerodynamic response. First, the induced normal velocity balances the surface-relative normal wash. Second, trailing-edge continuity connects the newly shed circulation to the bound circulation. Third, previously shed circulation is convected downstream. With bound circulation Γ_b , attached wake circulation Γ_0 , and free-wake state x_w , these relations are

$$\begin{aligned} A_b \Gamma_b + A_0 \Gamma_0 + A_w x_w &= \delta r, & \Gamma_0 &= K_{TE} \Gamma_b, \\ \partial_t \Gamma_w + U \partial_\ell \Gamma_w &= 0 \end{aligned} \quad (12)$$

The first two equations recover the bound circulation as a linear function of δr and x_w . Discretizing the transport equation on fixed physical-length rows then gives the wake-state equation. The linearized Bernoulli relation supplies the pressure jump from the potential jump $\Delta\Phi$,

$$\Delta p = \rho_f(\partial_t + U \partial_x)\Delta\Phi \quad (13)$$

where the jump orientation is chosen consistently with the panel normal. The discrete potential jump is determined by bound circulation. Substituting its recovered value and projecting pressure by Eq. (11) yields the continuous-time realization [21, 22]

$$\dot{x}_w = UA_1x_w + UB_1\delta r, \quad Q_a = UC_1x_w + UD_1\delta r + D_0\delta\dot{r} \quad (14)$$

The factors of U in the wake equation express convection and injection at the first free-wake row. In the force equation, the potential-jump time derivative generates the wash-rate term and part of the speed-weighted response; its streamwise derivative supplies the remaining convective contribution. Appendix B gives the explicit elimination and pressure operators.

At each fixed p , define

$$\begin{aligned} M_e &= M_s - D_0N_v, \\ C_e &= C_s - U(D_1N_v + D_0N_q), \\ K_e &= K_s - U^2D_1N_q \end{aligned} \quad (15)$$

For $x = (q, \dot{q}, x_w)$, the coupled linear system is

$$E_F\dot{x} = \mathcal{L}_F x, \quad E_F = \text{diag}(I, M_e, I), \quad \mathcal{L}_F = \begin{bmatrix} 0 & I & 0 \\ -K_e & -C_e & UC_1 \\ U^2B_1N_q & UB_1N_v & UA_1 \end{bmatrix} \quad (16)$$

The descriptor form retains aerodynamic acceleration terms explicitly. Each eigenproblem describes perturbations about one fixed configuration and flow speed.

III. Parameter Derivatives and Reduced Models

A. Exploiting the parameter dependence

The geometry, wash maps, wake operators, and force maps form a differentiable chain. For an algebraic relation $K(p)Z(p) = F(p)$, its derivative is evaluated by

$$KZ_j = F_j - K_jZ \quad (17)$$

Differentiating geometric primitives and reusing such factorizations avoids differencing an assembled eigenproblem. Analytical linearization of vortex-lattice loads is also developed in Ref. [23]; the present application additionally differentiates the reduction bases and the parameter-cell construction.

Two specific geometric properties simplify the Z-fold family. All folding axes are parallel to the free stream. Changing ϵ_A therefore translates every reference lifting surface and its fixed straight wake by the same vector $c \delta \epsilon_A e_x$. Relative aerodynamic geometry, panel normals, and the circulation-recovery operators remain unchanged under this translation. The velocity arms and virtual-work arms vary affinely. With $\delta = \epsilon_A - \epsilon_{A,0}$,

$$N_v = N_{v0} + \delta N_{v1}, \quad O = O_0 + \delta O_1, \quad \partial_{\epsilon_A} A_1 = \partial_{\epsilon_A} B_1 = 0 \quad (18)$$

For the flat reference segments, the offset-induced change in the perturbed span tangent has the form $c \delta \epsilon_A \partial_s(\delta R_s e_x)$. The chord tangent is parallel to e_x . Its cross product with that span-tangent change is perpendicular to e_x , so the wash contraction in Eq. (10) satisfies $\partial_{\epsilon_A} N_q = 0$. Fold-angle changes still alter intersegment induction and global structural assembly.

The prescribed physical spacing of wake nodes makes U an explicit transport coefficient. Its powers in Eqs. (14)–(16) can therefore be retained analytically throughout the derivative chain.

B. Balancing the wake input–output system

Balancing is applied to the dynamic wake channel. Its input includes both structural displacement and velocity, with a fixed metric so that changing the parameter does not redefine the reduction objective. Let

$$q = S_q \hat{q}, \quad \dot{q} = \omega_0 S_q \hat{v}, \quad u = \begin{bmatrix} \hat{q} \\ \hat{v} \end{bmatrix}, \quad y = S_q^\top Q_{a,w} \quad (19)$$

The constant S_q is the inverse transpose Cholesky factor of the structural mass at the reference configuration $\phi = 30$ deg. The two input channels are weighted by the identity in these coordinates. Their matrices are

$$A = U A_1, \quad B = [U^2 \bar{B}_q, U \bar{B}_v], \quad C = U \bar{C}, \quad \bar{B}_q = B_1 N_q S_q, \quad \bar{B}_v = B_1 N_v \omega_0 S_q, \quad \bar{C} = S_q^\top C_1 \quad (20)$$

Direct wash and acceleration terms are retained separately in the coupled model.

For a Hurwitz A , the Gramians satisfy [24, 25]

$$AP + PA^\top + BB^\top = 0, \quad A^\top Q + QA + C^\top C = 0 \quad (21)$$

The Gramians quantify reachability from past inputs and observability through future outputs. They are positive semidefinite and become positive definite on a controllable and observable realization. The specified input metric gives $BB^\top = U^4 \bar{B}_q \bar{B}_q^\top + U^2 \bar{B}_v \bar{B}_v^\top$.

The explicit speed dependence gives

$$P = U^3 P_q + U P_v(\delta), \quad Q = U Q_1(\delta), \quad \hat{P} = U^2 P_q + P_v(\delta), \quad \hat{Q} = Q_1(\delta) \quad (22)$$

Both P_v and Q_1 are quadratic polynomials in δ . Seven speed-independent Lyapunov coefficient equations per folding angle suffice: one for P_q , three for P_v , and three for Q_1 . Removing the common positive factor U from P, Q leaves the balanced bases unchanged. The remaining $U^2 P_q$ term shows why those bases still depend on speed.

Let $\hat{P} = R_c R_c^\top$, $\hat{Q} = R_o R_o^\top$, and $R_o^\top R_c = U_b \Sigma V_b^\top$. Retaining the first r positive singular values yields

$$T = R_c V_{b,r} \Sigma_r^{-1/2}, \quad W = R_o U_{b,r} \Sigma_r^{-1/2}, \quad W^\top T = I \quad (23)$$

Positive, simple retained singular values and a gap at the truncation boundary define the regular parameter region used for interpolation. The approximation $x_w = T z_w$ produces a reduced pencil by replacing A_1 with $W^\top A_1 T$, the wake inputs with $W^\top B_1 N_q, W^\top B_1 N_v$, and C_1 with $C_1 T$ in Eq. (16). The structural coordinates and direct aerodynamic terms are retained.

C. Retained-mode derivatives and interpolation

The equations defining the retained balanced modes directly provide the basis derivatives required by the reduced model. For a retained column t_i and singular value ζ_i ,

$$\hat{P} \hat{Q} t_i = \zeta_i^2 t_i, \quad t_i^\top \hat{Q} t_i = \zeta_i, \quad w_i = \hat{Q} t_i / \zeta_i \quad (24)$$

Differentiation produces a bordered linear system for $t_{i,j}, \zeta_{i,j}$; the same factorization supplies the required mixed parameter derivatives. Appendix C gives the equations and the sign convention. With simple retained singular values and real normalization, each paired column has an independent sign freedom. Both T and W , together with all derivatives of that column, must receive the same sign.

Each axis-aligned parameter cell uses tensor-product cubic Hermite interpolation. At its corners, the analytic-derivative construction, denoted H1, supplies

$$\mathcal{J}f = \{f, f_U, f_\phi, f_{\epsilon_A}, f_{U\phi}, f_{U\epsilon_A}, f_{\phi\epsilon_A}, f_{U\phi\epsilon_A}\} \quad (25)$$

The H1 construction uses derivatives of first degree in each parameter direction, including the mixed derivatives required for tricubic interpolation. Common signed corner data give neighboring cells matching face values and first derivatives.

The value-only comparison, denoted H0, starts from nodal values of the same model and zero-order balancing operation. Four-point tensor Lagrange interpolation provides the derivative data used in the Hermite polynomial. Neighboring support nodes are included as needed, and their shared derivative estimates produce a continuously differentiable interpolant. The comparison therefore isolates the effect of analytic derivative information within two smooth parameterized models.

After interpolation, the trial and test bases are paired by

$$G = \tilde{W}^\top \tilde{T}, \quad T = \tilde{T}, \quad W = \tilde{W} G^{-\top} \quad (26)$$

A nonsingular G guarantees $W^\top T = I$ and full retained rank. The parameter dependence of this pairing correction is differentiated as well. Simple ordered singular values, normalization, and consistent column signs provide a smooth local frame; sufficiently accurate interpolation preserves its rank. This permits direct interpolation of the basis components over the visited parameter region. Appendix C describes the sign transport and adjoining-cell consistency.

When a prediction or correction enters a new cell, only missing corner or value-support data are constructed. Neighboring cells share those data. Subsequent cell evaluations use interpolated aerodynamic coefficients and bases together with the explicit speed and offset dependence and structural assembly. Expensive geometry assembly, Lyapunov solves, and balancing are thus concentrated near the tracked transition curves.

IV. Stability Partitions and Numerical Verification

A. Critical conditions and numerical configuration

The reduced eigenproblem is $\mathcal{L}(p)v = \lambda E(p)v$. A simple eigenvalue has parameter derivative

$$\lambda_{,j} = \frac{w^*(\mathcal{L}_{,j} - \lambda E_{,j})v}{w^* E v} \quad (27)$$

Let $\mu = \lambda/\omega_0$. A flutter sheet has $\text{Re}\mu_H = 0$ with $\text{Im}\mu_H > 0$, and a divergence sheet has a real root $\mu_D = 0$. The two intersection conditions are

$$F_{\text{FF}}(p) = \begin{bmatrix} \text{Re}\mu_{H,1} \\ \text{Re}\mu_{H,2} \end{bmatrix} = 0, \quad F_{\text{FD}}(p) = \begin{bmatrix} \text{Re}\mu_H \\ \mu_D \end{bmatrix} = 0 \quad (28)$$

The flutter–flutter curve $\mathcal{C}_{\text{FF}} = \{p: F_{\text{FF}}(p) = 0\}$ involves two distinct oscillatory pairs. The flutter–divergence curve $\mathcal{C}_{\text{FD}} = \{p: F_{\text{FD}}(p) = 0\}$ couples a nonzero-frequency flutter condition to a real zero eigenvalue. Both curves identify changes in the mechanism governing linear stability and are labeled FF and FD, respectively, in the figures and tables.

Pseudo-arclength continuation adds a transverse phase condition to the two neutral conditions [14, 26]. It advances by a tangent predictor and corrects back to their common zero set. This allows the curve to turn in an individual physical parameter while arclength remains a usable local coordinate. In scaled coordinates

$$z = \text{diag}(0.3, 10, 0.1)^{-1} \left[\begin{pmatrix} V^* \\ \phi_{\text{deg}} \\ \epsilon_A \end{pmatrix} - \begin{pmatrix} 1.5 \\ 10 \\ -0.35 \end{pmatrix} \right], \quad t_k^\top (z - z_k - \Delta s t_k) = 0 \quad (29)$$

the tangent satisfies $F_z t_k = 0$ and $\| t_k \| = 1$. Eigenvalue predictions and structural eigenvector overlap maintain the identity of the continued modes.

Table 1 specifies the physical model and the main numerical choices. The full-wake realization is denoted F, its directly balanced reduction D, and the interpolated parameterized model P. The labels high- and low-frequency denote the continued flutter branches. Their identity is preserved through the computation even if the instantaneous frequencies approach one another. The lowest critical-speed surface is the lower envelope of all identified first-loss sheets,

$$U_{cr}(\phi, \epsilon_A) = \min\{U_{H,1}, U_{H,2}, U_D, \dots\} \quad (30)$$

with the active branches checked against the remaining spectrum and sampled lower-speed states.

Table 1 Physical model and continuation configuration.

Quantity	Specification
Segment span and chord	$L = 0.75$ m, $c = 0.18$ m, identical segments
Fluid density and reference frequency	$\rho_f = 1.225$ kg/m ³ , $f_0 = 4.5$ Hz; $\omega_0 = 2\pi f_0$
Section mass and frequency targets	$m_\ell = 40\pi\rho_f(c/2)^2$; $f_t/f_0 = 1.35$, $f_i/f_0 = 1.50$; $r_g/c = 0.30$
Structural discretization	Eight beam elements and three elastic modes per segment; two joint coordinates
Joint targets and damping	$r_j = (1.05, 1.15)$; damping ratio 0.005
Aerodynamic discretization	Six chordwise by eight spanwise panels per segment; all intersegment interactions
Free wake	24 rows per segment; fixed streamwise spacing $0.30c$; 576 states
Retained wake order	Four; coupled dimensions: 26 (P, D), 598 (F)
Parameter domain	$-0.60 \leq \epsilon_A \leq -0.10$, $0 \leq \phi \leq 130$ deg, $0.4 \leq V^* \leq 8$
Scaled arclength steps	Initial 0.1, minimum 10^{-4} , maximum 0.5
Correction tolerances	Neutral conditions 10^{-8} ; phase condition 10^{-10}
Iteration controls	Ten Newton iterations; at most eight backtracking trials
Curve comparison	21 ordered reference sections per intersection curve

The derivative construction is verified at its mathematical interfaces: independent geometric finite differences test section and surface derivatives, differentiated algebraic identities test circulation recovery and Lyapunov solves, and paired-mode identities test basis derivatives. Centered differences of the assembled pencil provide a final composition check at representative parameters. Resolution studies for the underlying structural and aerodynamic model are reported in Ref. [8].

B. Critical-speed surface and mechanism regions

Figure 2 presents the mechanism-resolved critical-speed surface and its parameter-plane partition. Panel (a) reconstructs each active sheet from the available critical-speed samples and adjoining transition curves. Panel (b) identifies the governing instability across the full parameter domain; panels (c) and (d) enlarge its narrow flutter–flutter and flutter–divergence transition regions. The computed curves are displayed separately in Fig. 3, with axis ranges suited to each curve.

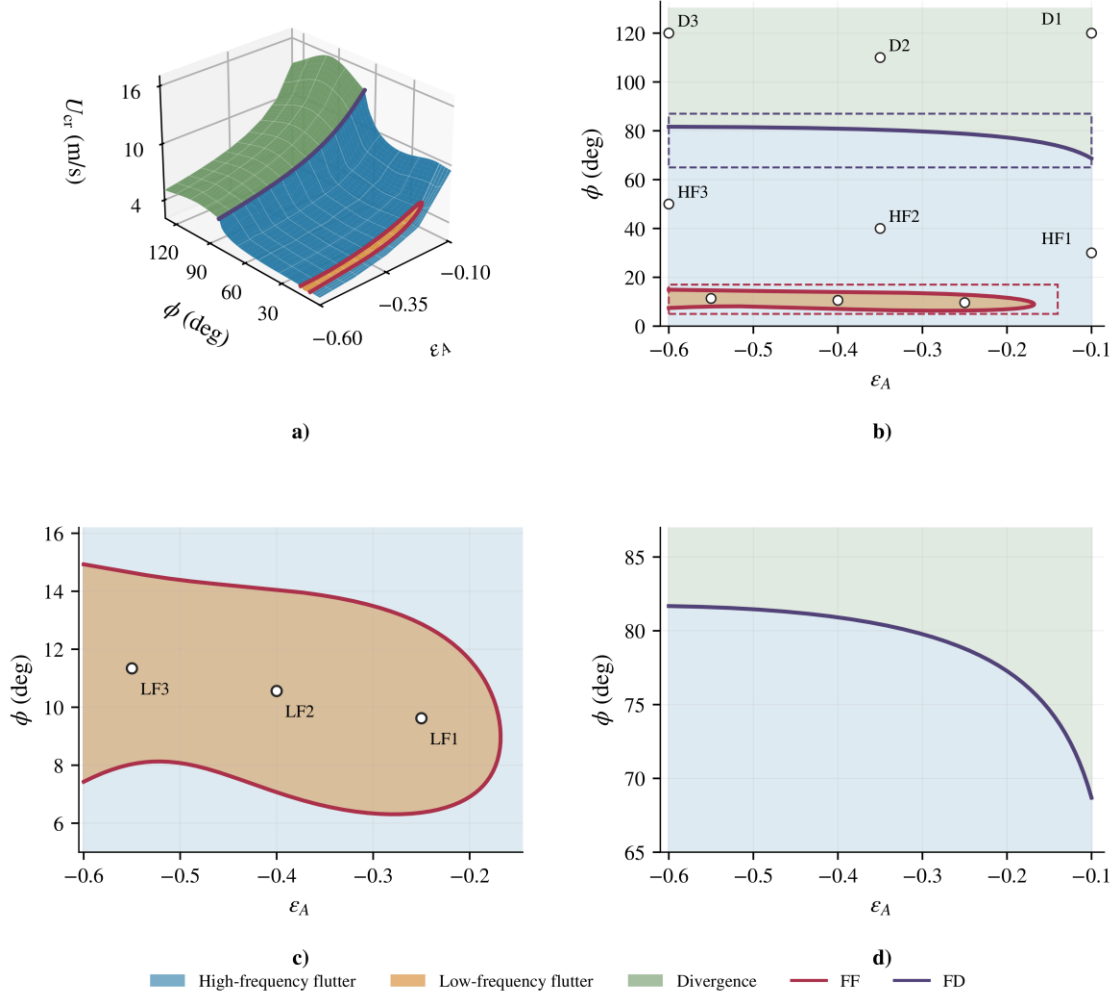


Fig. 2 Lowest critical-speed surface, governing-instability regions, and enlarged transition boundaries; labeled points identify the full-wake transition checks.

The flutter–flutter curve in Figs. 2b and 2c encloses a narrow low-frequency-controlled region within the high-frequency flutter domain. Its two arms meet near $\epsilon_A = -0.168$ and $\phi \approx 9$ deg. At $\epsilon_A = -0.35$, increasing ϕ crosses this boundary near 6.593 and 13.846 deg, producing a high–low–high sequence of controlling flutter modes. At $\epsilon_A =$

-0.60 , the corresponding angles are approximately 7.429 and 14.929 deg. Thus a smooth critical-speed envelope can contain transitions in the mode that must be stabilized.

At larger folding angles, Figs. 2b and 2d show a transition from flutter to static divergence. The boundary occurs at approximately 81.676 deg for $\epsilon_A = -0.60$, 80.443 deg for $\epsilon_A = -0.35$, and 68.682 deg for $\epsilon_A = -0.10$. A folding path at $\epsilon_A = -0.35$ therefore crosses two flutter-mode boundaries before encountering a static constraint. The $\epsilon_A = -0.10$ path remains in the high-frequency flutter region until it reaches the divergence boundary. These paths illustrate how the relative offset between the structural axis and quarter-chord affects the sequence of stability constraints during morphing.

The parameter-plane partition identifies which instability governs each configuration; the three-dimensional surface supplies its critical speed. Together, they support the choice of structural-axis location and morphing path. Surface visualization uses separate, boundary-constrained interpolation within each region. The curve comparisons below use the continuation coordinates and tangents.

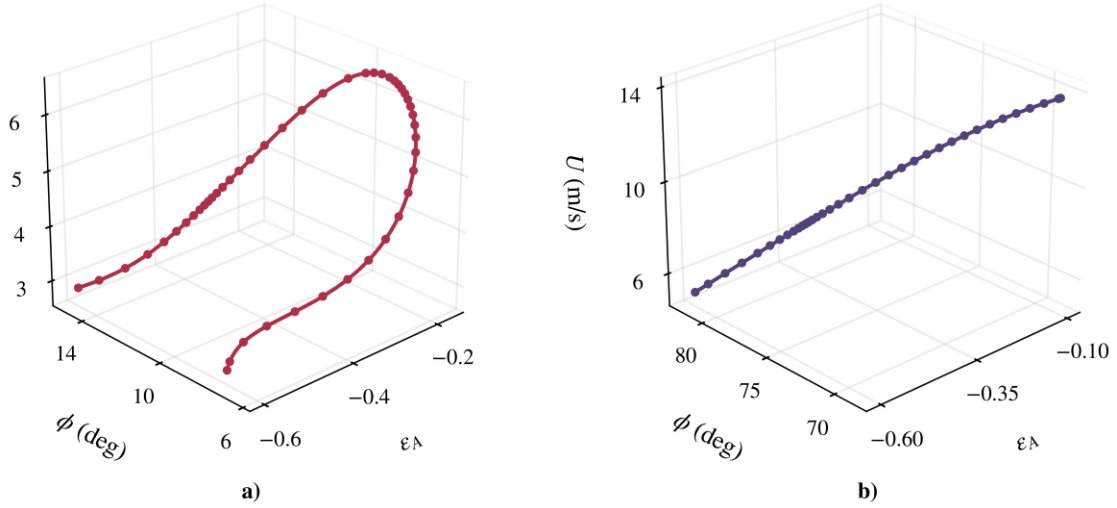


Fig. 3 Transition curves: a) flutter–flutter; b) flutter–divergence. Each panel uses its own parameter ranges, and markers show saved continuation points.

C. Comparing independently parameterized curves

Different continuation runs generally take different steps and have different arclength parameterizations. Pointwise subtraction at a common row index is therefore not meaningful. Matching by ϵ_A alone also becomes ambiguous near the turn of the FF curve. A global nearest-point distance can incorrectly switch between its two nearby arms. We instead compare corresponding intersections with ordered transverse sections of the fine H1 reference curve.

Let $z_r(s)$ be that reference in the scaled coordinates of Eq. (29). At 21 equally spaced reference-arclength locations, define

$$\Pi_j = \{z: t_r(s_j)^\top [z - z_r(s_j)] = 0\} \quad (31)$$

The candidate intersection $z_c^{(j)}$ is selected in the same order along the same oriented branch. Endpoints are compared on their actual parameter-domain faces. Multiple possible section intersections are resolved by this ordering and orientation. The recorded distance and its physical components are

$$d_j = \|z_c^{(j)} - z_r^{(j)}\|, \quad \Delta U_j, \quad \Delta \phi_j, \quad \Delta \epsilon_{A,j} \quad (32)$$

The maximum and root-mean-square of d_j summarize discrepancy relative to the fine parameterized reference. Ordered transverse sections preserve branch correspondence around the turning curve, while the accompanying physical components express the discrepancy in speed, angle, and offset units.

D. Effect of parameter-grid coarsening

Five parameter grids, G1–G5, are compared, with spacings listed in Table 2. G1/H1 supplies the fine reference for each boundary. H0 and H1 start from the same prescribed seed and independently construct their parameter data; reference-curve coordinates are used afterward for comparison. Angle intervals are truncated at the domain endpoint where required. These grids probe successively wider cells with different node locations.

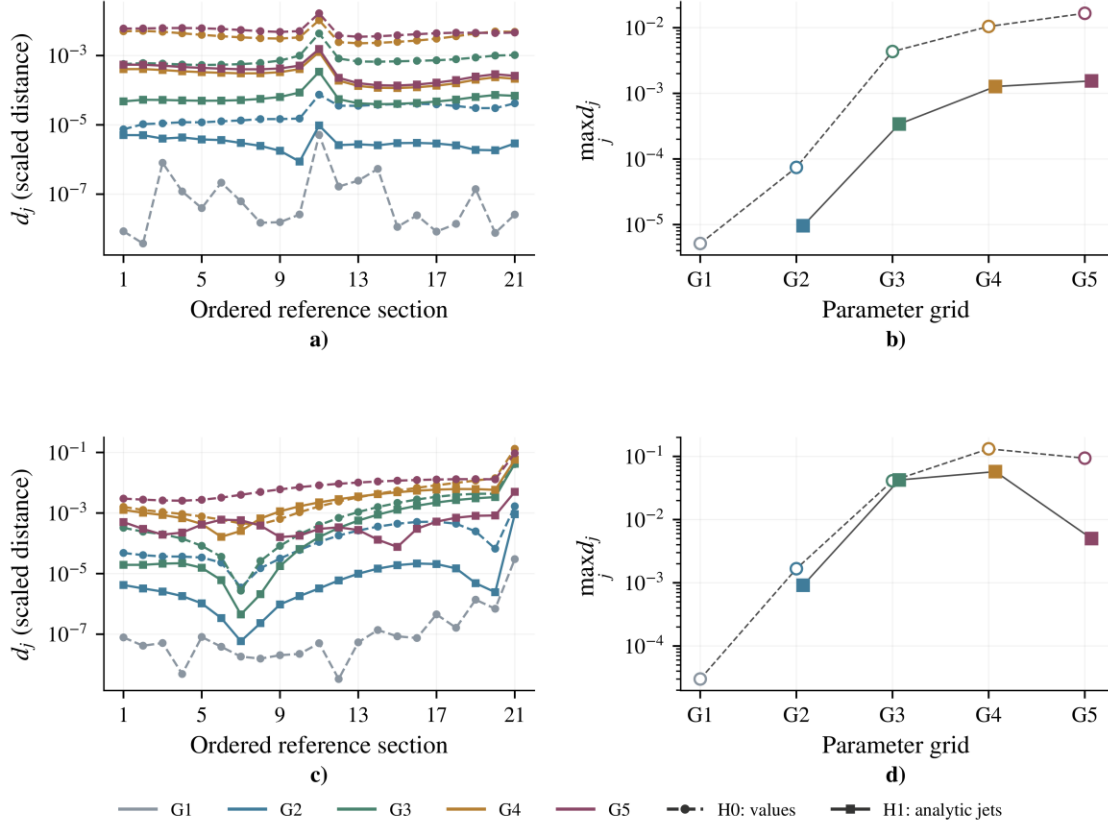


Fig. 4 Curve discrepancies relative to G1/H1: a, b) flutter–flutter; c, d) flutter–divergence. Left panels show sectionwise distances, and right panels show their maxima.

The fine-grid discrepancies in Fig. 4 are small for both methods, so the densely sampled constructions yield nearly coincident curves. Coarsening the grid separates the constructions more clearly. For FF on G5, the maximum scaled discrepancy decreases from 1.659×10^{-2} for H0 to 1.547×10^{-3} for H1, a factor of 10.7. The corresponding maximum absolute angle discrepancies are 0.1127 and 0.01058 deg; the maximum speed discrepancies are 0.00926 and 0.000848 m/s. Each component maximum is taken over all comparison sections.

For the flutter–divergence curve on G5, the maximum scaled discrepancy decreases from 9.371×10^{-2} to 5.012×10^{-3} , a factor of 18.7. The maximum absolute speed discrepancies are 0.0713 and 0.00382 m/s. Intermediate grids show a different ordering: on G3, the H1 and H0 maxima are 4.235×10^{-2} and 4.114×10^{-2} , respectively; on G4 they are 5.740×10^{-2} and 1.317×10^{-1} . The largest improvement occurs on the coarsest tested grid.

Local interpolation error depends on both cell size and placement. Changing cell faces and support nodes changes how a curved boundary samples the interpolant, and the transverse condition number varies along that boundary. The nonnested grids therefore reveal the combined effect of derivative information, cell placement, and conditioning. Appendix C relates the construction to local interpolation error under regularity assumptions.

E. Independent construction and continuation cost

Table 2 reports wall-clock time from the common boundary seed through construction and continuation to both domain endpoints. Each run begins with an empty parameter-model cache. Common structural preprocessing and seed discovery are excluded; the required aerodynamic coefficient families, balancing, interpolation support, parameter derivatives, and necessary output are included. Curve matching and posterior full-wake evaluation are excluded. All runs use the same physical model and continuation settings, sequentially in MATLAB R2024a on an Intel Core i7-10750H processor at 2.60 GHz with six numerical threads.

Table 2 Parameter spacings and independent boundary-continuation times in minutes.

Grid	ΔV^*	$\Delta \phi$, deg	$\Delta \epsilon_A$	FF		FD	
				H0	H1	H0	H1
G1	0.2	2	0.025	22.00	15.71	17.03	12.47
G2	0.4	10	0.05	13.87	5.97	9.54	6.00
G3	0.8	20	0.10	6.38	5.53	6.84	4.33
G4	1.6	30	1/6	5.49	4.22	4.95	3.00
G5	2.0	32.5	1/6	5.19	4.25	4.01	2.82

The endpoints are included explicitly. G3 and G4 have final angle intervals of 10 deg; speed intervals are similarly truncated at $V^* = 8$. For G4 and G5 the offset nodes are $-0.6, -13/30, -4/15, -0.1$.

H1 is faster in all ten paired comparisons, by factors of approximately 1.15–2.32. Its analytic jets add work at each corner, while H0 constructs the surrounding value-only stencil. The reported time includes these necessary support data, so it measures the cost of each complete interpolation-and-continuation method. Coarsening reduces total work substantially; the accuracy–cost balance varies with the transition curve.

F. Spectral defects at the computed intersection curves

The reference comparison in Fig. 4 isolates interpolation effects within the reduced-model family. To relate the computed boundaries to the unreduced wake, three models are evaluated at identical physical parameter points: the parameterized model P, a freshly constructed direct balanced model D of the same order, and the full-wake model F. All three share the same eleven structural coordinates. F restores the 576 free-wake states, giving 598 total states.

For a matched critical eigenvalue, define

$$\Delta_I = \mu_P - \mu_D, \quad \Delta_T = \mu_D - \mu_F, \quad \Delta = \mu_P - \mu_F = \Delta_I + \Delta_T \quad (33)$$

The complex differences separate interpolation and wake-reduction contributions. At a critical point of the fine parameterized model, the real parts of the matched full-wake roots measure the defect in its target neutral conditions;

the imaginary parts measure frequency disagreement. Real-root discrepancies are reported in absolute dimensionless units. To isolate the effect of grid construction, coarse models are also evaluated at these same reference parameters.

Table 3 Dimensionless spectral discrepancies at selected fine-model intersection points.

Point	ϵ_A	ϕ , deg	U , m/s	Mode	$ \Delta_I $	$ \Delta_T $	$\text{Re}\mu_F$
FD-1	-0.2985	79.749	7.4818	H	4.86×10^{-11}	2.26×10^{-4}	6.77×10^{-5}
FD-1	-0.2985	79.749	7.4818	D	7.61×10^{-11}	9.86×10^{-4}	-9.86×10^{-4}
FD-2	-0.1574	75.158	10.5539	H	8.43×10^{-10}	6.88×10^{-4}	1.78×10^{-5}
FD-2	-0.1574	75.158	10.5539	D	3.88×10^{-9}	3.93×10^{-3}	-3.93×10^{-3}
FF-1	-0.2499	12.872	4.8582	H1	1.40×10^{-12}	3.33×10^{-5}	-2.91×10^{-5}
FF-1	-0.2499	12.872	4.8582	H2	5.11×10^{-10}	2.79×10^{-4}	-2.73×10^{-4}
FF-2	-0.2515	6.358	4.8037	H1	6.49×10^{-12}	1.46×10^{-5}	-1.18×10^{-5}
FF-2	-0.2515	6.358	4.8037	H2	2.28×10^{-10}	3.52×10^{-4}	-3.45×10^{-4}

The listed parameters belong to G1/H1. H1 and H2 in the mode column denote the two distinct flutter pairs; H and D denote a flutter pair and a real divergence root. The differences are defined by Eq. (33), using $\mu = \lambda/\omega_0$.

Table 3 shows that interpolation differences at the selected fine-model curve points are much smaller than the direct-ROM/full-wake differences. The four-state wake reduction accounts for most of the remaining spectral discrepancy within the stated spatial and structural discretization. Table 4 holds the parameters fixed and compares the two coarsest-grid interpolants against the same directly reduced model. At the first flutter–flutter point, the second critical mode has an H0 interpolation discrepancy of 4.67×10^{-4} , comparable to the wake-reduction discrepancy 2.79×10^{-4} ; H1 reduces the interpolation contribution to 3.28×10^{-5} . The other selected modes show the same reduction in the interpolation contribution.

Table 4 Coarsest-grid interpolation and wake-reduction discrepancies at the common fine-model reference points of Table 3.

Point	Mode	$ \Delta_I $, G5/H0	$ \Delta_I $, G5/H1	$ \Delta_T $
FF-1	H1	1.10×10^{-5}	6.31×10^{-7}	3.33×10^{-5}
FF-1	H2	4.67×10^{-4}	3.28×10^{-5}	2.79×10^{-4}
FF-2	H1	6.22×10^{-6}	2.25×10^{-7}	1.46×10^{-5}
FF-2	H2	4.34×10^{-4}	1.66×10^{-5}	3.52×10^{-4}
FD-1	H	5.58×10^{-5}	3.41×10^{-6}	2.26×10^{-4}
FD-1	D	3.85×10^{-4}	4.76×10^{-5}	9.86×10^{-4}
FD-2	H	1.76×10^{-4}	2.54×10^{-7}	6.88×10^{-4}
FD-2	D	1.32×10^{-3}	1.41×10^{-6}	3.93×10^{-3}

Small neutral-condition defects and small parameter errors are different statements. In a transverse section, let $J_{\perp}$ be the two-by-two Jacobian of the F neutral conditions. If it is nonsingular and varies smoothly, the local displacement of the F intersection satisfies

$$\delta z_{\perp} = -J_{\perp}^{-1} F_F(p_p) + O(\|\delta z_{\perp}\|^2) \quad (34)$$

Equation (34) relates the neutral-condition residual to forward error through the inverse transverse Jacobian [27]. When that Jacobian is well conditioned, a small full-wake residual at the computed parameter implies a nearby full-wake intersection. The neighboring-speed tests below provide complementary evidence of the instability type inside each region.

G. Full-wake transitions inside the mechanism regions

To establish whether the mechanism partition predicts the same local stability transitions in the full-wake model, nine representative points are selected inside its three regions, away from the transition curves. Their positions are marked in Fig. 2. At each fixed (ϵ_A, ϕ) , the P critical speed U_P is evaluated in D and F. Spectra at nearby speeds locate an interval with a stable lower endpoint and the predicted instability at the upper endpoint. Lower-speed samples at $0.65U_P$ and $0.85U_P$ additionally check the stable approach to that interval.

Table 5 Regional critical speeds and neighboring full-wake stability-transition intervals.

Point	ϵ_A	ϕ , deg	U_P , m/s	Full-wake interval, m/s	Type
HF1	-0.10	30.00	9.21515	[9.21515, 9.30730]	HF flutter
HF2	-0.35	40.00	3.80830	[3.80830, 3.84638]	HF flutter
HF3	-0.60	50.00	3.05747	[3.05747, 3.08805]	HF flutter
LF1	-0.25	9.62	4.53746	[4.53746, 4.58284]	LF flutter
LF2	-0.40	10.56	3.29293	[3.29293, 3.32586]	LF flutter
LF3	-0.55	11.34	2.74377	[2.74377, 2.77121]	LF flutter
D1	-0.10	120.00	14.19294	[14.05101, 14.19294]	Divergence
D2	-0.35	110.00	7.36409	[7.29045, 7.36409]	Divergence
D3	-0.60	120.00	5.31439	[5.26125, 5.31439]	Divergence

The lower interval endpoint is stable and the upper endpoint has the listed unstable mode. No competing positive-real-part root is resolved in the inspected spectra. HF and LF retain their continued mode identities.

All nine cases recover the predicted transition within the neighboring one-percent speed intervals in Table 5. The sampled lower-speed states are stable. At the upper endpoint, the HF and LF cases have the corresponding unstable complex pair; the D cases have a positive real root with the other inspected roots stable. Direct-ROM evaluations give

nearby transitions, consistent with the small P–D spectral discrepancies. These checks connect the three regions of the parameterized stability map to the corresponding full-wake instability mechanisms.

V. A Two-Coordinate Explanation of Static Divergence

A. Geometric rotation transfer and diagonal elastic stiffness

The divergence region invites a simpler interpretation of the angle and offset dependence. Consider torsional perturbations of the three identical segments, with normalized local coordinate $\xi = s/L$ and shape

$$\psi(\xi) = 2\xi - \xi^2, \quad \theta_i(\xi) = \psi(\xi)\beta_i \quad (35)$$

This shape satisfies a clamped-root displacement condition and zero tip torque. A rigid rotation inherited from an upstream segment changes incidence but creates no additional torsional strain in the receiving segment. Projecting the rotations through the opposite folds gives the geometric incidence perturbations

$$\alpha_g(\xi) = \begin{bmatrix} \psi\beta_1 \\ g\beta_1 + \psi\beta_2 \\ \beta_1 + g\beta_2 + \psi\beta_3 \end{bmatrix}, \quad g = \cos\phi \quad (36)$$

The inner-to-outer inherited rotation has coefficient one because their span directions remain parallel. The middle segment receives the cosine projection of the upstream rotation.

A two-coordinate approximation is obtained by taking $\beta_3 = \beta_1/2$ and $b = (\beta_1, \beta_2)^\top$. This ratio follows from the dominant inner–outer torsional mode of the uncoupled projected model at $\phi = 90$ deg, as derived in Appendix D. The resulting incidence and stiffness matrices are

$$\alpha_g = F(\xi, \phi)b, \quad F = \begin{bmatrix} \psi & 0 \\ g & \psi \\ 1 + \psi/2 & g \end{bmatrix}, \quad K_b = kD_k, \quad k = \frac{4GJ}{3L}, \quad D_k = \text{diag}(5/4, 1) \quad (37)$$

The projection retains diagonal elastic stiffness, while the distributed incidence and aerodynamic response determine the generalized aerodynamic moment.

B. From lifting-line induction to a projected interaction model

Classical lifting-line theory supplies a useful interpretation of that aerodynamic response [28]. Linearizing about zero circulation, the local two-dimensional lift law and the circulation lift relation give

$$\rho_f U \delta\Gamma = 1/2 \rho_f U^2 c a_{2D} \delta\alpha_{\text{eff}}, \quad \delta\alpha_{\text{eff}} = \delta\alpha_g - \frac{\mathcal{D}\delta\Gamma}{U} \quad (38)$$

Here a_{2D} is the sectional lift slope and $\mathcal{D}$ maps circulation to downwash, with positive downwash reducing incidence. A spanwise variation of bound circulation sheds a trailing-vorticity density proportional to $-\partial_s \Gamma$; its Biot–Savart velocity, including the other segments, enters $\mathcal{D}$. Equivalently,

$$\mathcal{L}\delta\Gamma = U\delta\alpha_g, \quad \mathcal{L} = \frac{2}{ca_{2D}}I + \mathcal{D} \quad (39)$$

A circulation jump supplies a concentrated trailing filament; continuous spanwise variation supplies a distributed wake sheet. Their induced velocities enter the same no-penetration balance. In the steady vortex-lattice equations, trailing-edge continuity and wake transport eliminate the wake in favor of bound circulation. Spanwise projection of this circulation response gives the lifting-line interpretation.

For three identical isolated segments in the same local flow alignment, write

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{E}, \quad \mathcal{L}_0 = \text{diag}(\mathcal{A}, \mathcal{A}, \mathcal{A}), \quad \mathcal{L}^{-1} = \mathcal{L}_0^{-1} - \mathcal{L}_0^{-1}\mathcal{E}\mathcal{L}_0^{-1} + \mathcal{L}_0^{-1}\mathcal{E}\mathcal{L}_0^{-1}\mathcal{E}\mathcal{L}_0^{-1} + \dots \quad (40)$$

The diagonal blocks include self-induction and $\mathcal{E}$ contains intersegment induction. Its zero diagonal blocks make the first correction in Eq. (40) off diagonal; a return influence on the originating segment first appears at second order. For sufficiently weak normalized interaction, this convergent expansion separates a leading isolated-wing scale from a first-order cross-wing correction.

The following projection represents that response by a common scalar scale and one neighboring-segment coupling coefficient:

$$H_\chi = \begin{bmatrix} 1 & \chi & 0 \\ \chi & 1 & \chi \\ 0 & \chi & 1 \end{bmatrix}, \quad A_\chi(\phi) = \int_0^1 F^\top H_\chi F d\xi \quad (41)$$

H_χ acts on the incidence already generated by F . Its off-diagonal entries approximate the projected induced response, including repeated aerodynamic interaction. Equal neighboring coefficients provide a compact approximation for the equal-segment geometry; the projection incorporates detailed spanwise and distant interactions into its effective coefficients. The condition $|\chi| < 1/\sqrt{2}$ keeps this response matrix positive definite.

C. Closed-form divergence relation and fitted parameters

The integrals in Eq. (41) are elementary:

$$A_\chi = \begin{bmatrix} \frac{7}{3} + g^2 + 4\chi g & 2g + \chi\left(g^2 + \frac{22}{15}\right) \\ 2g + \chi\left(g^2 + \frac{22}{15}\right) & \frac{8}{15} + g^2 + \frac{4}{3}\chi g \end{bmatrix} \quad (42)$$

Define Λ_+ as the larger eigenvalue of $D_k^{-1/2} A_\chi D_k^{-1/2}$. The reduced static balance is

$$[D_k - U^2 a_0 (-\epsilon_A - s_0) \Lambda_+] b = 0, \quad U_D = [a_0 (-\epsilon_A - s_0) \Lambda_+(\phi, \chi)]^{-1/2} \quad (43)$$

The coefficient a_0 has units s^2/m^2 and combines lift-response scale, geometry, and torsional stiffness. The dimensionless s_0 adjusts the effective aerodynamic moment arm, incorporating the pressure projection and the

selected torsional shape relative to the quarter-chord approximation. The coefficient χ controls cross-segment response and its angle dependence. Both a_0 and χ enter the projected aerodynamic moment, so their fitted values jointly describe its overall scale and coupling.

Eighty-one corrected parameterized-ROM divergence samples on nine offset slices and thirteen angle stations are used for fitting. Each offset slice receives equal total weight. The fitted objective uses the relative error in U_D^{-2} , and the speed errors are evaluated afterward. Entire offset or angle groups are also held out, with parameters re-estimated from the remaining groups. Table 6 and Fig. 5 report the measured outcomes.

Table 6 Two-coordinate model parameters and absolute relative speed errors over 81 divergence samples.

Model	$a_0, s^2/m^2$	s_0	χ	Median, %	95th, %	Max., %
Neighbor coupling	0.029573	0.024409	0.159463	0.733	3.008	4.136
Uncoupled response	0.024895	0.014174	0	3.083	9.799	11.900

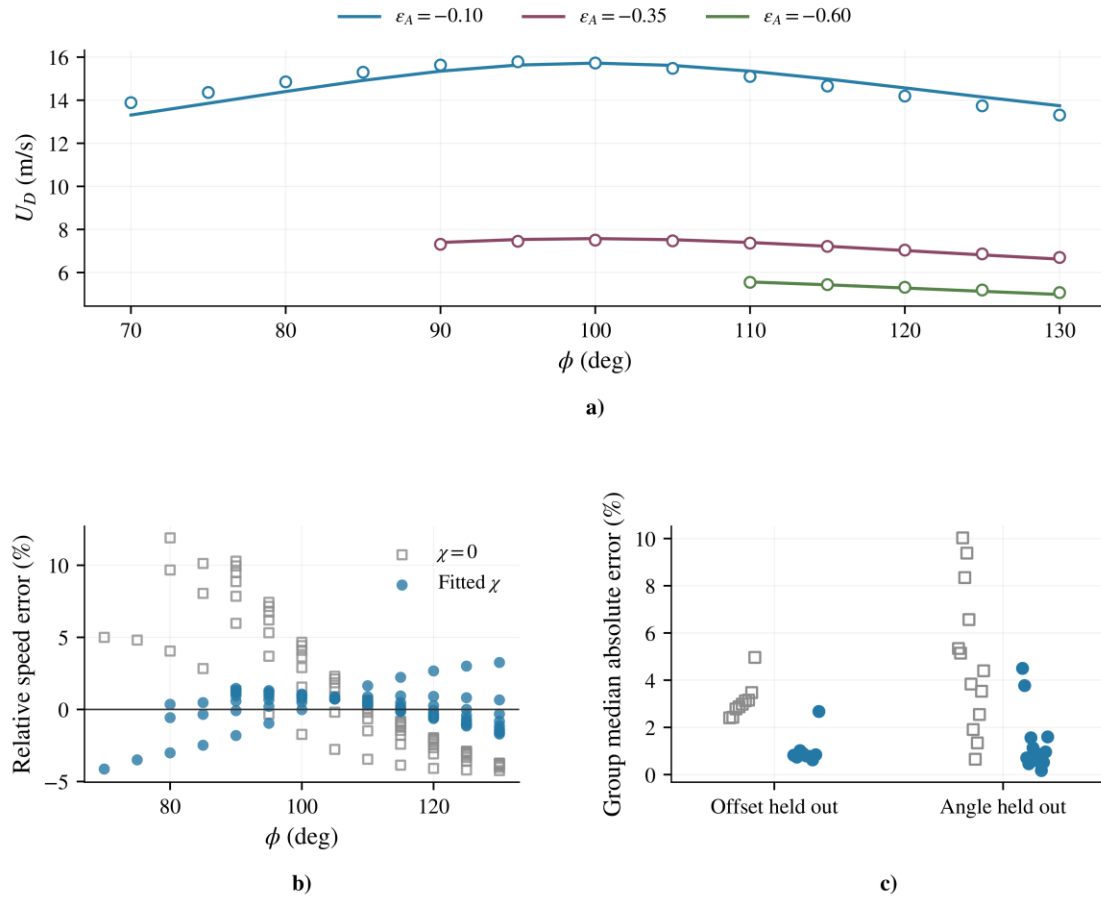


Fig. 5 Two-coordinate divergence model: a) fitted divergence speeds (lines) and corrected ROM samples (open circles); b) speed errors at all 81 fitting points; and c) grouped prediction checks. Open squares and filled circles in panels (b) and (c) distinguish uncoupled and coupled aerodynamic responses, respectively.

Figure 5a compares the fitted divergence curves with corrected ROM samples on three representative offset slices. Panels (b) and (c) show the fitting errors and the grouped prediction errors, respectively. The coupled model reduces both the central and upper-tail errors. Leaving out each offset slice gives group median absolute errors between 0.623 and 2.673%, with a largest individual error of 6.069%. Leaving out each angle station gives group medians between 0.171 and 4.499%. The grouped results show that the improvement extends to offset or angle stations excluded from fitting.

The angular effect is especially informative. With $\chi = 0$, the larger eigenvalue is unchanged under $g \mapsto -g$, giving divergence speeds symmetric about 90 deg. Terms linear in χg break that symmetry. The model thus represents the observed angular asymmetry through aerodynamic coupling while retaining diagonal structural stiffness.

A classical finite-wing estimate supplies an independent scale check. Taking a single segment's aspect ratio $AR = L/c = 4.167$ and the approximate lift slope $a_{FW} = 2\pi/(1 + 2/AR)$ gives $a_{FW} = 4.245 \text{ rad}^{-1}$. With $GJ = 1.20768 \text{ N m}^2$ and $k = 4GJ/(3L)$,

$$a_{0,FW} = \frac{\rho_f c^2 L}{2k} a_{FW} \approx 2.943 \times 10^{-2} \text{ s}^2/\text{m}^2 \quad (44)$$

The estimate differs from the fitted coupled value by approximately one-half percent. Its agreement has the structure suggested by Eq. (40): isolated-wing self-induction sets the principal response scale, while neighboring interactions first modify the cross-segment response. The projection $F^\top H_\chi F$ accounts for the distributed incidence and moment work through its torsional shape integrals.

VI. Conclusions

The Z-fold wing exhibits three governing instability mechanisms across its folding-angle and structural-axis-offset domain. Continued flutter–flutter and flutter–divergence curves locate the changes between them. For a fixed structural-axis design, a folding path can pass through a high–low–high flutter sequence and then enter a divergence region. The resulting partition identifies both the critical-speed margin and the mechanism that a configuration-selection or control analysis must address.

A consistent geometric model makes these boundaries accessible to derivative-based continuation. Section kinematics connect structural inertia, normal wash, and aerodynamic work; their parameter derivatives propagate through wake reduction and local interpolation. On the coarsest tested grid, analytic derivative information

substantially improves both transition-curve approximations. Complete construction-and-continuation timings also favor this construction, reflecting its use of corner derivatives in place of neighboring value-support data.

Comparisons at common critical parameters distinguish interpolation error from wake-reduction error. The four-state wake reduction dominates the selected fine-model spectral discrepancies. Neighboring-speed checks at interior regional points recover the predicted flutter or divergence transition in the full-wake system, linking the mechanism partition to the underlying model.

The divergence boundary admits a compact physical explanation. Parallel inner and outer segments motivate a two-coordinate torsional approximation, geometric rotation transfer determines incidence, and intersegment aerodynamic response produces the angular asymmetry. Its closed-form critical-speed relation captures the offset dependence and gives a fitted response scale close to a classical finite-wing estimate. Together, the numerical partition and this physical interpretation provide a basis for selecting morphing paths and structural-axis locations within the stated configuration-scheduled wing family.

Appendix A: Structural Coefficients and First-Order Beam Kinematics

A1. Reference section and spring specification

The section properties follow from the frequency and mass targets in Table 1. With $\beta_1 = 1.875104$, mass per unit length m_ℓ , and sectional polar mass moment I_ℓ ,

$$\begin{aligned} m_\ell &= 40\pi\rho_f(c/2)^2, & I_\ell &= m_\ell(0.30c)^2, \\ EI_f &= m_\ell L^4(\omega_0/\beta_1^2)^2, & EI_i &= 1.50^2 EI_f, \\ GJ &= I_\ell(2L \cdot 1.35\omega_0/\pi)^2, \\ EA &= 10^5 \max(EI_f, EI_i)/L^2, & GA &= 2 \times 10^4 \max(EI_f, EI_i)/L^2 \end{aligned} \quad (45)$$

The large axial and shear stiffnesses keep the selected low modes dominated by bending and torsion. The same I_ℓ is assigned to the three rotary-inertia entries of the idealized section. The modal basis contains the first selected out-of-plane bending, in-plane bending, and torsional mode of each clamped component. Modal coordinates are fixed in these local component frames while the assembly changes with ϕ .

The structural damping uses componentwise Rayleigh blocks $C_{ii} = \alpha_R M_{ii} + \beta_R K_{ii}$, where

$$\begin{bmatrix} 1/(2\omega_0) & \omega_0/2 \\ 1/(3\omega_0) & 3\omega_0/4 \end{bmatrix} \begin{bmatrix} \alpha_R \\ \beta_R \end{bmatrix} = \begin{bmatrix} 0.005 \\ 0.005 \end{bmatrix} \quad (46)$$

The joint diagonal terms are $C_{jj} = 2(0.005)\sqrt{M_{jj}k_j}$; other damping cross blocks are zero. The mass matrix retains the complete kinematic coupling. From Eq. (7), $k_{j,\phi} = (r_j\omega_0)^2 M_{jj,\phi}$, so the scheduled spring dependence is included in the parameter derivative.

A2. Endpoint perturbations and the section field

For $a \in \mathfrak{se}(3)$, define the right exponential Jacobian by

$$\exp(-\hat{a}) D \exp_a[\hat{b}] = J_r(\widehat{a})b, \quad J_r(a) = \int_0^1 \exp(-tad_a) dt \quad (47)$$

Using Eq. (4), the linear terms are

$$\begin{aligned} d^{(1)} &= J_r(d_0)^{-1}[\delta_b - \text{Ad}_{\exp(-\hat{d}_0)}\delta_a], \\ \sigma^{(1)}(\tau) &= \text{Ad}_{\exp(-\tau\hat{d}_0)}\delta_a + \tau J_r(\tau d_0)d^{(1)} \end{aligned} \quad (48)$$

The translational entries of d have units of length and its rotational entries are angles. With dimensionless $\tau = (s - s_a)/L_e$, Eq. (4) therefore defines a section pose. The strain d/L_e contains dimensionless extension and shear and curvature measured in inverse meters.

Let $d^{(1)} = D_d q$ and $\sigma^{(1)} = D_\sigma(s)q$ after endpoint modal projection. At a stress-free reference, the element contributions are

$$K_e^{(s)} = \frac{1}{L_e} D_d^\top \mathcal{K}_s D_d, \quad M_e^{(s)} = \int_{s_a}^{s_b} J(s)^\top \mathcal{M}_s J(s) ds \quad (49)$$

The velocity map J includes upstream root motion and the local D_σ . Differentiating the first integral with respect to ϕ gives, for a parameter-independent section inertia,

$$M_{s,\phi} = \sum_i \int (J_{i,\phi}^\top \mathcal{M}_i J_i + J_i^\top \mathcal{M}_i J_{i,\phi}) ds \quad (50)$$

For the unloaded reference, these first-order kinematic maps determine the linear mass and stiffness. Higher-order potential derivatives, prestress contributions, and nonlinear static condensation follow from the extended construction in Refs. [15, 8].

Appendix B: Circulation Elimination, Pressure, and Geometric Derivatives

B1. No penetration, trailing-edge continuity, and wake transport

Let Γ_b denote bound-ring circulation, Γ_0 the attached wake row, and x_w the free wake. With K_{TE} selecting the trailing-edge circulation, the algebraic system is

$$A_b \Gamma_b + A_0 \Gamma_0 + A_w x_w = \delta r, \quad \Gamma_0 = K_{TE} \Gamma_b \quad (51)$$

Writing $\bar{A}_b = A_b + A_0 K_{TE}$ gives

$$\Gamma_b = G_r \delta r + G_x x_w, \quad G_r = \bar{A}_b^{-1}, \quad G_x = -\bar{A}_b^{-1} A_w \quad (52)$$

Under inviscid material transport, the shed circulation remains constant along a prescribed wake trajectory. A straight reference wake with streamwise coordinate ℓ therefore satisfies $\partial_t \Gamma_w + U \partial_\ell \Gamma_w = 0$. Its spatial discretization on fixed physical-length rows is

$$\dot{x}_w = U(L_w x_w + B_0 \Gamma_0), \quad A_1 = L_w + B_0 K_{TE} G_x, \quad B_1 = B_0 K_{TE} G_r \quad (53)$$

This derivation identifies the origin of both speed factors in the wake-state equation. The row spacing enters L_w, B_0 and remains fixed as U changes.

B2. Pressure and virtual-work projection

In linear potential flow about a stationary reference, the pressure jump contains the time derivative and the streamwise derivative of the potential jump. Let M_r and D_r be their discrete circulation-to-pressure maps, including the adopted panel orientation. Then

$$\Delta p = \rho_f (M_r \dot{\Gamma}_b + U D_r \Gamma_b) \quad (54)$$

Using Eqs. (52) and (53), at fixed p ,

$$\begin{aligned} C_1 &= \rho_f O(M_r G_x A_1 + D_r G_x), \\ D_1 &= \rho_f O(M_r G_x B_1 + D_r G_r), \\ D_0 &= \rho_f O M_r G_r \end{aligned} \quad (55)$$

The $M_r G_x B_1$ contribution comes from the time derivative of the recovered bound circulation as newly shed circulation enters the wake. The $D_r G_r$ contribution is the convective pressure response. Both belong in D_1 . The wash-rate term $D_0 \delta \dot{r}$ gives the effective mass and part of the damping in Eq. (15). These relationships also show why the steady lifting-line discussion concerns circulation response after wake elimination, while the dynamical reduced model retains wake memory explicitly.

B3. Derivative chain and the offset specialization

The influence matrices depend on relative vortex and collocation positions and the receiving normals. Their parameter derivatives follow from the Biot–Savart kernel and the geometry supplied by Eq. (8). A typical scalar influence entry is $n_i^\top v_{ij}$, so

$$\partial_p (n_i^\top v_{ij}) = n_{i,p}^\top v_{ij} + n_i^\top D v_{ij} [X_{i,p}, X_{\text{vortex},p}] \quad (56)$$

The receiving point and inducing vortex must be expressed in one common frame. Structural velocity and force maps may still be evaluated locally before their final composition. This is why intercomponent induction and section-local wash have different convenient evaluation frames but share one geometric derivative chain.

At zero reference circulation, state linearization gives $A_b^0 \delta I_b$ together with the geometry-induced normal wash. Parameter differentiation acts on this reference operator $A_b^0(p)$, so changes in folding angle retain their effect on aerodynamic influence. For general attachment offsets, both ρ_s and $R_{S,s}\rho$ contribute through Eq. (9); the streamwise-translation invariance of Eq. (18) follows after contracting the resulting normal variation with the reference flow.

Appendix C: Gramian Jets, Basis Gauge, and Parameter-Cell Consistency

C1. Lyapunov coefficient equations and mixed derivatives

Write $\bar{B}_v = B_{v0} + \delta B_{v1}$ and $\bar{C} = C_0 + \delta C_1^o$, where C_1^o denotes the coefficient of the output offset dependence.

Define $\mathcal{L}_A(X) = A_1 X + X A_1^\top$ and $\mathcal{L}_A^*(X) = A_1^\top X + X A_1$. The coefficient equations are

$$\begin{aligned} \mathcal{L}_A(P_q) &= -\bar{B}_q \bar{B}_q^\top, \\ \mathcal{L}_A(P_{v0}) &= -B_{v0} B_{v0}^\top, \\ \mathcal{L}_A(P_{v1}) &= -(B_{v0} B_{v1}^\top + B_{v1} B_{v0}^\top), \\ \mathcal{L}_A(P_{v2}) &= -B_{v1} B_{v1}^\top, \quad P_v = P_{v0} + \delta P_{v1} + \delta^2 P_{v2} \end{aligned} \tag{57}$$

The three coefficients of Q_1 follow under $\mathcal{L}_A^*$ with right-hand sides $-C_0^\top C_0$, $-(C_0^\top C_1^o + (C_1^o)^\top C_0)$, and $-(C_1^o)^\top C_1^o$.

The middle coefficients retain the cross terms generated by each affine offset-dependent map.

For a generic equation $AP + PA^\top + S = 0$,

$$AP_{,j} + P_{,j}A^\top = -A_{,j}P - PA_{,j}^\top - S_{,j} \tag{58}$$

Higher mixed partials follow by repeated product rules. If D_J differentiates once in each parameter in a subset $J \subseteq \{U, \phi, \epsilon_A\}$, then

$$D_J(XY) = \sum_{K \subseteq J} (D_K X) (D_{J \setminus K} Y) \tag{59}$$

The derivatives in Eq. (59) use the ordinary, unscaled partial-derivative convention. For a solve $AX = B$, the unknown $D_J X$ is obtained using the same A after all terms with a nonempty derivative subset on A have been moved to the right-hand side. The affine and polynomial offset/speed reductions provide many of these terms directly.

C2. Differentiation on the retained spectral subspace

Let $S = \hat{P}\hat{Q}$. Differentiating Eq. (24) with respect to p_j gives

$$\begin{bmatrix} S - \varsigma_i^2 I & -2\varsigma_i t_i \\ 2t_i^\top \hat{Q} & -1 \end{bmatrix} \begin{bmatrix} t_{i,j} \\ \varsigma_{i,j} \end{bmatrix} = - \begin{bmatrix} S_j t_i \\ t_i^\top \hat{Q}_j t_i \end{bmatrix}, \quad w_{i,j} = \frac{\hat{Q}_{j,j} t_i + \hat{Q} t_{i,j} - w_i \varsigma_{i,j}}{\varsigma_i} \quad (60)$$

For mixed derivatives, the same bordered matrix is used and the right-hand side contains the known lower-order product terms from Eq. (59). The paired identities $\hat{P}w_i = \varsigma_i t_i$, $\hat{Q}t_i = \varsigma_i w_i$, and $W^\top T = I$ provide independent residual checks. An equivalent formulation differentiates the two paired identities and $w_i^\top t_i = 1$ directly.

The retained spectral derivatives require resolved positive Hankel values, separation between retained eigenlines, and a gap at the cutoff. Symmetric Gramian factorizations treat roundoff-scale tail eigenvalues according to a backward-error scale proportional to $n\epsilon_{\text{mach}} \|P\|$. The derivative equations are solved on the resolved retained modes.

C3. Independent signs, transport, and face consistency

For simple positive ordered Hankel singular values, the remaining real gauge is

$$T \mapsto TS_g, \quad W \mapsto WS_g, \quad S_g = \text{diag}(s_1, \dots, s_r), \quad s_i \in \{-1, 1\} \quad (61)$$

Each paired column has its own independently chosen sign. At a node, every derivative of that column is multiplied by the same s_i . Such a constant local sign change preserves the balanced identities and simply changes the reduced coordinate representation.

A useful transport indicator from node a to node b is

$$c_i(a, b) = w_{a,i}^\top t_{b,i} + w_{b,i}^\top t_{a,i} \quad (62)$$

For nearby points on the same smoothly normalized eigenline, it is near +2 for consistent signs and near -2 for opposite signs. A negative value determines the corresponding column sign change. The indicator must be examined column by column: a sum near zero can arise from cancellation of two well-resolved columns with opposite signs. A near-zero individual overlap can instead indicate a rotation too large for one coarse transport step.

Transport is anchored to an already consistent node or shared face. Intermediate parameter points resolve a retained eigenline when its direction turns substantially along an edge; these auxiliary points determine the sign convention. Agreement around used loops establishes a common convention over the visited cells. The construction assumes simple retained singular values on this region, so each transported object is a one-dimensional real eigenline.

An edge-sharing cell in two dimensions adds two corners, and a face-sharing cell in three dimensions adds four. Shared corners retain their signed jets; new corners are transported into that convention. Each shared-face polynomial and its paired basis therefore agree from both cells.

C4. Hermite construction and local error interpretation

For normalized cell coordinate $t \in [0,1]$, the value basis functions are $h_{00} = 2t^3 - 3t^2 + 1$ and $h_{10} = -2t^3 + 3t^2$, and the derivative basis functions are $h_{01} = t^3 - 2t^2 + t$ and $h_{11} = t^3 - t^2$. With cell widths Δp_k ,

$$\mathcal{H}f(p) = \sum_{a \in \{0,1\}^3} \sum_{b \in \{0,1\}^3} \left[\prod_{k=1}^3 h_{a_k b_k}(t_k) (\Delta p_k)^{b_k} \right] D_b f(p_a) \quad (63)$$

The same face polynomial is obtained from either cell when the corresponding corner jets agree. Equation (26) then gives a common paired basis and common derivatives on that face. For its derivative, $G_j = \tilde{W}_j^\top \tilde{T} + \tilde{W}^\top \tilde{T}_j$ and

$$W_j = \tilde{W}_j G^{-\top} - W G_j^\top G^{-\top} \quad (64)$$

These terms enter the derivative of every projected operator, for example

$$(W^\top A_1 T)_{,j} = W_{,j}^\top A_1 T + W^\top A_{1,j} T + W^\top A_1 T_{,j} \quad (65)$$

Both basis derivatives contribute to the projected-operator sensitivity.

On a shape-regular shrinking cell with sufficiently smooth compatible jets, cubic Hermite interpolation has local value error $O(h^4)$ and first-derivative error $O(h^3)$. Smooth pairing repair preserves these orders when G^{-1} is bounded. Uniformly simple critical eigenvalues and a full-rank transverse Jacobian then give nearby local intersection curves through the implicit-function theorem. Pseudo-arclength correction enforces the neutral conditions at each accepted point, with local error controlled by model accuracy and transverse conditioning. These regularity conditions provide the basis for interpreting the finite-grid comparisons in Sec. IV.

Appendix D: Two-Coordinate Projection and Fitting Relations

D1. Trailing-vorticity induction and segment interaction

Let $X_j(s)$ be the reference bound line of segment j , and use a fixed orientation for its circulation. For straight wake filaments convected in direction e_x , the trailing-vorticity contribution is

$$\delta u_w(x) = \frac{1}{4\pi} \sum_j \int [-\partial_s \delta \Gamma_j(s)] \int_0^\infty \frac{e_x \times [x - X_j(s) - \ell e_x]}{\|x - X_j(s) - \ell e_x\|^3} d\ell ds \quad (66)$$

Endpoint circulation jumps are included as concentrated strengths in the outer integral. The receiving normal projects this velocity into downwash, with the sign chosen as in Eq. (38); the bound-vortex contribution completes the induction operator. The self-interaction requires the usual lifting-line limiting interpretation, while distinct segments use their actual relative positions and normals. A discrete ring lattice represents the same circulation differences through the cancellation of shared edges and the remaining trailing filaments. This gives a physical route from a

circulation perturbation on one segment to the off-diagonal blocks of Eq. (40). Their signs and magnitudes are determined by the directed filaments, receiving normals, and projection.

D2. Torsional projection and parameter estimation

The torsional strain energy associated with Eq. (35) is

$$\mathcal{V} = \frac{GJ}{2L} \int_0^1 (\psi')^2 d\xi \sum_{i=1}^3 \beta_i^2 = \frac{1}{2} \frac{4GJ}{3L} \sum_{i=1}^3 \beta_i^2 \quad (67)$$

At $g = 0$ with no aerodynamic interaction, the inner-outer incidence fields are $\psi\beta_1$ and $\beta_1 + \psi\beta_3$. Their integrated aerodynamic matrix is

$$\begin{bmatrix} 23/15 & 2/3 \\ 2/3 & 8/15 \end{bmatrix}, \quad \lambda_{\max} = 28/15, \quad \begin{bmatrix} \beta_1 \\ \beta_3 \end{bmatrix} \propto \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \quad (68)$$

This supplies the two-coordinate choice in Eq. (37). The shape integrals $\int \psi d\xi = 2/3$ and $\int \psi^2 d\xi = 8/15$ generate all entries of Eq. (42).

Writing the normalized two-by-two matrix as $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$, its larger eigenvalue is

$$\Lambda_+ = \frac{a + d + \sqrt{(a - d)^2 + 4b^2}}{2} \quad (69)$$

The closed form makes the critical speed directly evaluable. In particular, the model predicts the offset law $U_D^{-2} \propto -\epsilon_A - s_0$ at fixed angle. Its coefficients contain the selected distributed mode shape through $F^\top H_\chi F$.

For sample i , let $y_i = U_i^{-2}$, $\hat{y}_i = a_0(-\epsilon_{A,i} - s_0)\Lambda_+(\phi_i, \chi)$, and let n_g be the number of points on its offset slice.

The fitted objective and reported speed error are

$$\min_{a_0, s_0, \chi} \frac{1}{2} \sum_i \frac{1}{N_g n_g} \left(\frac{\hat{y}_i - y_i}{y_i} \right)^2, \quad e_{U,i} = \frac{\hat{U}_i - U_i}{U_i} \quad (70)$$

The parametrization $a_0 = \exp(a)$ and $\chi = \tanh(\eta)/\sqrt{2}$ enforces positive scale and the stated response-matrix condition. The fitted domain has $-\epsilon_A - s_0 > 0$. Grouped prediction tests repeat this fit after excluding an entire offset or angle group and evaluate only the excluded points. The model is fitted to corrected parameterized-ROM divergence roots; the full-wake representative checks remain a separate level of evidence.

Appendix E: Fixed-Offset Stability Sections

Figure 6 shows fixed-offset sections of the lowest critical-speed boundary. Transition points connect the adjoining active branches. Comparable display spacing is used across the slices, and small-angle enlargements resolve the

narrow low-frequency regions. These sections complement the spatial view by showing the sequence of instability mechanisms along fixed-offset folding paths.

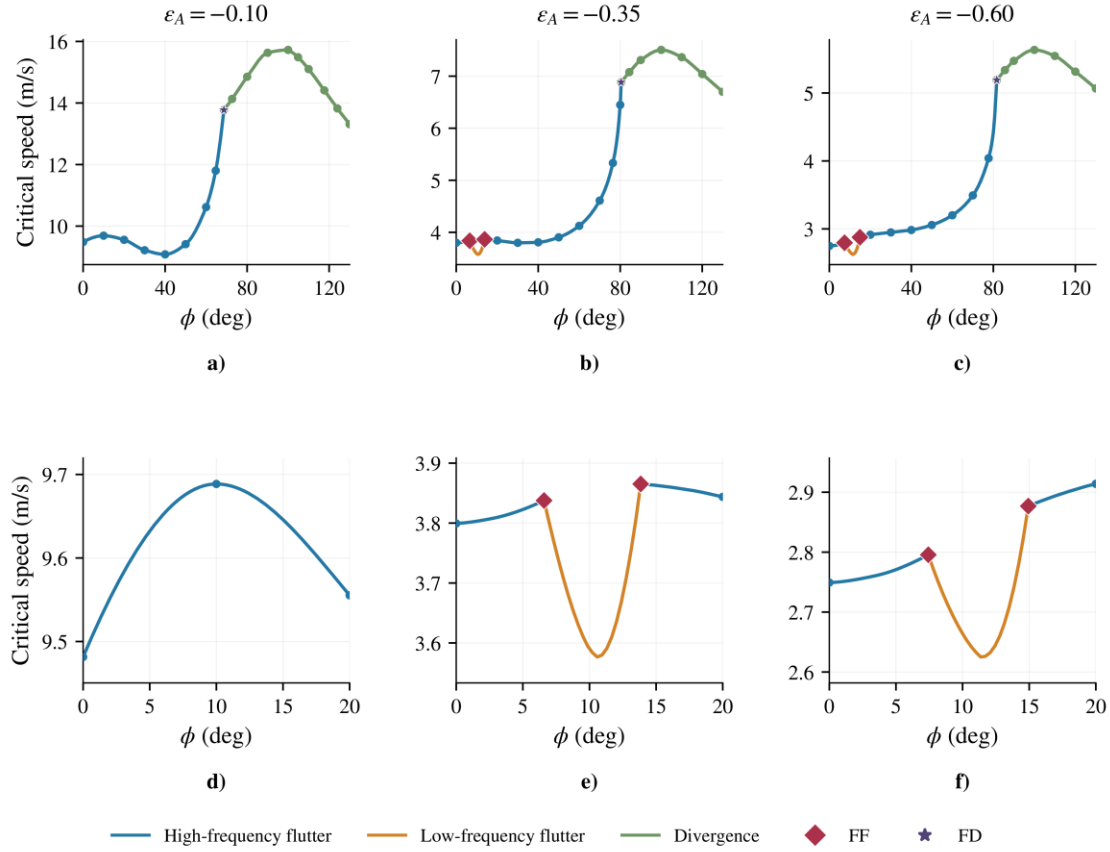


Fig. 6 Fixed-offset lowest critical-speed boundaries at $\epsilon_A = -0.10$, -0.35 , and -0.60 (columns): a–c) complete angle range; d–f) small-angle enlargements. Diamond and star markers identify flutter–flutter and flutter–divergence transitions, respectively.

Data Availability

The data supporting the findings of this study are available from the authors upon reasonable request.

Declaration of Competing Interests

The authors declare that they have no competing interests.

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Author Contributions

Gelin Chen developed the methodology, defined the problem and model configurations, derived the equations, and prepared the manuscript and figures. Chen Song and Haolin Yang contributed to technical discussions and advised on manuscript preparation. Chao Yang reviewed the technical content and presentation.

Use of Artificial Intelligence

ChatGPT and Codex (OpenAI) were used to assist with drafting and language editing, code review, plotting, and manuscript preparation. The authors are responsible for checking the scientific content, references, computations, and final presentation and take responsibility for the submitted work.

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