

High Resolution Analysis of Wing with Stiffness Cross-Coupling By John Cipolla

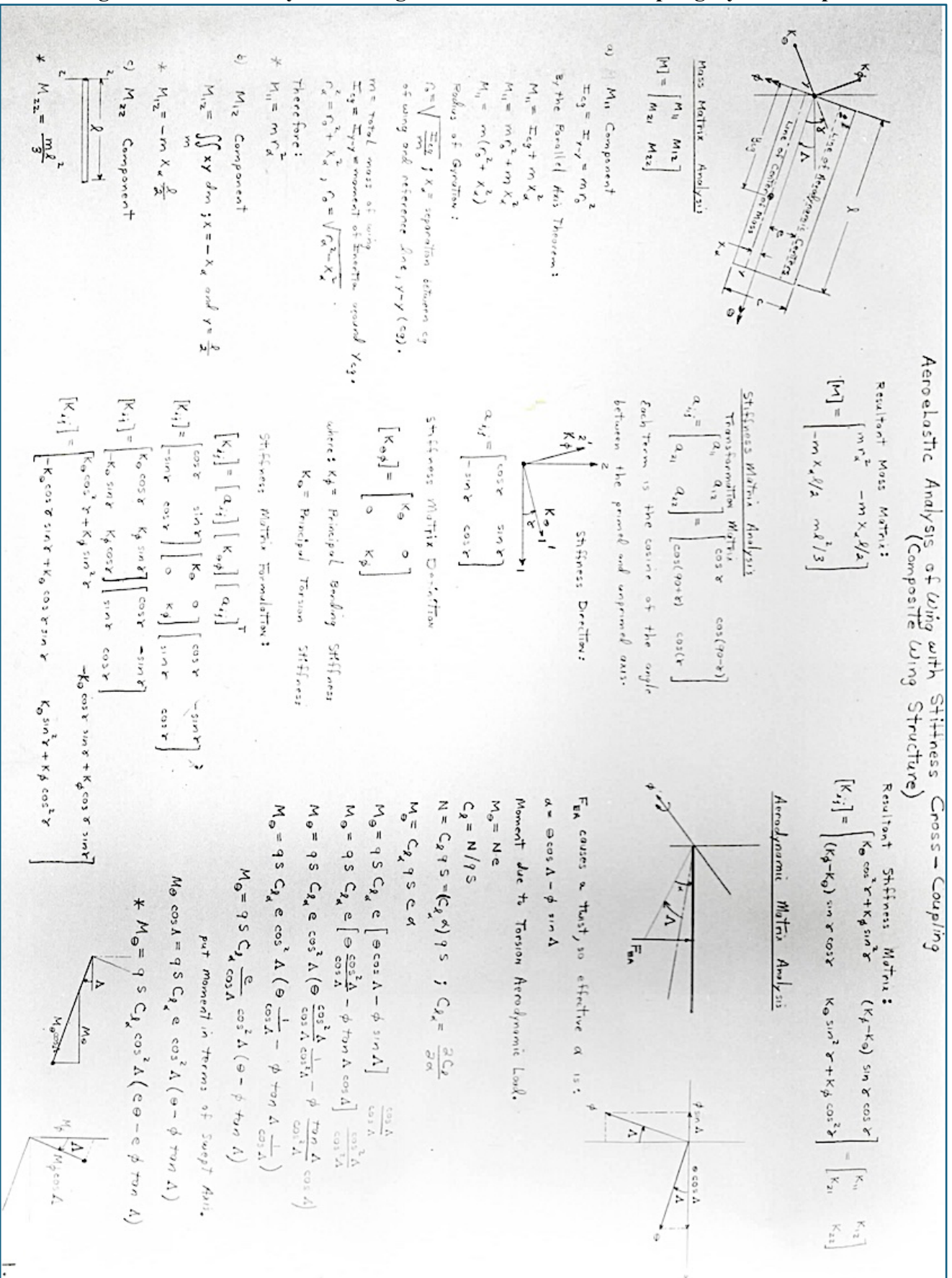


Figure 9. Historical project record documenting the original aeroelastic flutter analysis effort and its connection to the corrected stiffness cross-coupling study.

Moment due to Bending Aerodynamic Load

$$M_\phi = N Y_{cp}$$

$$C_L = \frac{N}{qS}$$

$$N = C_L q S = (C_{L_\alpha} \alpha) q S$$

$$M_\phi = C_{L_\alpha} q S Y_{cp} \alpha$$

$$M_\phi = q S C_{L_\alpha} Y_{cp} [\theta \cos \Lambda - \phi \sin \Lambda] \frac{\cos \Lambda}{\cos \Lambda}$$

$$M_\phi = q S C_{L_\alpha} Y_{cp} [\theta \frac{\cos^2 \Lambda}{\cos \Lambda} - \phi \tan \Lambda \cos \Lambda] \frac{\cos^2 \Lambda}{\cos^2 \Lambda}$$

$$M_\phi = q S C_{L_\alpha} Y_{cp} \cos^2 \Lambda (\theta \frac{1}{\cos \Lambda} - \phi \tan \Lambda \frac{1}{\cos \Lambda})$$

$$M_\phi = q S C_{L_\alpha} \frac{Y_{cp}}{\cos \Lambda} \cos^2 \Lambda (\theta - \phi \tan \Lambda)$$

put Moment in terms of Swept Axis:

$$M_\phi \cos \Lambda = q S C_{L_\alpha} Y_{cp} \cos^2 \Lambda (\theta - \phi \tan \Lambda)$$

$$M_\phi = q S C_{L_\alpha} \cos^2 \Lambda (Y_{cp} \theta - Y_{cp} \phi \tan \Lambda)$$

Put into Matrix Form (Matrix Equation)

$$\begin{Bmatrix} M_\theta \\ M_\phi \end{Bmatrix} = q S C_{L_\alpha} \cos^2 \Lambda \begin{bmatrix} 0 & -e \tan \Lambda \\ Y_{cp} & -Y_{cp} \tan \Lambda \end{bmatrix} \begin{Bmatrix} \theta \\ \phi \end{Bmatrix}$$

The Aerodynamic Matrix is:

$$[A] = \begin{bmatrix} 0 & -e \tan \Lambda \\ Y_{cp} & -Y_{cp} \tan \Lambda \end{bmatrix}$$

The Aerodynamic Load Equation can be expressed as:

$$\begin{Bmatrix} M_\theta \\ M_\phi \end{Bmatrix} = [A_{ij}] \begin{Bmatrix} \theta \\ \phi \end{Bmatrix}$$

Equation of Motion for Vibration of the airfoil in the presence of an Airstream:

$$\Sigma F = m a$$

$$m \ddot{a} - \Sigma F = 0$$

$$\begin{bmatrix} m C_{L_\alpha}^2 & -m X_{cg} Y_{cg} \\ -m X_{cg} Y_{cg} & m I_{cg}^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} K_\theta \cos^2 \Lambda + K_\phi \sin^2 \Lambda & (K_\phi - K_\theta) \sin \Lambda \cos \Lambda \\ (K_\phi - K_\theta) \sin \Lambda \cos \Lambda & K_\theta \sin^2 \Lambda + K_\phi \cos^2 \Lambda \end{bmatrix} \begin{Bmatrix} \theta \\ \phi \end{Bmatrix} + q S C_{L_\alpha} \cos^2 \Lambda \begin{bmatrix} -e \tan \Lambda \\ Y_{cp} \tan \Lambda \end{bmatrix} \begin{Bmatrix} \theta \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \theta \\ \phi \end{Bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} \theta \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assume Harmonic Sinusoidal Motion:

$$\begin{aligned} \theta &= \bar{\theta} e^{i\omega t} & \phi &= \bar{\phi} e^{i\omega t} & \ddot{\theta} &= i\omega \bar{\theta} e^{i\omega t} & \ddot{\phi} &= i\omega \bar{\phi} e^{i\omega t} \\ \omega_p &= \sqrt{\frac{K_{11}}{M_{11}}} & \omega_f &= \sqrt{\frac{K_{22}}{M_{22}}} & \eta &= \frac{\omega}{\omega_p} & \zeta &= \frac{\omega}{\omega_f} \end{aligned}$$

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} \bar{\theta} e^{i\omega t} \\ \bar{\phi} e^{i\omega t} \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \bar{\theta} e^{i\omega t} \\ \bar{\phi} e^{i\omega t} \end{Bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} \bar{\theta} e^{i\omega t} \\ \bar{\phi} e^{i\omega t} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} \bar{\theta}^2 M_{11} & \bar{\theta}^2 M_{12} \\ \bar{\phi}^2 M_{21} & \bar{\phi}^2 M_{22} \end{bmatrix} \begin{Bmatrix} \bar{\theta} \\ \bar{\phi} \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \bar{\theta} \\ \bar{\phi} \end{Bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} \bar{\theta} \\ \bar{\phi} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} \bar{\theta}^2 M_{11} + K_{11} & \bar{\theta}^2 M_{12} + K_{12} \\ \bar{\phi}^2 M_{21} + K_{21} & \bar{\phi}^2 M_{22} + K_{22} \end{bmatrix} \begin{Bmatrix} \bar{\theta} \\ \bar{\phi} \end{Bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} \bar{\theta} \\ \bar{\phi} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Substitution Variables For Mass/Stiffness Matrix:

$$a) \quad \bar{S}^2 = \frac{\bar{\theta}^2}{\omega_p^2} = \frac{\bar{\theta}^2}{1} \frac{M_{11}}{K_{11}} \Rightarrow \bar{S}^2 = \bar{S}^2 \frac{M_{11}}{K_{11}} : \text{assume } K_{11} = 1 \text{ (Normalized to Torsion)}$$

$$\bar{S}^2 = \bar{S}^2 M_{11}$$

$$b) \quad \psi = \frac{-K_{12}}{\sqrt{K_{11} K_{22}}} \quad \text{and} \quad R = \frac{K_{11}}{K_{22}}$$

$$\sqrt{K} = \frac{-\sqrt{K_{11} K_{22}}}{\sqrt{K_{11}}} = -\frac{\sqrt{K_{11} K_{22}}}{\sqrt{K_{11}}} \frac{\sqrt{K_{11}}}{\sqrt{K_{22}}} = -K_{12}$$

$$\frac{\psi}{\sqrt{K}} = -K_{12}$$

$$c) \quad R = \frac{K_{11}}{K_{22}} = \frac{1}{K_{22}} \quad \text{since } K_{11} = 1$$

Figure 10. Supporting historical material showing the analysis context and documentation trail for the forward-swept-wing flutter investigation.

$$d) M_{12} = -m x_a y_a$$

$$M_{11} = \frac{m^2 r_a^2}{2}$$

$$S^2 = \frac{m^2}{M_{11}} = \frac{\bar{S}^2}{m r_a^2}$$

$$S^2 M_{12} = \left\{ \frac{\bar{S}^2}{m r_a^2} \right\} \left\{ -m x_a y_a \right\}$$

$$S^2 M_{12} = \bar{S}^2 \left(-\frac{x_a y_a}{2 r_a^2} \right) = \bar{S}^2 \left(-\frac{x_a y_a}{2 r_a^2 / c^2} \right)$$

$$S^2 M_{12} = \bar{S}^2 \left(-\frac{x_a y_a}{2 r_a^2} \right)$$

$$S^2 M_{12} = \bar{S}^2 \left(-\frac{x_a y_a}{2 r_a^2} \right) \text{ where } \bar{A}R = y/c, \bar{x}_a = \frac{x_a}{c} \text{ and } \bar{r}_a = r_a/c$$

$$c) S^2 M_{22} = \frac{\bar{S}^2}{m r_a^2} \frac{m y_a^2}{3} = \frac{\bar{S}^2 y_a^2}{3 r_a^2}$$

$$S^2 M_{22} = \frac{\bar{S}^2 y_a^2}{3 r_a^2 / c^2}$$

$$S^2 M_{22} = \frac{\bar{S}^2 y_a^2}{3 r_a^2}$$

$$f. M_{11} = 1$$

Summary of Substitution Variables for Stiffness/Mass Matrix

$$S^2 M_{11} = \bar{S}^2; K_{11} = 1; S^2 M_{22} = \bar{S}^2 \left(-\frac{x_a y_a}{2 r_a^2} \right); K_{12} = -\frac{y_a}{r_a}$$

$$K_{22} = \frac{1}{r_a}; S^2 M_{22} = \frac{\bar{S}^2 y_a^2}{3 r_a^2}$$

Substitution variables for Aerodynamic Matrix:

$$[A] = q S C_{L_a} \cos^2 \Lambda \begin{bmatrix} -\bar{e} & e \tan \Lambda \\ -y_{cp} & y_{cp} \tan \Lambda \end{bmatrix}$$

$$q = \frac{1}{2} \rho V^2; S = l c; \bar{A}R = y/c$$

$$[A] = \frac{1}{2} \rho V^2 (l c) C_{L_a} \cos^2 \Lambda \begin{bmatrix} -\bar{e} & e \tan \Lambda \\ -y_{cp} & y_{cp} \tan \Lambda \end{bmatrix}$$

$$\bar{V} = \frac{V}{\omega_p c/l_2} \text{ and } \omega_p = \sqrt{\frac{K_{11}}{M_{11}}} \text{ and } K_{11} = 1 \Rightarrow \omega_p = \frac{1}{\sqrt{M_{11}}}$$

$$V = \omega_p \frac{c}{2} \bar{V}$$

$$V = \frac{1}{\sqrt{M_{11}}} \frac{c}{2} \bar{V}$$

$$V^2 = \left(\frac{c}{2} \right)^2 \frac{\bar{V}^2}{M_{11}}; M_{11} = m r_a^2$$

$$V^2 = \frac{c^2 \bar{V}^2}{4 m r_a^2} = \left(\frac{c}{2} \right)^2 \frac{\bar{V}^2}{m r_a^2}$$

$$[A] = \frac{1}{2} \rho \left[\left(\frac{c}{2} \right)^2 \frac{\bar{V}^2}{m r_a^2} \right] S C_{L_a} \cos^2 \Lambda \begin{bmatrix} -\bar{e} & e \tan \Lambda \\ -y_{cp} & y_{cp} \tan \Lambda \end{bmatrix} \begin{Bmatrix} \bar{\delta} \\ \bar{\phi} \end{Bmatrix}$$

$$[A] = \frac{1}{2} \rho \left(\frac{c}{2} \right)^2 \frac{\bar{V}^2}{m r_a^2} S C_{L_a} \cos^2 \Lambda \begin{bmatrix} -\bar{e} & e \tan \Lambda \\ -y_{cp} & y_{cp} \tan \Lambda \end{bmatrix} \begin{Bmatrix} \bar{\delta} \\ \bar{\phi} \end{Bmatrix}$$

$$[A] = \frac{1}{2} (\bar{V} \cos \Lambda)^2 C_{L_a} \frac{1}{\pi} \left\{ \frac{\pi \rho \frac{c}{2} S}{2m} \right\} \frac{c}{r_a^2} \frac{\bar{S}^2}{c^2} \begin{bmatrix} -\bar{e} & e \tan \Lambda \\ -y_{cp} & y_{cp} \tan \Lambda \end{bmatrix} \begin{Bmatrix} \bar{\delta} \\ \bar{\phi} \end{Bmatrix}$$

$$\text{assume: } \mu = \frac{2m}{\pi \rho \frac{c}{2} S}$$

$$[A] = \frac{1}{2} (\bar{V} \cos \Lambda)^2 C_{L_a} \frac{1}{\pi} \frac{1}{\mu} \begin{bmatrix} -\bar{e} & e \tan \Lambda \\ -y_{cp} & y_{cp} \tan \Lambda \end{bmatrix} \begin{Bmatrix} \bar{\delta} \\ \bar{\phi} \end{Bmatrix}$$

$$[A] = \frac{1}{2} (\bar{V} \cos \Lambda)^2 C_{L_a} \begin{bmatrix} -\frac{\bar{e}}{x} & \frac{e}{x} \tan \Lambda \\ -\frac{y_{cp}}{x} & \frac{y_{cp}}{x} \tan \Lambda \end{bmatrix} \begin{Bmatrix} \bar{\delta} \\ \bar{\phi} \end{Bmatrix}; \frac{y_{cp}}{x} = \bar{y}$$

$$[A] = \frac{1}{2} (\bar{V} \cos \Lambda)^2 C_{L_a} \begin{bmatrix} -\bar{e} & e \tan \Lambda \\ -\bar{y} & \bar{y} \tan \Lambda \end{bmatrix} \begin{Bmatrix} \bar{\delta} \\ \bar{\phi} \end{Bmatrix}$$

The above is the reduced form of the Aerodynamic Matrix:
The Equation of Motion becomes:

$$\begin{bmatrix} (\bar{S}^2 + 1) & (\bar{S}^2 \left[-\frac{\bar{x}_a \bar{A}R}{2 r_a^2} \right] - \frac{\bar{y}}{\bar{r}}) \\ (\bar{S}^2 \left[-\frac{\bar{x}_a \bar{A}R}{2 r_a^2} \right] - \frac{\bar{y}}{\bar{r}}) & (\frac{\bar{S}^2 \bar{A}R^2}{3 r_a^2} + \frac{1}{\bar{r}}) \end{bmatrix} \begin{Bmatrix} \bar{\delta} \\ \bar{\phi} \end{Bmatrix} + \begin{bmatrix} (\bar{V} \cos \Lambda)^2 C_{L_a} \\ -\bar{y} \bar{A}R \bar{y} \bar{A}R \tan \Lambda \end{bmatrix} \begin{Bmatrix} \bar{\delta} \\ \bar{\phi} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Equation 14

Figure 11. Archival material further documenting the computational basis and development history behind the corrected aeroelastic stability analysis.

$$\begin{bmatrix} (\bar{S}^2+1) & (\bar{S}^2 \left[-\frac{\bar{X}_A \bar{A} \bar{R}}{2 \bar{F}_A^2} - \frac{\psi}{\sqrt{R}} \right]) \\ (\bar{S}^2 \left[-\frac{\bar{X}_A \bar{A} \bar{R}}{2 \bar{F}_A^2} - \frac{\psi}{\sqrt{R}} \right]) & (\bar{S}^2 \frac{\bar{A} \bar{R}^2}{3 \bar{F}_A^2} + \frac{1}{R}) \end{bmatrix} \begin{Bmatrix} \bar{\psi} \\ \bar{\phi} \end{Bmatrix} + \frac{1}{2} \left(\frac{(\bar{V} \cos \Lambda)^2 C_{\theta_A}}{\tau \mu \bar{F}_A^2} \right) \begin{bmatrix} -\bar{e} & e \tan \Lambda \\ -\bar{y} \bar{A} \bar{R} & \bar{y} \bar{A} \bar{R} \tan \Lambda \end{bmatrix} \begin{Bmatrix} \bar{\psi} \\ \bar{\phi} \end{Bmatrix} = \begin{Bmatrix} \phi \\ \bar{\phi} \end{Bmatrix}$$

$$C_1 = -\frac{\bar{X}_A \bar{A} \bar{R}}{2 \bar{F}_A^2}, C_2 = \frac{\psi}{\sqrt{R}}, C_3 = \frac{\bar{A} \bar{R}^2}{3 \bar{F}_A^2}, C_4 = \frac{1}{R}, C_5 = \frac{1}{2} \left(\frac{(\bar{V} \cos \Lambda)^2 C_{\theta_A}}{\tau \mu \bar{F}_A^2} \right), C_6 = -\bar{e}$$

$$C_7 = \bar{e} \tan \Lambda, C_8 = -\bar{y} \bar{A} \bar{R}, C_9 = \bar{y} \bar{A} \bar{R} \tan \Lambda$$

$$\begin{bmatrix} (\bar{S}^2+1) & \bar{S}^2 C_1 - C_2 \\ \bar{S}^2 C_1 - C_2 & \bar{S}^2 C_3 + C_4 \end{bmatrix} \begin{Bmatrix} \bar{\psi} \\ \bar{\phi} \end{Bmatrix} + C_5 \begin{bmatrix} C_6 & C_7 \\ C_8 & C_9 \end{bmatrix} \begin{Bmatrix} \bar{\psi} \\ \bar{\phi} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency Form of The Equations:

$$\begin{bmatrix} (\bar{S}^2+1) + C_5 C_6 & \bar{S}^2 C_1 - C_2 + C_5 C_7 \\ \bar{S}^2 C_1 - C_2 + C_5 C_8 & \bar{S}^2 C_3 + C_4 + C_5 C_9 \end{bmatrix} \begin{Bmatrix} \bar{\psi} \\ \bar{\phi} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}; [\text{FM}] \begin{Bmatrix} \bar{\psi} \\ \bar{\phi} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

DET [Frequency Matrix] = 0

It has the following form:

$$a \bar{S}^4 - b \bar{S}^2 - d \bar{V}_A \bar{S}^2 + f + \bar{V}_A g = 0$$

Solve for a, b, d, f and g.

$$\left[\bar{S}^2+1 + C_5 C_6 \right] \left[\bar{S}^2 C_3 + C_4 + C_5 C_9 \right] - \left[\bar{S}^2 C_1 - C_2 + C_5 C_8 \right] \left[\bar{S}^2 C_1 - C_2 + C_5 C_7 \right] = 0$$

$$\begin{aligned} & a) \quad \bar{S}^2 C_3 + C_4 + C_5 C_9 \\ & \times \frac{\bar{S}^2+1 + C_5 C_6}{\bar{S}^4 C_3 + \bar{S}^2 C_4 + \bar{S}^2 C_5 C_9 + \bar{S}^2 C_3 + C_4 + C_5 C_9} \\ & + \frac{\bar{S}^2 C_1 - C_2 + C_5 C_8}{\bar{S}^4 C_3 + \bar{S}^2 C_4 + \bar{S}^2 C_5 C_9 + \bar{S}^2 C_3 + \bar{S}^2 C_5 C_6 + C_4 + C_5 C_6 + C_5 C_6 C_7 + C_5 C_8 + C_9} \end{aligned}$$

$$\begin{aligned} & b) \quad \bar{S}^2 C_1 - C_2 + C_5 C_8 \\ & \times \frac{\bar{S}^2 C_1 - C_2 + C_5 C_7}{\bar{S}^4 C_1^2 - \bar{S}^2 C_1 C_2 + \bar{S}^2 C_1 C_5 C_7 - \bar{S}^2 C_1 C_2 + C_2^2 - C_2 C_5 C_8} \\ & - \frac{\bar{S}^2 C_1 C_5 C_7 - C_2 C_5 C_7 + C_5^2 C_7 C_9}{\bar{S}^4 C_1^2 - \bar{S}^2 C_1 C_2 + \bar{S}^2 C_1 C_5 C_7 - \bar{S}^2 C_1 C_2 + \bar{S}^2 C_1 C_5 C_7 - C_2 C_5 C_7 - C_2 C_5 C_8 + C_5^2 C_7 C_9 + C_5^2} \end{aligned}$$

Substituting:

$$\begin{aligned} & \bar{S}^4 C_3 + \bar{S}^2 C_4 + \bar{S}^2 C_5 C_9 + \bar{S}^2 C_3 + \bar{S}^2 C_5 C_6 + C_4 + C_5 C_6 + C_5^2 C_6 C_7 + C_5 C_8 + C_9 \\ & - \bar{S}^4 C_1^2 + \bar{S}^2 C_1 C_2 - \bar{S}^2 C_1 C_5 C_7 + \bar{S}^2 C_1 C_2 - \bar{S}^2 C_1 C_5 C_7 + C_2 C_5 C_7 + C_2 C_5 C_8 - C_5^2 C_7 C_9 - C_5^2 = 0 \end{aligned}$$

$$\begin{aligned} & \bar{S}^4 C_3 - \bar{S}^4 C_1^2 + \left\{ \bar{S}^2 C_4 + \bar{S}^2 C_3 + \bar{S}^2 C_1 C_2 + \bar{S}^2 C_1 C_2 \right\} + \left\{ \bar{S}^2 C_5 C_9 + \bar{S}^2 C_5 C_6 - \bar{S}^2 C_1 C_5 C_7 - \bar{S}^2 C_1 C_5 C_8 \right\} \\ & + \left\{ C_4 + C_5 C_9 + C_5^2 C_6 C_9 + C_4 C_5 C_6 - C_5^2 C_7 C_9 + C_2 C_5 C_8 + C_2 C_5 C_7 - C_5^2 \right\} = 0 \end{aligned}$$

$$\begin{aligned} & \bar{S}^4 \left\{ C_3 - C_1^2 \right\} + \bar{S}^2 \left\{ C_4 + C_3 + 2 C_1 C_2 \right\} + \bar{S}^2 \left\{ C_5 C_9 + C_5 C_5 C_6 - C_1 C_5 C_7 - C_1 C_5 C_8 \right\} \\ & + \left\{ C_4 C_9 + C_5^2 C_6 C_9 + C_4 C_5 C_6 - C_5^2 C_7 C_9 + C_2 C_5 C_8 + C_2 C_5 C_7 - C_5^2 \right\} \\ & + \left\{ C_4 - C_2^2 \right\} = 0 \end{aligned}$$

Figure 12. Additional historical evidence showing how the original computational work supported later verification of the corrected flutter and divergence equations.

$$\begin{aligned}
& \bar{S}^2 \left\{ \frac{\bar{A}\bar{R}^2}{3\bar{C}_d^3} - \frac{\bar{X}_4 \bar{A}\bar{R}^2}{(2\bar{C}_d)^2} \right\} + \bar{S}^2 \left\{ \frac{1}{\bar{R}} + \frac{\bar{A}\bar{R}^2}{3\bar{C}_d^2} - \frac{\bar{X}_4 \bar{A}\bar{R}}{\bar{C}_d^2 \sqrt{\bar{R}}} \right\} + \bar{S}^2 \left\{ \frac{1}{2} \frac{\bar{V}_0^2 \bar{C}_{L\alpha}}{\pi \mu \bar{C}_d^2} \bar{y} \bar{A}\bar{R} \tan \Lambda + \frac{\bar{A}\bar{R}^2}{3\bar{C}_d^2} \frac{1}{2} \frac{\bar{V}_0^2 \bar{C}_{L\alpha}}{\pi \mu \bar{C}_d^2} (-\bar{e}) - \left(\frac{\bar{X}_4 \bar{A}\bar{R}}{2\bar{C}_d^2} \right) \frac{1}{2} \frac{\bar{V}_0^2 \bar{C}_{L\alpha}}{\pi \mu \bar{C}_d^2} \bar{e} \tan \Lambda - \frac{\bar{X}_4 \bar{A}\bar{R}}{2\bar{C}_d^2} \frac{1}{2} \frac{\bar{V}_0^2 \bar{C}_{L\alpha}}{\pi \mu \bar{C}_d^2} (\bar{y} \bar{A}\bar{R}) \right\} - \left\{ \frac{1}{\bar{R}} - \left(\frac{\psi}{\bar{R}} \right)^2 \right\} \\
& \bar{S}^2 \frac{\bar{A}\bar{R}^2}{3\bar{C}_d^3} \left\{ \frac{\bar{C}_d^2}{\bar{C}_d^2} + \frac{1}{2} \frac{\bar{X}_4}{\bar{C}_d} \right\} + \bar{S}^2 \frac{1}{\bar{R}} \left\{ 1 + \frac{\bar{R} \bar{A}\bar{R}^2}{3\bar{C}_d^2} - \frac{\bar{X}_4 \bar{A}\bar{R} \sqrt{\bar{R}}}{\bar{C}_d^2 \sqrt{\bar{R}}} \right\} + \bar{V}_0^2 \bar{S}^2 \left(\frac{\bar{C}_{L\alpha}}{2\pi \mu \bar{C}_d^2} \right) \bar{A}\bar{R} \left\{ \bar{y} \tan \Lambda - \frac{\bar{A}\bar{R}}{3\bar{C}_d^2} \bar{e} + \frac{\bar{X}_4}{2\bar{C}_d^2} \bar{e} \tan \Lambda - \frac{\bar{X}_4}{2\bar{C}_d^2} \bar{y} \bar{A}\bar{R} \right\} \\
& \text{(a)} \quad \downarrow \text{see NOTE \#1} \\
& \bar{S}^2 \frac{1}{\bar{R}} \left\{ 1 + \frac{\bar{R} \bar{A}\bar{R}^2}{3\bar{C}_d^2} - \frac{\bar{X}_4 \bar{A}\bar{R} \sqrt{\bar{R}}}{\bar{C}_d^2 \sqrt{\bar{R}}} \right\} + \bar{V}_0^2 \bar{S}^2 \left(\frac{\bar{C}_{L\alpha}}{2\pi \mu \bar{C}_d^2} \right) \bar{A}\bar{R} \left\{ \bar{y} \tan \Lambda - \frac{\bar{A}\bar{R}}{3\bar{C}_d^2} \bar{e} + \frac{\bar{X}_4}{2\bar{C}_d^2} \bar{e} \tan \Lambda - \frac{\bar{X}_4}{2\bar{C}_d^2} \bar{y} \bar{A}\bar{R} \right\} \\
& \text{(b)} \quad \downarrow \text{see NOTE \#2} \\
& \bar{S}^2 \frac{1}{\bar{R}} \left\{ 1 + \frac{\bar{R} \bar{A}\bar{R}^2}{3\bar{C}_d^2} - \frac{\bar{X}_4 \bar{A}\bar{R} \sqrt{\bar{R}}}{\bar{C}_d^2 \sqrt{\bar{R}}} \right\} + \bar{V}_0^2 \bar{S}^2 \left(\frac{\bar{C}_{L\alpha}}{2\pi \mu \bar{C}_d^2} \right) \bar{A}\bar{R} \left\{ \bar{y} \tan \Lambda - \frac{\bar{A}\bar{R}}{3\bar{C}_d^2} \bar{e} + \frac{\bar{X}_4}{2\bar{C}_d^2} \bar{e} \tan \Lambda - \frac{\bar{X}_4}{2\bar{C}_d^2} \bar{y} \bar{A}\bar{R} \right\} \\
& \text{(c)} \quad \downarrow \text{see NOTE \#3} \\
& \text{(d)} \\
& + \left(\frac{1}{2} \frac{\bar{V}_0^2 \bar{C}_{L\alpha}}{\pi \mu \bar{C}_d^2} \right) \bar{y} \bar{A}\bar{R} \tan \Lambda - \left(\frac{1}{2} \frac{\bar{V}_0^2 \bar{C}_{L\alpha}}{\pi \mu \bar{C}_d^2} \right) \bar{e} \bar{y} \bar{A}\bar{R} \tan \Lambda - \frac{1}{\bar{R}} \left(\frac{1}{2} \frac{\bar{V}_0^2 \bar{C}_{L\alpha}}{\pi \mu \bar{C}_d^2} \right) \bar{e} + \left(\frac{1}{2} \frac{\bar{V}_0^2 \bar{C}_{L\alpha}}{\pi \mu \bar{C}_d^2} \right) \bar{y} \bar{A}\bar{R} + \frac{\psi}{\sqrt{\bar{R}}} \left(\frac{1}{2} \frac{\bar{V}_0^2 \bar{C}_{L\alpha}}{\pi \mu \bar{C}_d^2} \right) \bar{y} \bar{A}\bar{R} + \frac{\psi}{\sqrt{\bar{R}}} \left(\frac{1}{2} \frac{\bar{V}_0^2 \bar{C}_{L\alpha}}{\pi \mu \bar{C}_d^2} \right) \bar{e} \tan \Lambda \\
& \left(\frac{1}{2} \frac{\bar{V}_0^2 \bar{C}_{L\alpha}}{\pi \mu \bar{C}_d^2} \right) \bar{y} \bar{A}\bar{R} \tan \Lambda - \frac{1}{\bar{R}} \left(\frac{1}{2} \frac{\bar{V}_0^2 \bar{C}_{L\alpha}}{\pi \mu \bar{C}_d^2} \right) \bar{e} - \frac{\psi}{\sqrt{\bar{R}}} \left(\frac{1}{2} \frac{\bar{V}_0^2 \bar{C}_{L\alpha}}{\pi \mu \bar{C}_d^2} \right) \bar{y} \bar{A}\bar{R} + \frac{\psi}{\sqrt{\bar{R}}} \left(\frac{1}{2} \frac{\bar{V}_0^2 \bar{C}_{L\alpha}}{\pi \mu \bar{C}_d^2} \right) \bar{e} \tan \Lambda \\
& \bar{V}_0^2 \frac{\bar{C}_{L\alpha}}{2\pi \mu \bar{C}_d^2} \left\{ \bar{y} \bar{A}\bar{R} \tan \Lambda - \frac{1}{\bar{R}} \bar{e} + \frac{\psi}{\sqrt{\bar{R}}} \left(\bar{e} \tan \Lambda - \bar{y} \bar{A}\bar{R} \right) \right\} \\
& \bar{V}_0^2 \frac{\bar{C}_{L\alpha}}{2\pi \mu \bar{C}_d^2} \left\{ -\frac{1}{\bar{R}} \bar{e} + \bar{y} \bar{A}\bar{R} \tan \Lambda + \frac{\psi}{\sqrt{\bar{R}}} \left(\bar{e} \tan \Lambda - \bar{y} \bar{A}\bar{R} \right) \right\} \\
& \bar{V}_0^2 \frac{\bar{C}_{L\alpha}}{2\pi \mu \bar{C}_d^2} \left\{ \frac{1}{\bar{R}} \left[-\bar{e} + \bar{R} \bar{y} \bar{A}\bar{R} \tan \Lambda + \psi \frac{\bar{R}}{\sqrt{\bar{R}}} \left(\bar{e} \tan \Lambda - \bar{y} \bar{A}\bar{R} \right) \right] \right\} \\
& \bar{V}_0^2 \frac{\bar{C}_{L\alpha}}{2\pi \mu \bar{C}_d^2} \left\{ \frac{1}{\bar{R}} \left[-\bar{e} + \bar{R} \bar{y} \bar{A}\bar{R} \tan \Lambda + \psi \sqrt{\bar{R}} \left(\bar{e} \tan \Lambda - \bar{y} \bar{A}\bar{R} \right) \right] \right\} \\
& \bar{V}_0^2 \epsilon \left\{ \frac{1}{\bar{R}} \left[-\bar{e} + \bar{R} \bar{y} \bar{A}\bar{R} \tan \Lambda + \psi \sqrt{\bar{R}} \left(\bar{e} \tan \Lambda - \bar{y} \bar{A}\bar{R} \right) \right] \right\} + \frac{1}{\bar{R}} - \left(\frac{\psi}{\sqrt{\bar{R}}} \right)^2 \\
& \text{(g)} \quad \frac{1}{\bar{R}} - \left(\frac{\psi}{\sqrt{\bar{R}}} \right)^2 \\
& \quad \quad \quad (1-\psi^2)/\bar{R} = 0 \\
& \quad \quad \quad f = (1-\psi^2)/\bar{R} \quad \text{and} \quad \epsilon = \frac{\bar{C}_{L\alpha}}{2\pi \mu \bar{C}_d^2} \\
& \quad \quad \quad g = \epsilon \left\{ \frac{1}{\bar{R}} \left[-\bar{e} + \bar{R} \bar{y} \bar{A}\bar{R} \tan \Lambda + \psi \sqrt{\bar{R}} \left(\bar{e} \tan \Lambda - \bar{y} \bar{A}\bar{R} \right) \right] \right\} \\
& \quad \quad \quad h = -\frac{1}{\bar{R}} \left\{ 1 + \frac{\bar{R} \bar{A}\bar{R}^2}{3\bar{C}_d^2} - \frac{\bar{X}_4 \bar{A}\bar{R} \sqrt{\bar{R}}}{\bar{C}_d^2 \sqrt{\bar{R}}} \right\} \\
& \quad \quad \quad d = -\epsilon \bar{A}\bar{R} \left\{ \bar{y} \tan \Lambda - \frac{\bar{A}\bar{R}}{3\bar{C}_d^2} \bar{e} + \frac{\bar{X}_4}{2\bar{C}_d^2} \left(\bar{e} \tan \Lambda - \bar{y} \bar{A}\bar{R} \right) \right\} \\
& \quad \quad \quad a = \frac{\bar{A}\bar{R}^2}{3\bar{C}_d^3} \left\{ \frac{\bar{C}_d^2}{\bar{C}_d^2} + \frac{1}{2} \frac{\bar{X}_4}{\bar{C}_d} \right\} \\
& \quad \quad \quad \text{using } a\bar{S}^2 - b\bar{S}^2 - d\bar{V}_0^2 \bar{S}^2 + \bar{V}_0^2 g + f = 0 \quad \text{the coefficients are given as:} \\
& \quad \quad \quad \text{Note \#1:} \\
& \quad \quad \quad \frac{\bar{A}\bar{R}^2}{3\bar{C}_d^3} - \frac{\bar{X}_4 \bar{A}\bar{R}^2}{(2\bar{C}_d)^2} = \frac{\bar{A}\bar{R}^2}{3\bar{C}_d^3} - \frac{3}{3} \frac{\bar{X}_4 \bar{A}\bar{R}^2}{(2\bar{C}_d)^2} = \frac{\bar{A}\bar{R}^2}{3\bar{C}_d^3} \left(1 - \frac{3\bar{X}_4}{2\bar{C}_d} \right) \quad \bar{g} = \bar{C}_d + \bar{X}_4 \\
& \quad \quad \quad \frac{\bar{A}\bar{R}^2}{3\bar{C}_d^3} \left(\frac{4\bar{C}_d^2}{4\bar{C}_d^2} - \frac{3}{4} \frac{\bar{X}_4}{\bar{C}_d} \right) = \frac{\bar{A}\bar{R}^2}{3\bar{C}_d^3} \left(\frac{4\bar{C}_d^2 - 3\bar{X}_4}{4\bar{C}_d^2} \right) = \frac{\bar{A}\bar{R}^2}{3\bar{C}_d^3} \left(\frac{4(\bar{C}_d^2 - \bar{X}_4) - 3\bar{X}_4}{4\bar{C}_d^2} \right) \\
& \quad \quad \quad = \frac{\bar{A}\bar{R}^2}{3\bar{C}_d^3} \left(\frac{4\bar{C}_d^2 + 4\bar{X}_4 - 3\bar{X}_4}{4\bar{C}_d^2} \right) = \frac{\bar{A}\bar{R}^2}{3\bar{C}_d^3} \left(\frac{4\bar{C}_d^2 + \bar{X}_4}{4\bar{C}_d^2} \right) \\
& \quad \quad \quad = \frac{\bar{A}\bar{R}^2}{3\bar{C}_d^3} \left(\frac{\bar{C}_d^2 + \frac{1}{4}\bar{X}_4}{\bar{C}_d^2} \right) \\
& \quad \quad \quad \text{Note \#2:} \\
& \quad \quad \quad \frac{\bar{R}}{\sqrt{\bar{R}}} = \frac{\bar{R}}{\sqrt{\bar{R}}} \quad \frac{\bar{V}\bar{R}}{\sqrt{\bar{R}}} = \frac{\bar{R}\sqrt{\bar{R}}}{\bar{R}} = \sqrt{\bar{R}} \\
& \quad \quad \quad \frac{\bar{R}}{\sqrt{\bar{R}}} = \sqrt{\bar{R}} \\
& \quad \quad \quad \text{Note \#3:} \\
& \quad \quad \quad \epsilon = \frac{\bar{C}_{L\alpha}}{2\pi \mu \bar{C}_d^2}
\end{aligned}$$

Figure 13. Historical documentation connecting the stiffness cross-coupling analysis to the author's broader aeroelastic and flutter-compatibility work.

Compute the Flutter Velocity from the previous derivations:

$$\bar{S} = j\Omega \quad \text{where } \Omega = \omega/\omega_p$$

eqn (19) becomes:

$$a\Omega^4 + (b + \bar{V}_0^2 d)\Omega^2 + (f + \bar{V}_0^2 g) = 0 \quad ; \text{ solving for } \Omega^2.$$

$$\Omega^2 = \frac{-[b + \bar{V}_0^2 d] \pm \sqrt{[b + \bar{V}_0^2 d]^2 - 4a(f + \bar{V}_0^2 g)}}{2a} \quad (\text{See Derivation}) \longrightarrow$$

The on-set of Flutter occurs when the expression under the radical is zero.

$$(b + \bar{V}_0^2 d)^2 = 4a(f + \bar{V}_0^2 g)$$

Solve for V_F

$$b^2 + 2bd\bar{V}_0^2 + (\bar{V}_0^2)^2 d^2 = 4af + 4ag\bar{V}_0^2$$

$$d^2(\bar{V}_0^2)^2 + 2bd\bar{V}_0^2 - 4ag\bar{V}_0^2 + (b^2 - 4af) = 0$$

$$d^2(\bar{V}_0^2)^2 + (2bd - 4ag)\bar{V}_0^2 + (b^2 - 4af) = 0$$

$$d^2(\bar{V}_0^2)^2 + 2(bd - 2ag)\bar{V}_0^2 + (b^2 - 4af) = 0$$

$$\bar{V}_0 = \bar{V} \cos \Delta$$

$$a = d^2, \quad b = 2(bd - 2ag), \quad c = b^2 - 4af$$

$$\bar{V}_0^2 = \frac{-2(bd - 2ag) \pm \sqrt{[2(bd - 2ag)]^2 - 4d^2(b^2 - 4af)}}{2d^2}$$

$$\bar{V}_0^2 = \frac{-2(bd - 2ag) \pm \sqrt{4(b^2d^2 - 4adbg + 4a^2g^2) - 4d^2b^2 + 16ad^2f}}{2d^2}$$

$$\bar{V}_0^2 = \frac{-2(bd - 2ag) \pm \sqrt{4b^2d^2 - 16adbg + 16a^2g^2 - 4d^2b^2 + 16ad^2f}}{2d^2}$$

$$\bar{V}_0^2 = \frac{-2(bd - 2ag) \pm \sqrt{16a[-bdg + ag^2 + d^2f]}}{2d^2}$$

$$\bar{V}_0^2 = \frac{-2(bd - 2ag) \pm 4\sqrt{a[-bdg + ag^2 + d^2f]}}{2d^2}$$

$$\bar{V}_0^2 = \frac{-(bd - 2ag) \pm 2\sqrt{a[-bdg + ag^2 + d^2f]}}{d^2}$$

$$\bar{V}_F^2 = \frac{-(bd - 2ag) \pm 2\sqrt{a[-bdg + ag^2 + d^2f]}}{d^2 \cos^2 \Delta}$$

For purely sinusoidal Motion:

$$a\bar{S}^4 - b\bar{S}^2 - d\bar{V}_0^2\bar{S}^2 + f + \bar{V}_0^2g = 0$$

$$\bar{S} = j\Omega \quad \bar{S}^2 = (j\Omega)(j\Omega) = -\Omega^2$$

$$\bar{S}^4 = \Omega^4$$

$$a\Omega^4 + b\Omega^2 + d\bar{V}_0^2\Omega^2 + f + \bar{V}_0^2g = 0$$

$$a\Omega^4 + (b + d\bar{V}_0^2)\Omega^2 + (f + \bar{V}_0^2g) = 0$$

$$a(\Omega^2)^2 + (b + d\bar{V}_0^2)(\Omega^2) + (f + \bar{V}_0^2g) = 0$$

$$\Omega^2 = \frac{-(b + d\bar{V}_0^2) \pm \sqrt{(b + d\bar{V}_0^2)^2 - 4a(f + \bar{V}_0^2g)}}{2a}$$

Determinant of Stiffness Matrix:

$$[K] = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$$\Delta = K_{11}K_{22} - K_{21}K_{12} \quad K_{12} = K_{21}$$

$$\Delta = K_{11}K_{22} - K_{12}^2$$

$$\Delta = K_{11}K_{22} \left[1 - \frac{K_{12}^2}{K_{11}K_{22}} \right]$$

since Stiffness Matrix must be positive definite:

$$\frac{K_{12}^2}{K_{11}K_{22}} < 1$$

Stiffness Cross-Coupling parameter is:

$$\psi = \frac{-K_{12}}{\sqrt{K_{11}K_{22}}}, \quad -1 < \psi < 1$$

Figure 14. Final archival figure supporting the paper's historical note and showing the continuity between the original analysis effort and the corrected results presented here.