# Aeroelastic Analysis of Wing with Stiffness Cross-Coupling (Composite Wing Structure)

Moment due to Bending Aerodynamic Load

$M_\phi = N\, y_{cp}$

$C_\ell = \frac{N}{qS}$

$N = C_\ell\, qS = (C_{\ell_\alpha}\, \alpha)\, qs$

$M_\phi = C_\ell\, qS\, y_{cp}\, \alpha$

$M_\phi = qS\, C_{\ell_\alpha}\, y_{cp} \left[\theta \cos\Lambda - \phi \sin\Lambda\right] \frac{\cos\Lambda}{\cos\Lambda}$

$M_\phi = qS\, C_{\ell_\alpha}\, y_{cp} \left[\theta \frac{\cos^2\Lambda}{\cos\Lambda} - \phi \tan\Lambda \cos\Lambda\right] \frac{\cos^2\Lambda}{\cos^2\Lambda}$

$M_\phi = qS\, C_{\ell_\alpha}\, y_{cp} \cos^2\Lambda \left(\theta \frac{1}{\cos\Lambda} - \phi \tan\Lambda \frac{1}{\cos\Lambda}\right)$

$M_\phi = qS\, C_{\ell_\alpha} \frac{y_{cp}}{\cos\Lambda} \cos^2\Lambda\, (\theta - \phi \tan\Lambda)$

put Moment in terms of Swept Axis:

$M_\phi \cos\Lambda = qS\, C_{\ell_\alpha}\, y_{cp} \cos^2\Lambda\, (\theta - \phi \tan\Lambda)$

$* \quad M_\phi = qS\, C_{\ell_\alpha} \cos^2\Lambda\, (y_{cp}\,\theta - y_{cp}\,\phi \tan\Lambda)$

Put into Matrix Form (Matrix Equation)

$$\begin{Bmatrix} M_\theta \\ M_\phi \end{Bmatrix} = qS\, C_{\ell_\alpha} \cos^2\Lambda \begin{bmatrix} e & -e\tan\Lambda \\ y_{cp} & -y_{cp}\tan\Lambda \end{bmatrix} \begin{Bmatrix} \theta \\ \phi \end{Bmatrix}$$

The Aerodynamic Matrix is:

$$[A] = \begin{bmatrix} e & -e\tan\Lambda \\ y_{cp} & -y_{cp}\tan\Lambda \end{bmatrix}$$

The Aerodynamic Loads Equation can be expressed as:

$$\begin{Bmatrix} M_\theta \\ M_\phi \end{Bmatrix} = [A_{ij}] \begin{Bmatrix} \theta \\ \phi \end{Bmatrix} .$$

Equation of Motion For Vibration of the airfoil in the presence of an Arstream.

$\Sigma F = ma$

$ma - \Sigma F = 0$

$$\begin{bmatrix} mr_\alpha^2 & -mx_\alpha b/2 \\ -mx_\alpha b/2 & mb^2/3 \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} K_\theta \cos^2\gamma + K_\phi \sin^2\gamma & (K_\phi - K_\theta)\sin\gamma\cos\gamma \\ (K_\phi - K_\theta)\sin\gamma\cos\gamma & K_\theta \sin^2\gamma + K_\phi \cos^2\gamma \end{bmatrix} \begin{Bmatrix} \theta \\ \phi \end{Bmatrix} + qS\, C_{\ell_\alpha} \cos^2\Lambda \begin{bmatrix} -e & e\tan\Lambda \\ -y_{cp} & y_{cp}\tan\Lambda \end{bmatrix} \begin{Bmatrix} \theta \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \theta \\ \phi \end{Bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} \theta \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assume Harmonic Sinusoidal Motion:

$\theta = \bar{\theta} e^{st}$ $\quad \phi = \bar{\phi} e^{st}$ $\quad b/c = AR$ $\quad \omega_\phi = \sqrt{\frac{K_{11}}{M_{11}}}$

$\dot{\theta} = s\bar{\theta} e^{st}$ $\quad \dot{\phi} = s\bar{\phi} e^{st}$ $\quad \psi = \frac{-K_{12}}{\sqrt{K_{11} K_{12}}}$ $\quad ; R = K_{11}/K_{22}$

$\ddot{\theta} = s^2\bar{\theta} e^{st}$ $\quad \ddot{\phi} = s^2\bar{\phi} e^{st}$

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{Bmatrix} s^2\bar{\theta} e^{st} \\ s^2\bar{\phi} e^{st} \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \bar{\theta} e^{st} \\ \bar{\phi} e^{st} \end{Bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} \bar{\theta} e^{st} \\ \bar{\phi} e^{st} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} s^2 M_{11} & s^2 M_{12} \\ s^2 M_{21} & s^2 M_{22} \end{bmatrix} \begin{Bmatrix} \bar{\theta} \\ \bar{\phi} \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \bar{\theta} \\ \bar{\phi} \end{Bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} \bar{\theta} \\ \bar{\phi} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} s^2 M_{11} + K_{11} & s^2 M_{12} + K_{12} \\ s^2 M_{21} + K_{21} & s^2 M_{22} + K_{22} \end{bmatrix} \begin{Bmatrix} \bar{\theta} \\ \bar{\phi} \end{Bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} \bar{\theta} \\ \bar{\phi} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Substitution Variables For Mass/Stiffness Matrix:

a) $\bar{s}^2 = \frac{s^2}{\omega_\phi^2} = \frac{s^2}{1} \frac{M_{11}}{K_{11}} \Rightarrow \bar{s}^2 = s^2 \frac{M_{11}}{K_{11}}$ : assume $K_{11} = 1$ (Normalized to Torsion)

$\bar{s}^2 = s^2 M_{11}$

b) $\psi = \frac{-K_{12}}{\sqrt{K_{11} K_{22}}}$ and $R = \frac{K_{11}}{K_{22}}$

$$\frac{\psi}{\sqrt{R}} = \frac{-\frac{K_{12}}{\sqrt{K_{11} K_{22}}}}{\sqrt{\frac{K_{11}}{K_{22}}}} = \frac{-\frac{K_{12}}{\sqrt{K_{11} K_{22}}}}{\frac{\sqrt{K_{11}}}{\sqrt{K_{22}}}} \frac{\sqrt{K_{22}}}{\sqrt{K_{22}}} = -K_{12}$$

$\frac{\psi}{\sqrt{R}} = -K_{12}$

c) $R = \frac{K_{11}}{K_{22}} = \frac{1}{K_{22}}$ since $K_{11} = 1$

$K_{22} = \frac{1}{R}$

d) $M_{12} = -m x_\alpha \ell/2$

$M_{11} = m r_\alpha^2$

$s^2 = \frac{\bar{s}^2}{M_{11}} = \frac{\bar{s}^2}{m r_\alpha^2}$

$s^2 M_{12} = \left\{\frac{\bar{s}^2}{m r_\alpha^2}\right\}\left\{-m x_\alpha \ell/2\right\}$

$s^2 M_{12} = \bar{s}^2\left(-\frac{x_\alpha \ell}{2 r_\alpha^2}\right) = \bar{s}^2\left(\frac{-x_\alpha/c \;\; \ell/c}{2 r_\alpha^2/c^2}\right)$

$s^2 M_{12} = \bar{s}^2\left(\frac{-\bar{x}_\alpha \ell/c}{2\bar{r}_\alpha^2}\right)$

$s^2 M_{12} = \bar{s}^2\left(\frac{-\bar{x}_\alpha \overline{AR}}{2\bar{r}_\alpha^2}\right)$ where : $\overline{AR} = \ell/c$, $\bar{x}_\alpha = \frac{x_\alpha}{c}$ and $\bar{r}_\alpha = r_\alpha/c$

e) $s^2 M_{22} = \frac{\bar{s}^2}{m r_\alpha^2}\frac{m\ell^2}{3} = \frac{\bar{s}^2 \ell^2}{3 r_\alpha^2}$

$s^2 M_{22} = \frac{\bar{s}^2 \ell^2/c^2}{3 r_\alpha^2/c^2}$

$s^2 M_{22} = \frac{\bar{s}^2 \overline{AR}^2}{3\bar{r}_\alpha^2}$

f. $M_{11} = I$

Summary of Substitution Variables for Stiffness/Mass Matrix

$s^2 M_{11} = \bar{s}^2$ ; $K_{11} = 1$ ; $s^2 M_{12} = \bar{s}^2\left(\frac{-\bar{x}_\alpha \overline{AR}}{2\bar{r}_\alpha^2}\right)$ ; $K_{12} = -\frac{\Psi}{R}$

$K_{22} = \frac{1}{R}$ ; $s^2 M_{22} = \frac{\bar{s}^2 \overline{AR}^2}{3\bar{r}_\alpha^2}$

Substitution variables for Aerodynamic Matrix:

$$[A] = q S C_{\ell_\alpha} \cos^2\Lambda \begin{bmatrix} -e & e\tan\Lambda \\ -y_{cp} & y_{cp}\tan\Lambda \end{bmatrix}$$

$q = \frac{1}{2}\rho V^2$ ; $S = \ell c$ $\overline{AR} = \ell/c$

$$[A] = \tfrac{1}{2}\rho V^2(\ell c) C_{\ell_\alpha}\cos^2\Lambda \begin{bmatrix} -e & e\tan\Lambda \\ -y_{cp} & y_{cp}\tan\Lambda \end{bmatrix}$$

$\bar{V} = \frac{V}{\omega_p c/2}$ and $\omega_p = \sqrt{\frac{K_{11}}{M_{11}}}$ and $K_{11} = 1 \Rightarrow \omega_p = \frac{1}{\sqrt{M_{11}}}$

$V = \omega_p \frac{c}{2}\bar{V}$

$\therefore V = \frac{1}{\sqrt{M_{11}}}\frac{c}{2}\bar{V}$

$V^2 = \left(\frac{c}{2}\right)^2 \frac{\bar{V}^2}{M_{11}}$ ; $M_{11} = m r_\alpha^2$

$V^2 = \frac{c^2\bar{V}^2}{4 m r_\alpha^2} = \left(\frac{c}{2}\right)^2 \frac{\bar{V}^2}{m r_\alpha^2}$

$$[A] = \tfrac{1}{2}\rho\left[\left(\frac{c}{2}\right)^2\frac{\bar{V}^2}{m r_\alpha^2}\right] S C_{\ell_\alpha}\cos^2\Lambda \begin{bmatrix} -e & e\tan\Lambda \\ -y_{cp} & y_{cp}\tan\Lambda \end{bmatrix}\begin{Bmatrix}\bar{\theta}\\ \bar{\phi}\end{Bmatrix}$$

$$[A] = \tfrac{1}{2}\rho\left(\tfrac{c}{2}\right)\left(\tfrac{c}{2}\right)\frac{\bar{V}^2}{m r_\alpha^2} S C_{\ell_\alpha}\cos^2\Lambda \begin{bmatrix} -e & e\tan\Lambda \\ -y_{cp} & y_{cp}\tan\Lambda \end{bmatrix}\begin{Bmatrix}\bar{\theta}\\ \bar{\phi}\end{Bmatrix}$$

$$[A] = \tfrac{1}{2}(\bar{V}\cos\Lambda)^2 C_{\ell_\alpha}\frac{1}{\pi}\left\{\frac{\pi\rho\frac{c}{2}S}{2m}\right\}\frac{c}{r_\alpha^2}\frac{\frac{c^2}{4}}{c^2}\begin{bmatrix} -e & e\tan\Lambda \\ -y_{cp} & y_{cp}\tan\Lambda \end{bmatrix}\begin{Bmatrix}\bar{\theta}\\ \bar{\phi}\end{Bmatrix}$$

assume: $\mu = \frac{2m}{\pi\rho\frac{c}{2}S}$

$$[A] = \frac{(\bar{V}\cos\Lambda)^2 C_{\ell_\alpha}}{\pi\mu\frac{r_\alpha^2}{c^2}}\frac{1}{c}\begin{bmatrix} -e & e\tan\Lambda \\ -y_{cp} & y_{cp}\tan\Lambda \end{bmatrix}\begin{Bmatrix}\bar{\theta}\\ \bar{\phi}\end{Bmatrix}$$

$$[A] = \frac{(\bar{V}\cos\Lambda)^2}{\pi\mu\bar{r}_\alpha^2} C_{\ell_\alpha}\begin{bmatrix} -\frac{e}{c} & \frac{e}{c}\tan\Lambda \\ -\frac{y_{cp}}{\ell}\frac{\ell}{c} & \frac{y_{cp}}{\ell}\frac{\ell}{c}\tan\Lambda \end{bmatrix}\begin{Bmatrix}\bar{\theta}\\ \bar{\phi}\end{Bmatrix}$$ ; $\ell/c = \overline{AR}$, $\frac{e}{c} = \bar{e}$, $\frac{y_{cp}}{\ell} = \bar{y}$

($\frac{\ell}{c} = \overline{AR}$, $\frac{\ell}{c}\tan\Lambda = \overline{AR}\tan\Lambda$)

$$[A] = \frac{(\bar{V}\cos\Lambda)^2}{\pi\mu\bar{r}_\alpha^2} C_{\ell_\alpha}\begin{bmatrix} -\bar{e} & \bar{e}\tan\Lambda \\ -\bar{y}\,\overline{AR} & \bar{y}\,\overline{AR}\tan\Lambda \end{bmatrix}\begin{Bmatrix}\bar{\theta}\\ \bar{\phi}\end{Bmatrix}$$

The above is the reduced form of the Aerodynamic Matrix:

The Equation of Motion Becomes:

$$\begin{bmatrix} (\bar{s}^2+1) & \left(\bar{s}^2\left[\frac{-\bar{x}_\alpha\overline{AR}}{2\bar{r}_\alpha^2}\right]-\frac{\Psi}{\sqrt{R}}\right) \\ \left(\bar{s}^2\left[\frac{-\bar{x}_\alpha\overline{AR}}{2\bar{r}_\alpha^2}\right]-\frac{\Psi}{\sqrt{R}}\right) & \left(\frac{\bar{s}^2\overline{AR}^2}{3\bar{r}_\alpha^2}+\frac{1}{R}\right)\end{bmatrix}\begin{Bmatrix}\bar{\theta}\\ \bar{\phi}\end{Bmatrix} + \frac{1}{2}\left(\frac{(\bar{V}\cos\Lambda)^2 C_{\ell_\alpha}}{\pi\mu\bar{r}_\alpha^2}\right)\begin{bmatrix} -\bar{e} & \bar{e}\tan\Lambda \\ -\bar{y}\,\overline{AR} & \bar{y}\,\overline{AR}\tan\Lambda \end{bmatrix}\begin{Bmatrix}\bar{\theta}\\ \bar{\phi}\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

Equation 14

$$\begin{bmatrix} (\bar{s}^2+1) & (\bar{s}^2\left[-\frac{\bar{x}_\alpha \overline{AR}}{2\bar{r}_\alpha^2}\right]-\frac{\Psi}{\sqrt{R}}) \\ (\bar{s}^2\left[-\frac{\bar{x}_\alpha \overline{AR}}{2\bar{r}_\alpha^2}-\frac{\Psi}{\sqrt{R}}\right)(\frac{\bar{s}^2\overline{AR}^2}{3\bar{r}_\alpha^2}+\frac{1}{R}) \end{bmatrix} \begin{Bmatrix} \bar{\theta} \\ \bar{\phi} \end{Bmatrix} + \frac{1}{2}\left(\frac{(\bar{V}\cos\Lambda)^2 C_{l_\alpha}}{\mu \bar{r}_\alpha^2}\right) \begin{bmatrix} -\bar{e} & e\tan\Lambda \\ -\bar{y}\overline{AR} & \bar{y}\overline{AR}\tan\Lambda \end{bmatrix} \begin{Bmatrix} \bar{\theta} \\ \bar{\phi} \end{Bmatrix} = \begin{Bmatrix} \phi \\ \phi \end{Bmatrix}$$

$C_1 = -\frac{\bar{x}_\alpha \overline{AR}}{2\bar{r}_\alpha^2}$ , $C_2 = \frac{\Psi}{\sqrt{R}}$ , $C_3 = \frac{\overline{AR}^2}{3\bar{r}_\alpha^2}$ , $C_4 = \frac{1}{R}$ , $C_5 = \frac{1}{2}\left(\frac{(\bar{V}\cos\Lambda)^2 C_{l_\alpha}}{\mu \bar{r}_\alpha^2}\right)$ , $C_6 = -\bar{e}$

$C_7 = \bar{e}\tan\Lambda$ , $C_8 = -\bar{y}\overline{AR}$ , $C_9 = \bar{y}\overline{AR}\tan\Lambda$

$$\begin{bmatrix} (\bar{s}^2+1) & \bar{s}^2 C_1 - C_2 \\ \bar{s}^2 C_1 - C_2 & \bar{s}^2 C_3 + C_4 \end{bmatrix} \begin{Bmatrix} \bar{\theta} \\ \bar{\phi} \end{Bmatrix} + C_5 \begin{bmatrix} C_6 & C_7 \\ C_8 & C_9 \end{bmatrix} \begin{Bmatrix} \bar{\theta} \\ \bar{\phi} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency Form of The Equations:

$$\begin{bmatrix} (\bar{s}^2+1)+C_5 C_6 & \bar{s}^2 C_1 - C_2 + C_5 C_7 \\ \bar{s}^2 C_1 - C_2 + C_5 C_8 & \bar{s}^2 C_3 + C_4 + C_5 C_9 \end{bmatrix} \begin{Bmatrix} \bar{\theta} \\ \bar{\phi} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \; ; \; [FM] \begin{Bmatrix} \bar{\theta} \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

DET [Frequency Matrix] = 0

It has the following form:

$a\bar{s}^4 - b\bar{s}^2 - d\bar{V}_n \bar{s}^2 + f + \bar{V}_n g = 0$

Solve for $a, b, d, f$ and $g$.

$$\underbrace{\left[\bar{s}^2+1+C_5C_6\right]}_{(a)}\left[\bar{s}^2C_3+C_4+C_5C_9\right] - \underbrace{\left[\bar{s}^2C_1-C_2+C_5C_8\right]}_{(b)}\left[\bar{s}^2C_1-C_2+C_5C_7\right] = 0$$

a)
$$\begin{array}{l} \bar{s}^2C_3+C_4+C_5C_9 \\ \times\ \dfrac{\bar{s}^2+1+C_5C_6}{\bar{s}^4C_3+\bar{s}^2C_4+\bar{s}^2C_5C_9} \\ \qquad\qquad +\bar{s}^2C_3+C_4+C_5C_9 \\ +\bar{s}^2C_3C_5C_6+C_4C_5C_6+C_5^2C_6C_9 \end{array}$$

$$\frac{}{\bar{s}^4C_3+\bar{s}^2C_4+\bar{s}^2C_5C_9+\bar{s}^2C_3+\bar{s}^2C_3C_5C_6+C_4C_5C_6+C_5^2C_6C_9+C_5C_9+C_4}$$

b)
$$\begin{array}{l} \bar{s}^2C_1-C_2+C_5C_8 \\ \times\ \dfrac{\bar{s}^2C_1-C_2+C_5C_7}{\bar{s}^4C_1^2-\bar{s}^2C_1C_2+\bar{s}^2C_1C_5C_7} \\ \qquad\qquad -\bar{s}^2C_1C_2+C_2^2-C_2C_5C_8 \\ \bar{s}^2C_1C_5C_7-C_2C_5C_7+C_5^2C_7C_8 \end{array}$$

$$\frac{}{\bar{s}^4C_1^2-\bar{s}^2C_1C_2+\bar{s}^2C_1C_5C_7-\bar{s}^2C_1C_2+\bar{s}^2C_1C_5C_7-C_2C_5C_7-C_2C_5C_8+C_5^2C_7C_8+C_2^2}$$

Substituting:

$$\bar{s}^4C_3+\bar{s}^2C_4+\bar{s}^2C_5C_9+\bar{s}^2C_3+\bar{s}^2C_3C_5C_6+C_4C_5C_6+C_5^2C_6C_9+C_5C_9+C_4$$
$$-\bar{s}^4C_1^2+\bar{s}^2C_1C_2-\bar{s}^2C_1C_5C_9+\bar{s}^2C_1C_2-\bar{s}^2C_1C_5C_7+C_2C_5C_7+C_2C_5C_8-C_5^2C_7C_8-C_2^2=0$$

$$\bar{s}^4C_3-\bar{s}^4C_1^2+\left\{\bar{s}^2C_4+\bar{s}^2C_3+\bar{s}^2C_1C_4+\bar{s}^2C_1C_2\right\}+\left\{\bar{s}^2C_5C_9+\bar{s}^2C_3C_5C_6-\bar{s}^2C_1C_5C_7-\bar{s}^2C_1C_5C_8\right\}$$
$$+\left\{C_4+C_5C_9+C_5^2C_6C_9+C_4C_5C_6-C_5^2C_7C_8+C_2C_5C_8+C_2C_5C_7-C_2^2\right\}=0$$

$$\bar{s}^4\left\{C_3-C_1^2\right\}+\bar{s}\left\{C_4+C_3+2C_1C_2\right\}+\bar{s}^2\left\{C_5C_9+C_3C_5C_6-C_1C_5C_7-C_1C_5C_8\right\}$$
$$+\left\{C_5C_9+C_5^2C_6C_9+C_4C_5C_6-C_5^2C_7C_8+C_2C_5C_8+C_2C_5C_7\right\}$$
$$+\left\{C_4-C_2^2\right\}=0$$

$$\bar{s}^4\left\{\frac{\overline{AR}^2}{3\bar{r}_\alpha^2}-\frac{\bar{x}_\alpha^2\overline{AR}^2}{(2\bar{r}_\alpha^2)^2}\right\}+\bar{s}^2\left\{\frac{1}{R}+\frac{\overline{AR}^2}{3\bar{r}_\alpha^2}-\frac{\bar{x}_\alpha\overline{AR}}{\bar{r}_\alpha^2}\frac{\psi}{\sqrt{R}}\right\}+\bar{s}^2\left\{\frac{1}{2}\frac{\bar{V}_n^2C_{l_\alpha}}{\pi\mu\bar{r}_\alpha^2}\bar{y}\,\overline{AR}\tan\Lambda+\frac{\overline{AR}^2}{3\bar{r}_\alpha^2}\frac{1}{2}\frac{\bar{V}_n^2C_{l_\alpha}}{\pi\mu\bar{r}_\alpha^2}(-\bar{e})-\left(\frac{-\bar{x}_\alpha\overline{AR}}{2\bar{r}_\alpha^2}\right)\frac{1}{2}\frac{\bar{V}_n^2C_{l_\alpha}}{\pi\mu\bar{r}_\alpha^2}\bar{e}\tan\Lambda-\frac{\bar{x}_\alpha\overline{AR}}{2\bar{r}_\alpha^2}\frac{1}{2}\frac{\bar{V}_n^2C_{l_\alpha}}{\pi\mu\bar{r}_\alpha^2}(\bar{y}\,\overline{AR})\right\}-\left\{\frac{1}{R}-\left(\frac{\psi}{\sqrt{R}}\right)^2\right\}$$

see Note #1

$$\bar{s}^4\frac{\overline{AR}^2}{3\bar{r}_\alpha^2}\left\{\frac{\bar{r}_0^2+\frac{1}{4}\bar{x}_\alpha^2}{\bar{r}_\alpha^2}\right\}+\bar{s}^2\frac{1}{R}\left\{1+\frac{R\overline{AR}^2}{3\bar{r}_\alpha^2}-\frac{\bar{x}_\alpha R\overline{AR}\psi}{\bar{r}_\alpha^2\sqrt{R}}\right\}+\bar{V}_n^2\bar{s}^2\left(\frac{C_{l_\alpha}}{2\pi\mu\bar{r}_\alpha^2}\right)\overline{AR}\left\{\bar{y}\tan\Lambda-\frac{\overline{AR}}{3\bar{r}_\alpha^2}\bar{e}+\frac{\bar{x}_\alpha}{2\bar{r}_\alpha^2}\bar{e}\tan\Lambda-\frac{\bar{x}_\alpha}{2\bar{r}_\alpha^2}\bar{y}\,\overline{AR}\right\}$$

(a)

see Note #2

$$\bar{s}^2\frac{1}{R}\left\{1+\frac{R\overline{AR}^2}{3\bar{r}_\alpha^2}-\frac{\bar{x}_\alpha\overline{AR}\sqrt{R}\psi}{\bar{r}_\alpha^2}\right\}+\bar{V}_n^2\bar{s}^2\left(\frac{C_{l_\alpha}}{2\pi\mu\bar{r}_\alpha^2}\right)\overline{AR}\left\{\bar{y}\tan\Lambda-\frac{\overline{AR}}{3\bar{r}_\alpha^2}\bar{e}+\frac{\bar{x}_\alpha}{2\bar{r}_\alpha^2}(\bar{e}\tan\Lambda-\bar{y}\,\overline{AR})\right\}$$

(b) (d)

see Note #3

$\varepsilon$

$$+\left(\frac{1}{2}\frac{\tilde{V}_n^2C_{L_\alpha}}{\pi\mu\bar{r}_\alpha^2}\right)\bar{y}\,AR\tan\Lambda-\left(\frac{1}{2}\frac{\tilde{V}_n^2C_{L_\alpha}}{\pi\mu\bar{r}_\alpha^2}\right)^2\bar{e}\,y\,AR\tan\Lambda-\frac{1}{R}\left(\frac{1}{2}\frac{\tilde{V}_n^2C_{L_\alpha}}{\pi\mu\bar{r}_\alpha^2}\right)\bar{e}+\left(\frac{1}{2}\frac{\tilde{V}_n^2C_{L_\alpha}}{\pi\mu\bar{r}_\alpha^2}\right)^2\bar{e}\tan\Lambda\,y\,AR-\frac{\psi}{\sqrt{R}}\left(\frac{1}{2}\frac{\tilde{V}_n^2C_{L_\alpha}}{\pi\mu\bar{r}_\alpha^2}\right)y\,AR+\frac{\psi}{\sqrt{R}}\left(\frac{1}{2}\frac{\tilde{V}_n^2C_{L_\alpha}}{\pi\mu\bar{r}_\alpha^2}\right)*\bar{e}\tan\Lambda$$

$$\left(\frac{1}{2}\frac{\bar{V}_n^2C_{l_\alpha}}{\pi\mu\bar{r}_\alpha^2}\right)\bar{y}\,\overline{AR}\tan\Lambda-\frac{1}{R}\left(\frac{1}{2}\frac{\bar{V}_n^2C_{l_\alpha}}{\pi\mu\bar{r}_\alpha^2}\right)\bar{e}-\frac{\psi}{\sqrt{R}}\left(\frac{1}{2}\frac{\bar{V}_n^2C_{l_\alpha}}{\pi\mu\bar{r}_\alpha^2}\right)\bar{y}\,\overline{AR}+\frac{\psi}{\sqrt{R}}\left(\frac{1}{2}\frac{\bar{V}_n^2C_{l_\alpha}}{\pi\mu\bar{r}_\alpha^2}\right)\bar{e}\tan\Lambda$$

$$\bar{V}_n^2\cdot\frac{C_{l_\alpha}}{2\pi\mu\bar{r}_\alpha^2}\left\{\bar{y}\,\overline{AR}\tan\Lambda-\frac{1}{R}\bar{e}+\frac{\psi}{\sqrt{R}}\left(\bar{e}\tan\Lambda-\bar{y}\,\overline{AR}\right)\right\}$$

$$\bar{V}_n^2\cdot\frac{C_{l_\alpha}}{2\pi\mu\bar{r}_\alpha^2}\left\{-\frac{1}{R}\bar{e}+\bar{y}\,\overline{AR}\tan\Lambda+\frac{\psi}{\sqrt{R}}(\bar{e}\tan\Lambda-\bar{y}\,\overline{AR})\right\}$$

$$\bar{V}_n^2\cdot\frac{C_{l_\alpha}}{2\pi\mu\bar{r}_\alpha^2}\left\{\frac{1}{R}\left[-\bar{e}+R\,\bar{y}\,\overline{AR}\tan\Lambda+\psi\frac{R}{\sqrt{R}}(\bar{e}\tan\Lambda-\bar{y}\,\overline{AR})\right]\right\}$$

$$\bar{V}_n^2\cdot\frac{C_{l_\alpha}}{2\pi\mu\bar{r}_\alpha^2}\left\{\frac{1}{R}\left[-\bar{e}+R\,\bar{y}\,\overline{AR}\tan\Lambda+\psi\sqrt{R}\,(\bar{e}\tan\Lambda-\bar{y}\,\overline{AR})\right]\right\}$$

$$\bar{V}_n^2\,\varepsilon\left\{\frac{1}{R}\left[-\bar{e}+R\,\bar{y}\,\overline{AR}\tan\Lambda+\psi\sqrt{R}\,(\bar{e}\tan\Lambda-\bar{y}\,\overline{AR})\right]\right\}+\frac{1}{R}-\left(\frac{\psi}{\sqrt{R}}\right)^2$$

(g)

$$\frac{1}{R}-\frac{\psi^2}{R}$$

$$(1-\psi^2)/R=0$$

using $a\bar{s}^4-b\bar{s}^2-d\bar{V}_n^2\bar{s}^2+\bar{V}_n^2g+f=0$ the coefficients are given as:

$$a=\frac{\overline{AR}^2}{3\bar{r}_\alpha^2}\left\{\frac{\bar{r}_0^2+\frac{1}{4}\bar{x}_\alpha^2}{\bar{r}_\alpha^2}\right\}$$

$$b=-\frac{1}{R}\left\{1+\frac{R\overline{AR}^2}{3\bar{r}_\alpha^2}-\frac{\bar{x}_\alpha\overline{AR}\sqrt{R}\psi}{\bar{r}_\alpha^2}\right\}$$

$$d=-\varepsilon\,\overline{AR}\left\{\bar{y}\tan\Lambda-\frac{\overline{AR}}{3\bar{r}_\alpha^2}\bar{e}+\frac{\bar{x}_\alpha}{2\bar{r}_\alpha^2}(\bar{e}\tan\Lambda-\bar{y}\,\overline{AR})\right\}$$

$$g=\varepsilon\left\{\frac{1}{R}\left[-\bar{e}+R\,\bar{y}\,\overline{AR}\tan\Lambda+\psi\sqrt{R}\,(\bar{e}\tan\Lambda-\bar{y}\,\overline{AR})\right]\right\}$$

$$f=(1-\psi^2)/R \quad \text{and} \quad \varepsilon=\frac{C_{l_\alpha}}{2\pi\mu\bar{r}_\alpha^2}$$

Note #1:

$$\frac{\overline{AR}^2}{3\bar{r}_\alpha^2}-\frac{\bar{x}_\alpha^2\overline{AR}^2}{(2\bar{r}_\alpha^2)^2}=\frac{\overline{AR}^2}{3\bar{r}_\alpha^2}-\frac{3\bar{x}_\alpha^2\overline{AR}^2}{3(2\bar{r}_\alpha^2)^2}=\frac{\overline{AR}^2}{3\bar{r}_\alpha^2}\left(1-\frac{3\bar{x}_\alpha^2}{(2)^2\bar{r}_\alpha^2}\right);\quad \bar{r}_\alpha^2=\bar{r}_0^2+\bar{x}_\alpha^2$$

$$=\frac{\overline{AR}^2}{3\bar{r}_\alpha^2}\left(\frac{4\bar{r}_\alpha^2}{4\bar{r}_\alpha^2}-\frac{3\bar{x}_\alpha^2}{4\bar{r}_\alpha^2}\right)=\frac{\overline{AR}^2}{3\bar{r}_\alpha^2}\left(\frac{4\bar{r}_\alpha^2-3\bar{x}_\alpha^2}{4\bar{r}_\alpha^2}\right)=\frac{\overline{AR}^2}{3\bar{r}_\alpha^2}\left(\frac{4(\bar{r}_0^2+\bar{x}_\alpha^2)-3\bar{x}_\alpha^2}{4\bar{r}_\alpha^2}\right)$$

$$=\frac{\overline{AR}^2}{3\bar{r}_\alpha^2}\left(\frac{4\bar{r}_0^2+4\bar{x}_\alpha^2-3\bar{x}_\alpha^2}{4\bar{r}_\alpha^2}\right)=\frac{\overline{AR}^2}{3\bar{r}_\alpha^2}\left(\frac{4\bar{r}_0^2+\bar{x}_\alpha^2}{4\bar{r}_\alpha^2}\right)$$

$$=\frac{\overline{AR}^2}{3\bar{r}_\alpha^2}\left(\frac{\bar{r}_0^2+\frac{1}{4}\bar{x}_\alpha^2}{\bar{r}_\alpha^2}\right)$$

Note #2:

$$\frac{R}{\sqrt{R}}=\frac{R}{\sqrt{R}}\frac{\sqrt{R}}{\sqrt{R}}=\frac{R\sqrt{R}}{R}=\sqrt{R}$$

$$\frac{R}{\sqrt{R}}=\sqrt{R}$$

Note #3:

$$\varepsilon=\frac{C_{l_\alpha}}{2\pi\mu\bar{r}_\alpha^2}$$

5.

Compute the Flutter Velocity From the previous derivation:

$\bar{s} = i\Omega$ where $\Omega = \omega/\omega_\theta$

eqn (19) becomes:

$a\Omega^4 + (b + \bar{V}_n^2 d)\Omega^2 + (f + \bar{V}_n^2 g) = 0$ ; solving for $\Omega$.

$\Omega^2 = \{-(b + \bar{V}_n^2 d) \pm [(b + \bar{V}_n^2 d)^2 - 4a(f + \bar{V}_n^2 g)]^{1/2}\}/2a$ (See Derivation) →

The on-set of Flutter occurs when the expression under the radical is zero.

$(b + \bar{V}_n^2 d)^2 = 4a(f + \bar{V}_n^2 g)$

Solve for $V_F$

$b^2 + 2bd\bar{V}_n^2 + (\bar{V}_n^2)^2 d^2 = 4af + 4ag\bar{V}_n^2$

$d^2(\bar{V}_n^2)^2 + 2bd\bar{V}_n^2 - 4ag\bar{V}_n^2 + (b^2 - 4af) = 0$

$d^2(\bar{V}_n^2)^2 + (2bd - 4ag)\bar{V}_n^2 + (b^2 - 4af) = 0$

$d^2(\bar{V}_n^2)^2 + 2(bd - 2ag)\bar{V}_n^2 + (b^2 - 4af) = 0$

$\bar{V}_n = \bar{V}\cos\Lambda$

$a = d^2, \quad b = 2(bd - 2ag), \quad c = b^2 - 4af$

$$\bar{V}_N^2 = \frac{-2(bd - 2ag) \pm \sqrt{[2(bd - 2ag)]^2 - 4d^2(b^2 - 4af)}}{2d^2}$$

$$\bar{V}_n^2 = \frac{-2(bd - 2ag) \pm \sqrt{4(b^2d^2 - 4abdg + 4a^2g^2) - 4d^2b^2 + 16ad^2f}}{2d^2}$$

$$\bar{V}_n^2 = \frac{-2(bd - 2ag) \pm \sqrt{4b^2d^2 - 16abdg + 16a^2g^2 - 4d^2b^2 + 16ad^2f}}{2d^2}$$

$$\bar{V}_n^2 = \frac{-2(bd - 2ag) \pm \sqrt{16a[-bdg + ag^2 + d^2f]}}{2d^2}$$

$$\bar{V}_n^2 = \frac{-2(bd - 2ag) \pm 4\sqrt{a[-bdg + ag^2 + d^2f]}}{2d^2}$$

$$\bar{V}_N^2 = \frac{-(bd - 2ag) \pm 2[a(-bdg + ag^2 + d^2f)]^{1/2}}{d^2}$$

$$\bar{V}_F^2 = \frac{-(bd - 2ag) \pm 2[a(-bdg + ag^2 + d^2f)]^{1/2}}{d^2\cos^2\Lambda}$$

For purely Sinusoidal Motion:

$a\bar{s}^4 - b\bar{s}^2 - d\bar{V}_n^2\bar{s}^2 + f + \bar{V}_n^2 g = 0$

$\bar{s} = i\Omega$

$\bar{s}^2 = (i\Omega)(i\Omega) = -\Omega^2$

$\bar{s}^4 = \Omega^4$

$a\Omega^4 + b\Omega^2 + d\bar{V}_n^2\Omega^2 + f + \bar{V}_n^2 g = 0$

$a\Omega^4 + (b + d\bar{V}_n^2)\Omega^2 + (f + \bar{V}_n^2 g) = 0$

$a(\Omega^2)^2 + (b + d\bar{V}_n^2)(\Omega^2) + (f + \bar{V}_n^2 g) = 0$

$$\Omega^2 = \frac{-(b + d\bar{V}_n^2) \pm \sqrt{(b + d\bar{V}_n^2) - 4a(f + \bar{V}_n^2 g)}}{2a}$$

Determinant of Stiffness Matrix:

$$[K] = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$\Delta = K_{11}K_{22} - K_{21}K_{12}$ $\qquad K_{12} = K_{21}$

$\Delta = K_{11}K_{22} - K_{12}^2$

$$\Delta = K_{11}K_{22}\left[1 - \frac{K_{12}^2}{K_{11}K_{22}}\right]$$

since Stiffness Matrix must be positive definite:

$$\frac{K_{12}^2}{K_{11}K_{22}} < 1$$

Stiffness Cross-Coupling parameter is:

$$\psi = \frac{-K_{12}}{\sqrt{K_{11}K_{22}}}, \quad -1 < \psi < 1$$