

Amplitude-Dependent Modes for VIV Fatigue of ZRB Pipeline Spans

Ralf Peek^a and Mohammed Abdelsalam^b

^a Peek Solutions, Seattle, WA, USA; ^b School of Earth and Oceans, The University of Western Australia, Perth, Australia

Abstract

Thermal expansion of subsea pipelines often leads to lateral buckling. Zero-Radius Bend (ZRB) buckle triggers were invented by the first author around 2004, to ensure that lateral buckles form at a controlled spacing, so that no single lateral buckle absorbs the thermal expansion of an excessive length of pipeline and thereby becomes overstressed. (Not only maximum stresses and strains but also fatigue loading at lateral buckles must be addressed.) Since then, ZRB buckle triggers have been extensively used in the offshore pipelines and flowlines industry and proved very successful. However, the trigger also generates a span to each side of the support that can be subject to Vortex-Induced Vibrations (VIV). This paper addresses the nonlinearity and inelasticity in the in-line VIV response of such spans.

Although it is long been recognized that nonlinearities and inelasticity of the soil are important at typical VIV amplitudes, the current practice for VIV fatigue assessment is based on linear behavior, so that modal analysis and empirical VIV response functions can be used to estimate the amplitude of the response of each mode to VIV. Without abandoning this approximation, this paper provides a way to improve it by a solution-dependent linearization, leading to a nonlinear eigenvalue problem for the amplitude-dependent modes and vibration frequencies, for which an efficient algorithm is described and applied.

Geometric nonlinearity of the pipe response is accounted for as well as the nonlinearity of the springs used to model pipe-soil interaction. The inelastic energy dissipation of the soil springs and ZRB support are used to calculate amplitude-dependent modal damping ratios. The accuracy of the linearization is verified by time-domain analysis.

A limitation of the method is that it does not capture the nonlinear interactions that can develop between closely-spaced modes. However, it works well for the pure in-line VIV response of the ZRB span considered, and shows a strong effect the nonlinearity and inelasticity for that case.

Keywords: Pipeline; free span, VIV; modal analysis; amplitude-dependent modes; ZRB; lateral buckle; dynamics; nonlinear eigenvalue; inelasticity; damping.

1 Nomenclature

All symbols are defined where they are first used but for convenience key notation is also summarized here.

Symbols

A	= amplitude of vibration; y or z component of the dynamic displacements in $\mathbf{u}_d$ that is largest in absolute value; the larger of A_y and A_z
A_y, A_z	= maximum lateral, and vertical vibration amplitude over all locations x
C_a	= added-mass coefficient
D	= diameter of pipe including coatings
E_d	= energy dissipation per cycle
EA_p	= axial rigidity of the pipe
EI	= flexural rigidity of the pipe
f	= natural frequency of free vibrations
f_e	= frequency of excitation

L	= spacing of ZRB triggers, also spacing between “virtual anchor points” corresponding to the midpoints between triggers where displacements during operation are assumed to be zero
$\mathbf{M}$	= mass matrix (diagonal)
N_{lay}	= lay tension
t	= time
U	= current velocity at the elevation of the pipe centroid
$\mathbf{u}$	= column vector containing the nodal displacements (i.e. total static + dynamic parts)
$\mathbf{u}_d$	= column vector containing the dynamic nodal displacements (i.e. the increment in displacement from the reference static equilibrium configuration (RSEC) that varies harmonically) as in Eq. (2) for each node
u_d, v_d, w_d	= components of the dynamic displacements $\mathbf{u}_d$ at a given location in the x, y, z directions, respectively, except that in Figure 5 and Eq. (10), u_d represents an axial or transverse component of displacement
x, y, z	= cartesian coordinates in the axial, lateral and vertical upwards directions (Figure 3)
α	= angle of deviation of pipe lay from mean direction, due to zig-zag pipelay from trigger to trigger
ϕ	= mode shape for infinitesimal-amplitude vibrations (column vector containing displacement components for all nodes in the finite element model)
ψ	= the elements of the column vector ψ are the same as those for ϕ for the degrees of freedom representing lateral (y) displacements, and zero for all other degrees of freedom.
ξ	= material or component damping ratio
ζ	= modal damping ratio or contribution of a component to the modal damping ratio
<i>Subscripts</i>	
$(.)_0$	= value of $(.)$ at the start of an increment used to track the solution branch
$(.)_b$	= value of $(.)$ at bifurcation point; corresponds to infinitesimal-amplitude vibration mode
$(.)_n$	= value of $(.)$ obtained by tracking the bifurcated solution branch starting at the bifurcation point in the negative ϕ direction, i.e. in the direction opposite to ϕ
$(.)_p$	= value of $(.)$ obtained by tracking the bifurcated solution branch starting at the bifurcation point in the ϕ direction
$(.)_{\text{rsec}}$	= value of $(.)$ at the reference static equilibrium configuration (RSEC) about which vibrations occur
$\ (\cdot)\ _{\infty}$	= maximum absolute value of all elements of the vector $(\cdot)$

Abbreviations

ADM	Amplitude-Dependent Modes or Modal (e.g. ADM properties refers to amplitude dependent mode shape, natural frequency, modal damping ratio, and stress ranges)
CFD	Computational Fluid Dynamics
CF	Cross-flow
CRA	Corrosion Resistance Alloy
DNS	Direct solution of Navier-Stokes equations
DNV	Det Norske Veritas
FE	Finite Element
IL	In-line
OOS	Out Of Straightness of the pipe profile, can also refer to the OOS of the profile projected onto the vertical (x - z) or horizontal (x - y) plane
PSI	Pipe-Soil Interaction
RSEC	Reference Static Equilibrium Configurations about which free vibrations take place
RP	Recommended Practice
TDZ	Touch-Down Zone where the vertical PSI force differs from the weight of the pipe
VIV	Vortex Induced Vibrations
ZRB	Zero Radius Bend

2 Introduction

Spans are ubiquitous in subsea pipelines, and they can become a threat to integrity if they suffer fatigue due to vortex induced vibrations (VIV). They arise from pipelay on an uneven seabed or on a buckle trigger, but they can also form and/or grow by scour. This calls for regular monitoring, and remediation when required, typically by installing intermediate supports, or by using strakes to suppress VIV.

The importance of nonlinearity in the dynamic behavior of spans was already well-recognized 33 years ago [1]¹. Since then, much effort has gone into addressing this problem for risers as well as pipelines, but for pipelines the impact on the engineering practice has been limited. In principle accounting for nonlinearities by time-domain simulations is straight forward, but it is difficult to find phenomenological empirical or semi empirical models that accurately capture the pipe-water and pipe-soil interactions (PSI). CFD does provide an alternative to such phenomenological models, but it is computationally very demanding and relies on engineering approximations to capture sub-grid turbulence.

One source of nonlinearity and inelastic energy dissipation is the Pipe-Soil Interaction (PSI). This is generally modelled with soil springs. A “Recommended Practice” for the spring stiffnesses may be found in [2] together with estimates of the contribution to the modal damping ratio from “soil damping”. These estimates are neither amplitude- nor frequency-dependent, and there is no reference to any theory or experiments on which those estimates are based. However, in [3,4], it is found that the spring properties should be both amplitude- and frequency-dependent, and involve radiation damping as well as material damping from inelasticity of the soil. Even at the low amplitudes of pluck tests on an operating pipeline, significant amplitude dependence of the natural frequency was observed [5], and attributed to nonlinear soil behavior. The PSI work in [6,7] appears extensive, but with limited reporting in the open literature. More recently, [8] investigates the drained PSI in a calcareous sand by centrifuge testing, and [9] a time-domain model for the vertical PSI at the touch down point for a catenary riser.

If computational fluid dynamics (CFD) is used, as in [10-19], the structural and seabed nonlinearity could readily be included, since the analysis is in the time domain. This appears² to have been done in [17] to account for the nonlinearity associated with loss of contact with the seabed. However, CFD is computationally demanding. A typical VIV fatigue assessment involves considering a large number of different metocean states. To do that with CFD would require a CFD simulation for each one of them. Further, approximations are needed to capture sub-grid turbulence, as direct solution of the Navier-Stokes equations (DNS) resolving the smallest scales of turbulence remains unfeasible for typical pipe VIV problems. This calls for validation of CFD results against experimental results.

More efficient, but approximate alternatives to CFD, are (semi) empirical models formulated in the time domain, so that they also allow structural and soil nonlinearity to be included: This includes wake oscillator models [20-35] as well as models in [36-48] that build upon the work of Thorsen [36,38].

The wake oscillator models are reviewed in [34,35]. The idea appears to have arisen in 1957 [20] and been further articulated in [21]. Although a variety of wake oscillators have been put forward, following the conclusion of Faccinetti et. al. [28] that “inertial coupling” of the wake with the oscillating structure provided a better fit to VIV test results, the wake-oscillator models are almost (but not quite) equivalent to having a mass attached to the pipe via a linear spring and a nonlinear dashpot. The dashpot has negative damping up to a certain amplitude, as in a Van der Pol oscillator. This excites the VIV but the damping turns positive if that amplitude is exceeded. The attached mass represents the wake and the hydrodynamic force exerted on the pipe.

¹ Numbers in square brackets refer to those listed in the “References” section.

² The word “appears” is used because the results in [17] strongly suggest that the springs used to model the seabed resist only downward and not upward forces, but the description of the model therein does not make this clear.

In the last 10 years, a class of time-domain empirical VIV models based on the work of Thorsen et al. [36,38] has received considerable attention [36-48], mostly in Norway. They are implemented in VIVANA-TD [44]. The pipe motion is represented by an instantaneous phase angle ϕ_{pipe} and an amplitude, both of which vary as a function of time. The corresponding phase angle and amplitude of the hydrodynamic force due to vortex shedding is then calculated from intuitively-motivated and empirically-calibrated relations. The models are relatively simple, and like any time-domain model can readily be used in a nonlinear structural model.

A challenge in the Thorsen type models is the calculation of the instantaneous phase angle ϕ_{pipe} and amplitude for the pipe velocity. This cannot be done by means of the Hilbert transform, because the instantaneous phase is needed at every increment before the entire time-series has been calculated. Instead, the “phase portrait”, described in [36,38] is used. This amounts to assuming the pipe motion is harmonic in order to determine the phase at time t from the velocity and acceleration values at time t , with the amplitudes of the velocity and acceleration taken as the maximum absolute values in the time interval $\tau \in [t-5/f_{\text{st}}, t]$, where f_{st} denotes the Strouhal frequency.

The VIV excitation force in the Thorsen models is then calculated using a lift coefficient to get the amplitude of the excitation force, and the phase angle ϕ_{exc} , defined by a first order differential equation of the form $\dot{\phi}_{\text{exc}} = 2\pi f_{\text{st}} + H(\phi_{\text{pipe}} - \phi_{\text{exc}})$, where a dot denotes the time derivative, the function $H(\cdot)$ satisfies $H(0)=0$, and is monotonically increasing on $[-\pi/2, \pi/2]$. Thus, in that range of phase difference, ϕ_{exc} moves towards ϕ_{pipe} , but it can only move fast or slow enough to get closer if the pipe frequency $\dot{\phi}_{\text{pipe}}/(2\pi)$ is within a specified range of excitation frequencies for which lock-in is observed in VIV tests. Even within the excitation frequency range, the phase difference does not converge to zero unless the structural frequency is equal to the Strouhal frequency. This affects the energy input from the hydrodynamic force, which is highest when $\phi_{\text{exc}} = \phi_{\text{pipe}}$ is maintained at all times.

Finally, Morison’s equation is used for hydrodynamic damping and added-mass effects. In what appears to be its simplest form [40], the Thorsen model has 6 parameters, which also define the function $H(\cdot)$, and the amplitude of the lift force does not depend on the amplitude of the structural motion. However, different versions of the model are used in [36-48], e.g. by adding an oscillating force in the in-line direction with its own phase angle [47], or adjusting the function $H(\cdot)$ to match the lift force from forced vibration tests at a given amplitude [36].

A time-domain model that is closer to CFD is the vortex tracking model of Sarpkaya and Schoaff [49]. Like CFD, this model captures the creation and evolution of vortices and their effect on the pipeline, though it does so in a simplified, approximate manner based on the inviscid, irrotational flow solution for vortices in the neighborhood of a cylinder. With some modifications, this model has been implemented in Orcflex [30]. Although this concept seems appealing and originated from considerable insights and experience with such flows, we find the model difficult to understand³, and it did not fare well in the blind testing of VIV models against test results reported in [51], as it tended to overpredict VIV. This was confirmed by Orcina [52], who also indicated that this model is not being used by their clients. Other time-domain models have been implemented in ABAQUS and named “ABAVIV” [53,54].

All of the time-domain models are quite powerful: even if initially applied only to simple linear one or two degrees of freedom systems, they can also be applied to multi-degree-of-freedom flexible pipes, structural and soil nonlinearity, and changing flow velocity, as arises from wave effects. However, the effort to validate the models for such a wide set of applications is large. Although the total validation effort in [10-49] is indeed large, inevitably, the proliferation of models dilutes such validation efforts. If reliable guidance is developed to indicate which time-domain model to use, time domain models will become more accepted in engineering practice.

³ For instance, it appears that the irrotational flow solution in Eq. 8 of [49] does not satisfy the flow velocity boundary condition at the surface of the cylinder. The same equation is also stated in [50]. Whereas the flow field used may still be a reasonable approximation, it is difficult to ascertain this.

Meanwhile, most span assessments are based on the empirical response functions in [55], for which linear dynamic behavior is assumed to calculate the mode shapes and natural frequencies that do not depend on the amplitude of vibration. Without leaving this framework, it is possible to account for nonlinearity approximately by allowing the mode shapes and natural frequencies to depend on the amplitude of vibration. This is the approach taken in [58] and here.

The approach still involves linearization, but the linearization is based on the secant stiffness, which is amplitude dependent. This leads to a nonlinear eigenvalue problem for the mode shapes and natural frequencies. The eigenproblem has the same form as a bifurcation buckling problem. Thus, it can be solved with the same algorithm [56,57] used to track the bifurcated solution branches that describe the postbuckling behavior of a structure.

Compared to the in line VIV response problem of [58], the ZRB span problem addressed here includes the following additional challenges: both the dynamic and static parts of the solution need to be included; there are asymmetries in the dynamic structural resistances, e.g. due to loss of contact with the seabed; the dynamic response for all 3 coordinate directions is important instead of only the lateral response; geometric nonlinearity in the pipe response is included; finally, there are closely-spaced in-line (IL) and cross-flow (CF) modes that can interact nonlinearly.

3 ADM Theory and Approximations

The method of [58] is used to solve the nonlinear eigenvalue problem that gives the Amplitude-Dependent Modes (ADMs) for the span vibrations. However, for the ZRB span considered here, the following complicating factors arise: (a) The solution for the Reference Static Equilibrium Configuration (RSEC) about which the vibrations occur is non-trivial: it involves simulating pipelay, and the operating cycles (temperature and pressure) that can have an influence on the RSEC. (b) Nonlinear soil springs are needed for all three coordinate directions rather than just for the lateral directions, and springs are also needed to model the ZRB sleeper, and post. For simplicity, coupling between the springs in different directions (axial, lateral, and vertical) is neglected. (c) Geometric nonlinearity is also included and becomes important due to the axial restraint from the soil and the ZRB sleeper. Finally (d), the forces that tend to restore the RSEC are not necessarily an antisymmetric function of the dynamic displacements. For example, upward displacement can involve loss of contact with the seabed, which is not the case for downward displacement. As a result, the dynamic properties for the half cycle to one side of the RSEC may differ from those for the half cycle to the other side.

The finite element discretization used is based on the (geometrically nonlinear) Von Kármán type beam theory for moderately large deflections with shear deformations included. The soil and ZRB sleeper and post are represented by uncoupled nonlinear, inelastic springs in the axial, lateral and vertical directions. The pipe is assumed to remain elastic, and thus is fully characterized by its flexural stiffness EI , axial stiffness EA_p , and shear stiffness GA_s , which is calculated using $G = E/[2(1+\nu)]$, and $A_s = A_p/2$, where E is Young's modulus for the pipe steel, ν Poisson's ratio, and A_p the cross-sectional area of the pipe.

In reality the springs to represent the soil and the ZRB sleeper and post are inelastic, allowing for frictional slip, but for simplicity the theory is developed first for nonlinear but elastic behavior.

Under these assumptions, the nonlinear equations that lead to the solution for the Reference Static Equilibrium Configuration (RSEC) can be written as

$$\mathbf{s}(\mathbf{u}) = 0 \quad \text{for } \mathbf{u} = \mathbf{u}_{\text{rsec}} \quad (1)$$

where $\mathbf{s}$ is the vector containing the nodal out-of-balance force and moments, $\mathbf{u}$ is the vector containing the modal displacements and rotations, and $\mathbf{u}_{\text{rsec}}$ is the value of $\mathbf{u}$ for the static equilibrium configuration.

The dynamic free vibrations response of the linearized system is assumed to be of the form

$$\mathbf{u} = \mathbf{u}_{\text{rsec}} + \mathbf{u}_d \sin(2\pi ft) \quad (2)$$

where $\mathbf{u}_d$ represents the dynamic part of the displacement, and f denotes the natural frequency of the mode considered. The momentum balance condition at the time when $\sin(2\pi ft) = 1$ (i.e. at maximum amplitude of vibration) can then be written as

$$\mathbf{s}(\mathbf{u}_{\text{rsec}} + \mathbf{u}_d) = \lambda \mathbf{M} \mathbf{u}_d \quad (3)$$

in which $\lambda = (2\pi f)^2$ is the eigenvalue and $\mathbf{M}$ the mass matrix. For simplicity, a diagonal, lumped mass matrix is used. Inertia is included only on the translational (not rotational) degrees of freedom. In the axial direction the mass per unit length m_a corresponds to the mass per unit length of the pipe, coatings and content, whereas in the transverse (lateral or vertical) direction, the effective mass m_t used also includes the hydrodynamic added-mass with and added-mass coefficient of 1.

Eq. (3) is the nonlinear eigenvalue problem to be solved. It can be rewritten as

$$\mathbf{s}_d(\mathbf{u}_d, \lambda) = 0 \quad (4)$$

where

$$\mathbf{s}_d(\mathbf{u}_d, \lambda) = \mathbf{s}(\mathbf{u}_{\text{rsec}} + \mathbf{u}_d) - \lambda \mathbf{M} \mathbf{u}_d \quad (5)$$

For any λ , $(\mathbf{u}_d, \lambda) = (\mathbf{0}, \lambda)$ is a solution to Eq. (4), since by static equilibrium [Eq. (1)], $\mathbf{s}(\mathbf{u}_{\text{rsec}}) = \mathbf{0}$. This is the principal solution branch. At certain values of λ denoted by $\lambda = \lambda_b$ the principal branch is crossed by bifurcated solution branches. To find these, the linear eigenvalue problem

$$(\mathbf{K}_T - \lambda_b \mathbf{M}) \boldsymbol{\phi} = 0 \quad (6)$$

is solved, for the eigenvalues λ_b and corresponding mode shapes $\boldsymbol{\phi}$, which are normalized so that the maximum absolute value of the nodal displacement components is 1. The degree of freedom at which this occurs will be referred to as the “pivotal” degree of freedom, and the node where it occurs as the pivotal node. Further, $\mathbf{K}_T = \partial \mathbf{s} / \partial \mathbf{u}$ denotes the tangent stiffness matrix evaluated at the RSEC. It is assumed here that the null space of $\mathbf{K}_T - \lambda_b \mathbf{M}$ is one-dimensional, so that there is only one eigenvector for which Eq. (6) is satisfied for a given λ_b .

The modes $\boldsymbol{\phi}$ and natural frequencies $\lambda_b = (2\pi f_b)^2$ correspond to the infinitesimal-amplitude free vibration modes. Each one of them also indicates the bifurcation point and direction of the bifurcated solution branch. These are tracked to obtain the modes and natural frequencies for the secant linearization as a function of amplitude. The harmonic free vibration response of Eq. (2) applies under the assumption that the restoring force $\mathbf{s}_d(\mathbf{u}_d, \lambda)$ varies linearly in $\mathbf{u}_d$ following the secant stiffness from $\mathbf{0}$ to $\mathbf{u}_d$. This is the underlying linearization assumption. Thus, the linearization used depends on the dynamic displacements $\mathbf{u}_d$.

To incrementally track the bifurcated solution branches, Riks’ “Bordered Equations” method [56] is used as implemented in Npex [57]. For this the above-defined pivotal degree of freedom is used. Whereas the method requires the selection of a pivotal degree of freedom, the results do not depend on the selection, except that one should avoid selecting a degree of freedom for which the displacement for the mode to be calculated is near zero. This is achieved by the above selection of the pivotal degree of freedom. Although Riks’ “Bordered Equations” method is described in [57], a brief outline follows.

A symmetric bilinear form $\{.,.\}$ is defined on the space of the vectors $\mathbf{u}$ containing all nodal displacement and rotation components for the finite element model. This may be an inner product, but it need not be. It is sufficient for the symmetric, bilinear form to satisfy $\{\boldsymbol{\phi}, \boldsymbol{\phi}\} > 0$. The bifurcated solution branch does not depend on the selection of this bilinear form. Only its parametrization does. Here $\{.,.\}$ is defined by selecting the node at which the modal displacement ϕ is highest as the pivotal node, and then calculating $\{.,.\}$ as the dot product of the displacement vectors at the pivotal node as follows:

$$\{\hat{\mathbf{u}}, \tilde{\mathbf{u}}\} = \hat{v}_{\text{piv}} \tilde{v}_{\text{piv}} + \hat{w}_{\text{piv}} \tilde{w}_{\text{piv}} \quad \text{for any } (\hat{\mathbf{u}}, \tilde{\mathbf{u}}) \quad (7)$$

Thus, $\{\hat{\mathbf{u}}, \hat{\mathbf{u}}\}$ is the dot product of the displacements projected onto the plane of Figure 1. It follows that η converges to the arclength along the projected bifurcated branch as $\Delta\eta \rightarrow 0$.

We seek bifurcated solutions $(\mathbf{u}_d(\eta), \lambda(\eta))$ as a function of an arclength parameter η along the bifurcated branch. That the solution sought should be on the dashed line through point D in Figure 1 means that

$$\{\mathbf{u}_d', \mathbf{u}_d - \mathbf{u}_{d0}\} = \eta - \eta_0 \quad (8)$$

where a prime denotes differentiation with respect to the arclength parameter η , $(\cdot)_0$ denotes the known value of $(\cdot)$ at the start of the increment, and no subscripts 0 in Eq. (8) means the value at the end of the increment to be calculated, except that $\mathbf{u}_d'$ in Eq. (8) denotes the value of the derivative at the end of the previous increment, so that it is tangent to the solution branch at the start of the current increment. By a Taylor series expansion for $\mathbf{u}_d(\eta)$ about $\eta = \eta_0$, it is readily verified that $\{\mathbf{u}_d', \mathbf{u}_d'\} \cong 1$ for small increments $\Delta\eta = \eta - \eta_0$, so that η is indeed approximately an arclength parameter along the bifurcated branch, as expected from Figure 1.

The solution $(\mathbf{u}_d, \lambda)$ for the increment is calculated by enforcing Eq. (8) together with Eq. (4) at the end of the increment. This is done by Newton Iteration, with Riks' "Bordered Equations" method [56,57]. The solution procedure also yields the derivatives $\mathbf{u}_d'$ and λ' at the end of the increment. These are needed in Eq. (8) for the next increment, and are also used to calculate the initial guess to the solution for $(\mathbf{u}_d, \lambda)$.

For the first increment (starting at the bifurcation point), it can be shown that $\mathbf{u}_d' = \boldsymbol{\phi}/\{\boldsymbol{\phi}, \boldsymbol{\phi}\}^{1/2}$ and $\lambda' = 0$, which is used to initiate the stepping procedure. The bifurcated solution branch, then gives the natural frequency $f = \lambda^{1/2}/(2\pi)$, and the mode shape $\mathbf{u}_d$ as a function of the amplitude of vibration A , given by

$$A = \|\mathbf{u}_d\|_\infty \quad (9)$$

i.e. A is the largest of the absolute values of the nodal displacement components in $\mathbf{u}_d$.

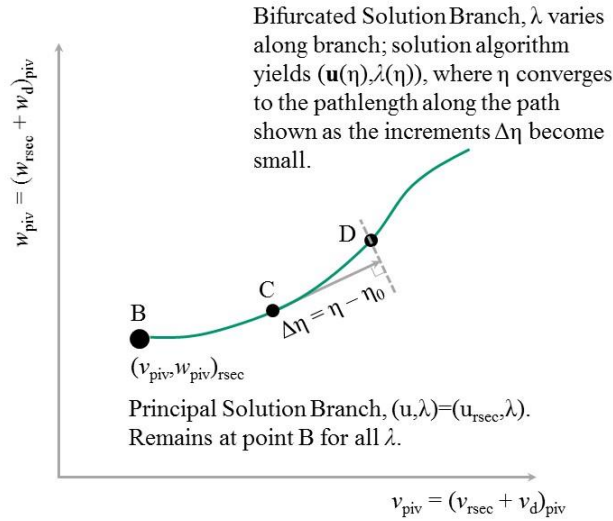


Figure 1 Illustration of arclength-control condition of Eq. (8) used to track the bifurcated solution path by calculating the solution for successive increments in arclength $\Delta\eta$ starting from the bifurcation point B. According to Eq. (8) the solution for a typical increment from point C to point D is sought on the dashed line through point D.

For each bifurcation point, there is a bifurcated branch in the $\boldsymbol{\phi}$ direction, and another in the opposite direction. The latter is calculated by specifying a negative value for the increment in η from one solution to the next. (In Npex [57] calculating each branch requires a separate branch-switching restart.)

If the function $\mathbf{s}_d(\mathbf{u}_d, \lambda)$ is antisymmetric in $\mathbf{u}_d$, in the sense that $\mathbf{s}_d(-\mathbf{u}_d, \lambda) = -\mathbf{s}_d(\mathbf{u}_d, \lambda)$, the solution branches will be similar. Otherwise, the secant stiffness that applies for one half of the vibration cycle (for which $\sin(2\pi ft) > 0$) is not the same at that for the other half (for which $\sin(2\pi ft) < 0$). Because of loss of

contact with the seabed such asymmetry is stronger for the cross-flow (vertical) mode. Attention is therefore focused on the in-line (horizontal) mode, which is also mostly the governing one since it can be excited at lower current velocities (with in-line VIV starting at a current velocity of one pipe diameter per natural period instead of 2 pipe diameters for cross-flow according to the empirical response functions in [55]).

4 Definition of ZRB Span Considered

This section describes the spans considered, the modelling assumptions, and provides the reference static equilibrium configuration (RSEC) about which the vibrations take place.

4.1 ZRB Triggers Description and Purpose

ZRB lateral buckle triggers were invented by the first author in 2004 to deal with the challenges of a high-temperature, high-pressure flowline on a soft clay seabed. Publication of the method followed in 2009 [62], after it had been successfully applied. The publication includes finite element analyses that confirm the effectiveness of the method predicted by a simple analytical model. The ZRB lateral buckle trigger creates lateral as well as vertical out of straightness (OOS) at the sleeper, so that buckles are triggered at a low axial force. This ensures that the buckle will accommodate its share of the thermal expansion. (Otherwise, that thermal expansion might result in excessive expansion going to a different buckle which thereby could become overstressed.)

Sleepers had already been used to apply a vertical OOS to trigger buckles when ZRBs were invented. The sleepers consist of a horizontal length of pipe raised slightly above the seabed by a support structure based on a mud mat, and placed normal to the pipe route. The pipe is then laid over this sleeper. The thermal compression acting through the vertical OOS results in an upward force. Thereby the vertical reaction at the sleeper and the associated frictional lateral resistance are reduced until it slips laterally resulting in a lateral buckle. Typically, the sleeper is also made long enough so the pipe remains on the sleeper when it buckles laterally. This reduces the effect of pipe-soil interaction uncertainty on the pipe stresses and strains due to lateral buckling.

The ZRB trigger structure consists of a sleeper with a vertical post on one side, as shown in Figure 2 and Figure 3. Pipelay proceeds in a straight line until the pipe is firmly seated on the sleeper but not yet in contact with the seabed on the down-lay side of the support. Then the barge moves laterally causing the pipe to slip laterally on the sleeper until stopped by the post. Thus, the lay barge movement is as if it were trying to lay a Zero-Radius Bend, hence the name “ZRB”. The result is lateral as well as vertical OOS at the sleeper which has proven very effective in triggering lateral buckles [60], as had been predicted by the simple analytical methods that motivated the invention and were finally published in [62] together with finite element (FE) analyses to verify the analytical approximations. Since then, ZRB triggers have been extensively used. (See “ZRB pipeline” online.) However, the spans created thereby need to be assessed for VIV fatigue, and they may grow by scour.

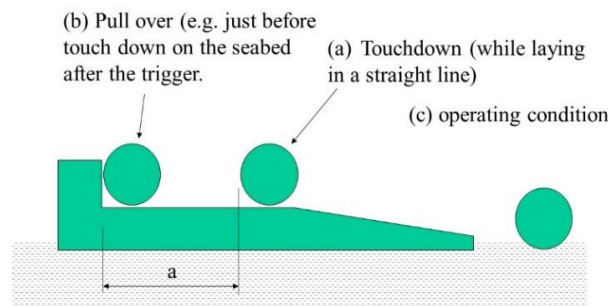


Figure 2 Sketch of ZRB trigger from December 2nd, 2004 note proposing further consideration of the concept for a high-temperature, high-pressure, CRA-clad pipeline on a clay seabed on the South China Sea. The original 2004 caption: “Cross section of a trigger, showing location of pipe at various stages of laying through to operation. (It is also possible to extend the trigger so that the pipe remains on the trigger during operation. This has the advantage a possibly high resistance to lateral movement from the soil is avoided, but it also creates a span.)”

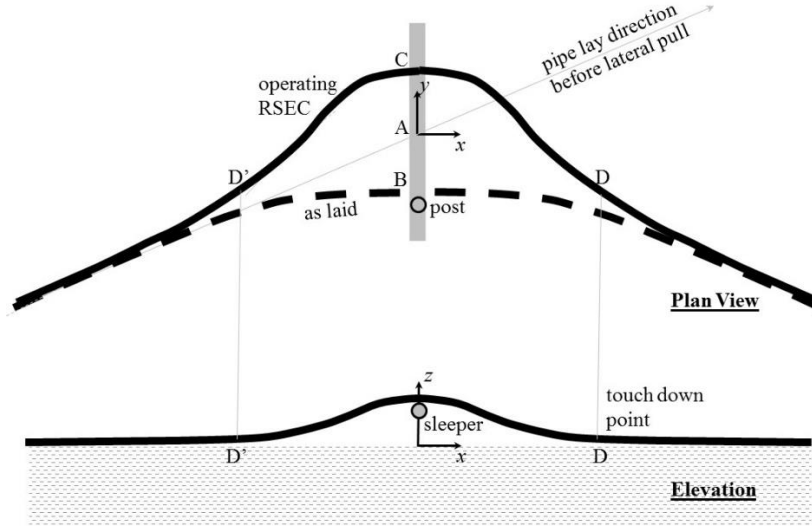


Figure 3 ZRB trigger with coordinate system used (as in [63])

4.2 Pipe Properties, Discretization and Operational Loading

A 20-inch pipe is considered with an outer diameter of the steel pipe of 0.508 m and a wall thickness of 18mm. With the coatings, the outer diameter increases to $D = 0.594$ m. The pipe material is assumed to remain-linearly elastic, with flexural and axial rigidities of $EI = 17.38 \text{ MNm}^2$ and $EA = 5736 \text{ MN}$, respectively, and a coefficient of thermal expansion of 1.17×10^{-5} per $^{\circ}\text{C}$. (For simplicity the stiffness of the coatings has been neglected, and they are also not considered in calculating the axial and bending stresses in the steel.) The submerged weight during operation is 1.22 kN/m. For simplicity, the same weight is assumed throughout installation, operation, and shut down. The mass per unit length of the pipe and contents is $m_a = 408 \text{ kg/m}$, and including the added-mass with $C_a = 1$, the effective mass in the transverse (vertical and lateral) directions is $m_t = 692 \text{ kg/m}$.

The operational loading consists of an internal pressure of 15 MPa, and a temperature rise of 60°C . Both are applied after pipelay, removed, and reapplied to obtain the RSEC about which the vibrations occur. For simplicity the temperature and pressure vary only as a function of time, and are the same for all locations along the pipeline.

The FE analysis is performed with Npex [57], using 2-noded pipe elements (Npex, Element 4), which are based on the geometrically nonlinear, small-rotations theory with shear deformations included. The element length is 0.5 m up to 200 m from the ZRB support (where the pipe experiences significant bending in operation) and 5 m beyond that. Npex has been used extensively for pipeline lateral buckling and upheaval buckling problems and validated against Abaqus, as well as field observations. It also has special capabilities for bifurcation buckling problems which enable the solution of the ADM problem with minimal adaptations. For further information see [57] and references cited therein.

4.3 Geometry and Quasi Static Loading till RSEC

The ZRB sleeper rises 0.5m above a flat seabed, and is assumed to extend far enough so the pipe does not fall off during lateral buckling.

The FE model captures the length of pipe from the midpoint between two ZRB triggers, over the trigger, and on to the next midpoint. Thus, the length of the model is equal to the spacing of ZRB triggers, which is taken to be $L = 2 \text{ km}$. By symmetry, at the midpoints the displacements are assumed to be zero during operation.

Pipelay is simulated in 80 m water depth, by considering a “lay barge” that is long enough to hold the entire modeled length of pipeline with a “firing line” at the water level and 100m-radius stinger. Approximate modelling of the “lay barge” as a bed of movable vertical springs may have some

influence on the stresses and strains on and near the “lay barge”, but the influence of this approximation and the water depth used on the as-laid pipe configuration is expected to be minimal.

The following steps are involved in modelling pipelay:

- The pipe model initially extends from $(x,y) = (-L/2, -L \tan(\alpha)/2)$ to $(x,y) = (L/2, L \tan(\alpha)/2)$. In the vertical (z) direction, it is entirely supported by the “lay barge” at the elevation of the firing line. Here α is the angle of deviation from the mean pipe direction, which in this case is such that $\tan(\alpha) = 0.1$. At the start, the x and y displacements are set to zero, and at the end of the line the lay tension is applied as a force of $N_{lay} = 0.4$ MN in the x direction, and $\tan(\alpha)$ times that amount in the y direction, so that the lay tension is in the direction of the initial orientation of the pipe.
- Pipelay proceeds by moving the “lay barge” forward until the pipe comes into contact with the trigger, which occurs at the landing point $(x,y) = (0,0)$, and the slope of the pipe at the trigger is zero. This occurs before the pipe comes into contact with the seabed on the down-lay side of the trigger.
- The lateral pull is performed by gradually changing the y component of force applied at the end of the line from $N_{lay} \tan(\alpha)$ to the same amount in the opposite (i.e. negative y) direction. In that process the pipe slides over the trigger until it reaches the post, which is positioned to allow a movement of $y_{BA} = \tan(\alpha) (EI/N_{lay})^{1/2} = 2.08$ m. This corresponds to the distance BA in Figure 3. It has been chosen from the analytical solution in [62] for the pipe pulled around the post at constant tension, to result in an approximately symmetrical as-laid configuration. That this is achieved for the example considered can be seen from the “ y as laid” pipe profile in Figure 4.
- Pipelay continues until the lay barge is no longer supporting the pipe anywhere resulting in the “ y as laid” profile of Figure 4.
- The node at the end of the model is fixed against any further translations, by applying a high (100 GN/m) stiffness in the x and y directions with Npex Element 17 [57].
- The two cycles of operating loads are applied. The RSEC about which the vibrations are considered, is reached when temperature and pressure are fully applied for the 2nd cycle, resulting in the pipe profiles “ z RSEC” (vertical) and “ y RSEC” (horizontal) of Figure 4.

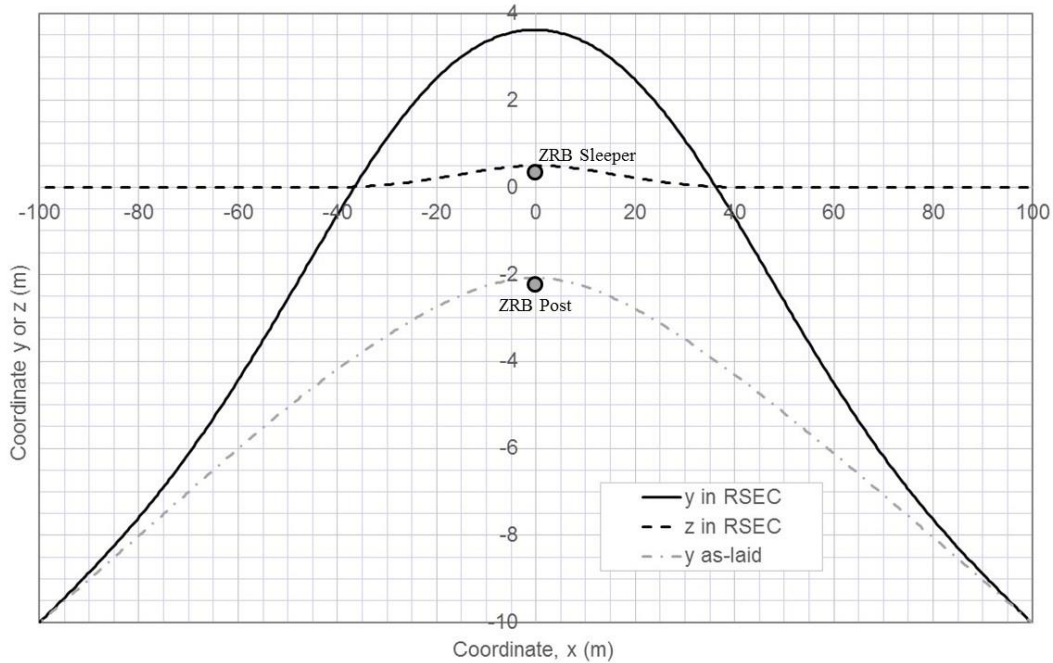


Figure 4 Pipe profile in the Reference Static Equilibrium Configuration (RSEC) about which vibrations take place, based the 2nd operating cycle with pressure and temperature applied

4.4 Seabed and Support Modelling

A simple seabed-pipe interaction model is used using Npex Element 43 [57]: In the normal (z) direction, a linear elastic spring of stiffness 0.8 MN/m^2 resists penetration into the seabed, but the pipe is free to lift off the seabed without any soil suction.

In the axial (x) direction the force-displacement relation is captured by a linear elastic spring of stiffness 0.4 MN/m^2 until the frictional resistance is reached, which is calculated by multiplying the normal force by a friction coefficient of 0.6. Once this frictional resistance is reached, the pipe slips at constant force.

Similar seabed springs apply for the transverse (y) direction, except elastic stiffness is 0.5 MN/m^2 and the friction coefficient 0.7.

The ZRB sleeper is modeled in a similar way to the seabed, except that in this case a single spring is applied to the node at the sleeper, rather than a bed of springs. The ZRB support spring stiffnesses are 50 MN/m in the normal (z) direction, 30 MN/m in the axial (x) direction and 40 MN/m in the transverse (y) direction. The friction coefficient is 0.5 for both the axial (x) and lateral (y) directions.

The same properties are used for the ZRB post, except that in this case the normal, axial and transverse directions are the y , x , and z directions, respectively. Further, the post is not active in the RSEC, or the vibrations about it, since the pipe has buckled away from the post under the temperature and pressure loads.

The pipe and support properties of this section apply to pipelay, operational (pressure and temperature) loads, and the time-domain analysis. The next section describes how they are modified for the calculation of the amplitude-dependent modes.

5 Dynamic Seabed and Support Spring Properties

The seabed and support models defined above will be referred to as the “actual” system. They also apply to the dynamic response in the time-domain validation analysis, but, for the ADM analysis, a suitable approximation is needed for the stiffness of the linearized system and the energy dissipation per cycle. This is done via relationship $s_d = s_d(u_d)$ between the dynamic spring force s_d and the dynamic displacement u_d . The stiffness of the linearized system is then s_d/u_d . (Recall that the dynamic force and displacement represent the increment from the RSEC about which the dynamic response occurs.) A separate function $E_d = E_d(u_d)$ is needed for the energy dissipated per cycle to determine the equivalent viscous damping ratio by postprocessing the finite element analysis results. This section describes the approximations to define the functions $s_d(u_d)$ and $E_d(u_d)$ for the springs representing the seabed and the ZRB support.

Where the “actual” system is nonlinear elastic, the same force-deflection relation is used for the dynamic increment. This applies to the geometric nonlinearity associated with the pipe, and the vertical seabed and ZRB support springs, but for the springs that capture frictional slip on the seabed and ZRB support a different approach is needed.

As an example, consider the axial springs that represent the seabed: During heating of the pipeline to reach the RSEC the pipe slips axially to feed into the buckle. Thus, in the RSEC, these springs are at the slip limit. If one simply used the “actual” behavior for a further monotonic increment in displacement u_d , the linear spring used for the ADM analysis would have zero stiffness for increasing axial feed into the span, and initially the elastic stiffness for the opposite direction until slip occurs. A better approximation for the dynamic properties of the soil spring is obtained by assuming cyclic motion about the mean position. For that, the behavior of the elastic-perfectly plastic soil spring is the same as it would be starting from zero force and displacement.

Specifically, the dynamic force-displacement relation is trilinear. It consists of a middle segment through the origin $(u_d, s_d) = (0, 0)$ at the slope of the linear elastic stiffness, which extends to each side of the origin until the slip force is reached. This slip force is the normal (i.e. vertical) force in the RSEC multiplied by the applicable friction coefficient. Thereafter the dynamic force is assumed to remain constant with increasing absolute value of the dynamic displacement. This approximation is used for

the ZRB sleeper, and the seabed, away from the Touch-Down Zone (TDZ), where the vertical PSI force is equal to the weight of the pipe.

Based on the pipe submerged weight of 1.22 kN/m and the vertical sleeper reaction in the RSEC of 68.9 kN, the mobilization displacements for the sleeper resistance are 1.1 mm for the axial (x) direction, and 0.8mm for the transverse (y) direction. For the seabed, away from the TDZ, they are 1.8 mm for the axial direction, and 1.7 mm for the lateral direction.

For the seabed springs in the TDZ, where the vertical PSI force in the RSEC is not equal to the weight of the pipe, the mobilization displacement is assumed to be the same as outside the TDZ and the slip force is calculated in the same way. Thus, not only the slip force, but also the elastic stiffness is proportional to the normal (i.e. vertical) PSI force in the RSEC. A similar approach is used in the vertical direction so that the dynamic upward displacement for the pipe to lose contact with the seabed is the same inside and outside the TDZ.

Where frictional slip can occur, elastic-perfectly plastic hysteresis loops are assumed to calculate the energy dissipated per cycle E_d and the damping ratio ζ as follows:

$$\begin{aligned} E_d &= 4 s_{\text{slip}} (|u_d| - u_m), \quad \zeta = (2/\pi)(1 - u_m/|u_d|) \quad \text{for } |u_d| > u_m \\ E_d &= \zeta = 0 \quad \text{otherwise} \end{aligned} \quad (10)$$

where s_{slip} is the frictional resistance based on the normal force in the RSEC, u_d denotes the dynamic component of displacement for the particular transverse or axial spring, and u_m is the displacement to mobilize the frictional resistance. For the vertical direction there is no slip and therefore no damping.

The approximations where slip can occur are illustrated in Figure 5. The upper figure shows the “assumed” behavior of the spring for the ADM analysis, whereas the lower one shows what the “actual” behavior for an elastic-perfectly-plastic spring undergoing the same displacement history would be.

In the upper figure, the dynamic displacement u_d is the distance CJ. The stiffness for the linearization used is the slope of the line GE. The energy dissipated per cycle E_d the area of the parallelepiped GFEH. The equivalent damping ratio ζ is $1/(4\pi)$ times the energy dissipated per cycle divided by the maximum potential energy of the linear elastic system, which is the area of the triangle CJE.

Turning to the difference between the assumed and actual behavior in Figure 5, it is seen that the hysteresis loops are the same, but they are translated: in the assumed behavior, they are centered about the RSEC, which is at Point C, but in the actual behavior, they are centered about Point C*.

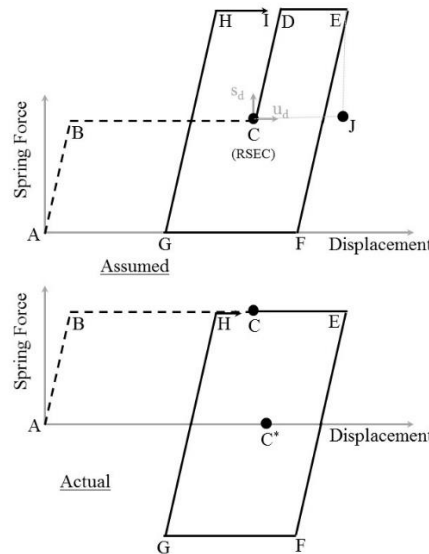


Figure 5 Assumed and actual behavior of seabed or support springs in lateral or axial directions

Further, in constructing the actual behavior in Figure 5, it has been assumed that the total displacement history is unaffected by the approximation made. That is not the case. The drop in the force at the center of the hysteresis loop makes the configuration about which the vibrations occur drift. This will be seen in the time domain simulations. The most obvious drift or ratchetting mechanisms occur where frictional forces in the RSEC are resisting lateral buckling displacements or axial displacements feeding into the buckle. Drift then leads to an increase in such displacements.

6 ADM Results

The first two modes from the linear, infinitesimal-amplitude modal analysis (according to Eq. (6)) are a cross-flow mode at natural frequency of $f_b = 0.522$ Hz, and an in-line mode at $f_b = 0.771$ Hz. The in-line mode includes a small component of cross-flow motion and vice versa, but, at infinitesimal amplitude, this is negligible.

Four bifurcated solution branches are calculated: for the in-line mode, Branch IL1+ by tracking by tracking in the ϕ direction from the bifurcation point, and Branch IL1- by tracking in the opposite direction. Similarly, there are branches CF1+ and CF1- for the cross-flow mode. How the natural frequencies and dynamic displacements at the pivotal node change as a function of the amplitude of vibration is shown in Figure 6 and Figure 7. The mode shape, $\mathbf{u}_d/A$, for branch IL1+ is shown in Figure 8, and the stress amplitudes in Figure 9.

The modal damping ratio due to the frictional slip captured by the springs is included in Figure 6. It is calculated by summing the energy dissipated per cycle, E_d from Eq. (10), for all nodal springs for which frictional slip can occur and dividing that by 4π times the total free vibration energy, which is $\frac{1}{2} (2\pi f)^2 \mathbf{u}_d^T \mathbf{M} \mathbf{u}_d$.

Remarks regarding ADM results in Figure 6 to Figure 9 follow:

- 1) Before doing the reported case, a verification case confirmed that the solution process gave the expected results when the seabed and support properties were the same as for the approximate analytical solution in [63].
- 2) The results confirm that $\mathbf{u}_d/A \rightarrow \phi$ as $A \rightarrow 0$, as expected. This can be seen in Figure 8, where the infinitesimal-amplitude mode ϕ is plotted as the grey (" $A/D \rightarrow 0$ ") line, which is found to be very close to the dashed line for $\mathbf{u}_d/A$ at the lowest nonzero amplitude of $A/D = 0.008$.
- 3) At low amplitudes of vibration ($A/D \leq 0.017$), there is no axial slip at the ZRB support, so arch action [63,59] develops mainly in the horizontal direction. The associated stiffening effect is much stronger in the horizontal than in the vertical direction, because of higher horizontal OOS compared to the vertical. Once axial dynamic slip at the ZRB support starts, damping increases rapidly in Figure 6 as is also predicted analytically in [63], but arch action is lost causing the in-line and cross-flow frequencies to converge. Finally, beyond $A/D \cong 0.15$, lateral slip on the seabed for the in-line mode makes the in-line frequency drop below the cross-flow one, since such slip is not possible for the cross-flow mode.
- 4) The damping ratio for the cross-flow mode is zero up to $A/D \cong 0.89$, because no frictional slip develops, and that is the only damping mechanism included. It is for this reason that the cross-flow damping ratio is not included in Figure 6.
- 5) From Figure 7, the cross-flow mode remains close to vertical up to the amplitudes shown ($A/D \leq 0.2$), but for the in-line mode, the initially small vertical component starts to grow rapidly at $A/D \cong 0.08$. See also Figure 8.
- 6) This change in direction of the modal displacements is not surprising in view of the crossing natural frequency plots in Figure 6: Although they only cross at $A/D \cong 0.15$, at $A/D \cong 0.1$, the in-line natural frequency already coincides with the cross-flow one at low amplitude.
- 7) Tracking a given bifurcated solution branch in the positive or negative ϕ direction gives similar results. This can be seen in Figure 6 and Figure 9, where they are essentially the same, and in Figure 7, where they are almost exactly opposite to each other. This is due to near symmetry of the RSEC about $x = 0$.
- 8) There are two reasons why the mode shape plots for the IL1+ branch in Figure 8 are not exactly antisymmetric about $x = 0$. One is the slight asymmetry of the RSEC. This makes the $A/D \rightarrow 0$ plots

not exactly antisymmetric. The other is the geometric nonlinearity in the pipe modelling. It makes the deviation from antisymmetry increase with increasing amplitude of vibration.

- 9) In Figure 8, the modal displacements near the touch-down points ($x \cong \pm 40$ m) show the effect of softening of the lateral seabed springs due to cyclic lateral slip as the amplitude of vibration increases.
- 10) Except for the slight asymmetry in the RSEC, the mode shape plots for the IL1- branch would be the mirror image of those in Figure 8 for the IL1+ branch. The IL1- mode shape plots have not been included, since they reveal nothing that cannot already be seen from the IL1+ plot. (Further, the IL1- plot is more difficult to interpret, because the asymmetry arising from the RSEC, and the geometric nonlinearity go in opposite directions for the IL1- branch.)

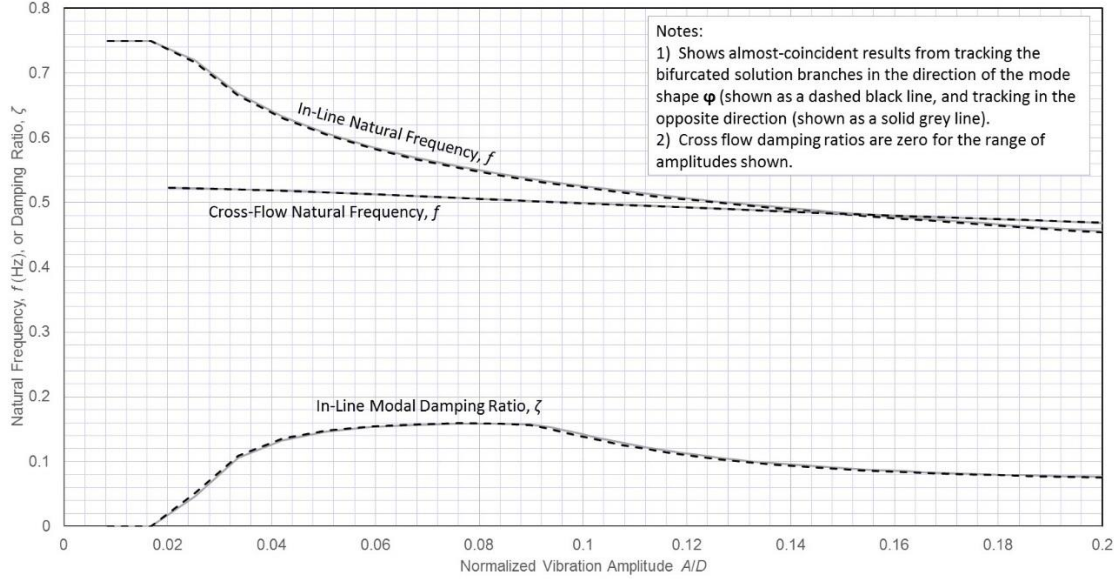


Figure 6 Natural frequencies f and modal damping ratio ζ as a function of the normalized amplitude of vibration A/D

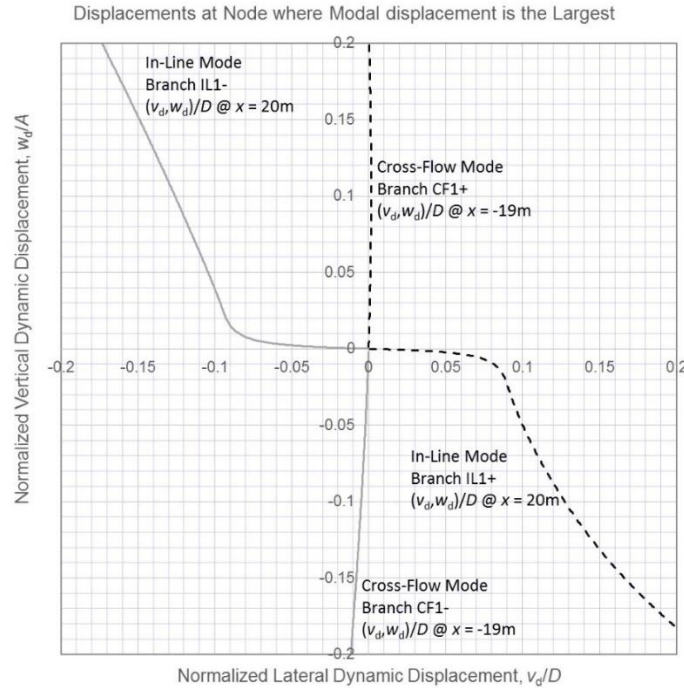


Figure 7 Normalized Lateral (y) and Vertical (z) Dynamic Displacements, $(v_d, w_d)/D$, at $x = 20$ m for in-line mode, and at $x = -19$ m for cross-flow mode (in both cases the x location chosen is where the infinitesimal-amplitude modal displacement component is largest in absolute value)

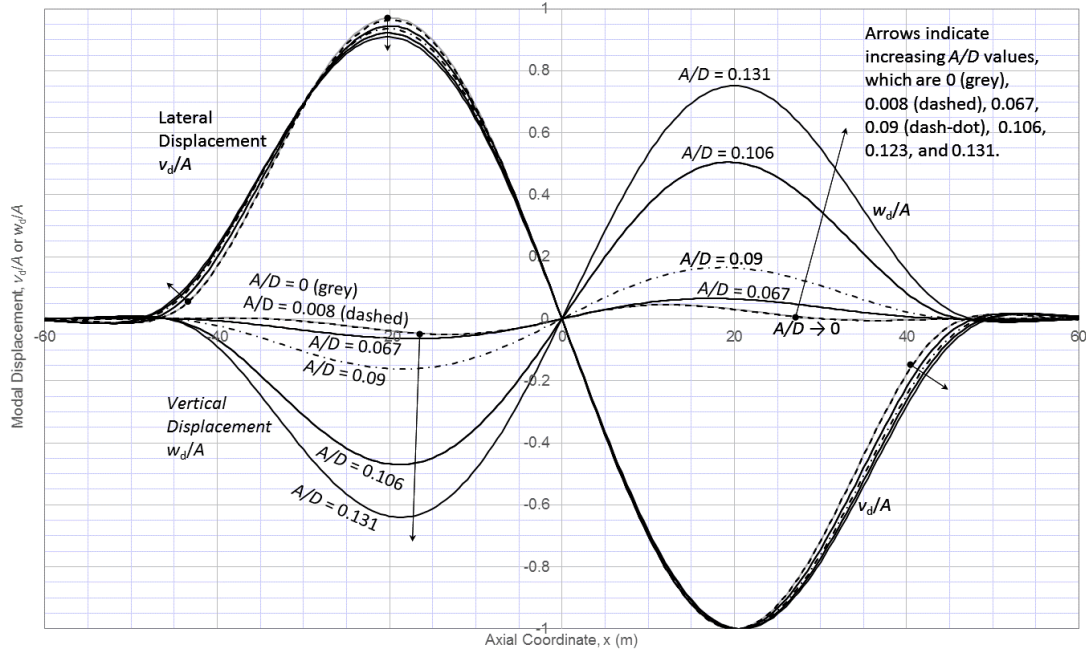


Figure 8 Normalized shape of first in-line mode (v_d/A and w_d/A), branch IL1+ (in direction of infinitesimal-amplitude mode ϕ)

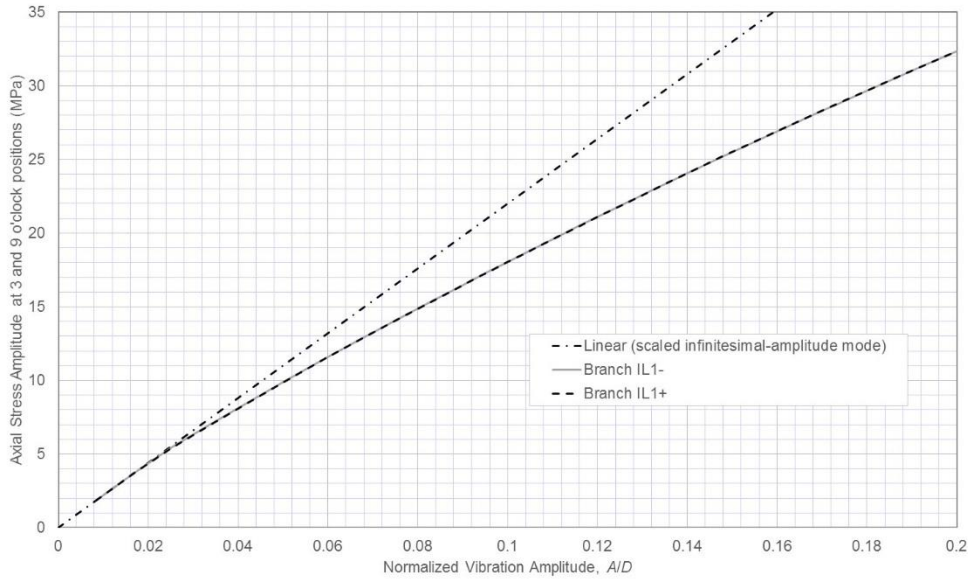


Figure 9 Stress amplitudes for in-line mode

7 VIV Fatigue Assessment

With the ADM results in place, the VIV fatigue assessment is done following DNV-RP-F105 [55]. According to this the normalized modal response amplitude A/D is an empirical function of the current velocity expressed in pipe diameters per natural period of oscillation and the modal damping ratio. All details are described in [55]. To provide the estimates of the actual amplitude and fatigue damage rates, all safety factors set to 1. The first in-line mode and the first cross-flow mode are considered. The current velocity is U at the elevation of the pipeline and perpendicular to it, with no turbulence. For the fatigue resistance, the “F1” curve “in air” from [61] is used, with stresses calculated at the inside of the pipe wall, no stress concentration factor, and no thickness factor.

The only complication in following [55] with the ADMs, is lack of an explicit solution for the vibration amplitude A of VIV. The semi-empirical equations for the amplitude become implicit, since the

properties used to determine this amplitude A/D are themselves dependent on the amplitude. Further, non-unique solutions are conceivable. To be conservative, the search for a solution is started at the highest possible A/D amplitude according to the response function in [55], and moves downward until a solution is found. The results are shown in Figure 10 and Figure 11. Only pure in-line VIV is considered, no cross-flow or cross-flow induced in-line.

For comparison with the ADM analysis, a standard VIV fatigue assessment according to [55,2] is performed with the following assumptions: Seabed stiffness and “soil damping” are determined by Sections 7.2.2 and 7.2.4 of [2] for “soft clay” with a Poisson’s ratio of 0.5 and span length of 40 m (from ZRB support to touch down point). This yields seabed spring stiffnesses of 1.79 MN/m² laterally, and 2.79 MN/m² vertically. For the axial springs (not included in [2]) a stiffness of 1 MN/m² is used. The contribution of soil damping to the modal damping ratio from Section 7.2.2 of [2] is halved since the spans on each side of the ZRB support are supported by the seabed on only one side, to obtain damping ratios of 1.54% for in-line, and 1.27% for cross-flow. For the ZRB support, the lateral and vertical elastic stiffnesses defined above are used, and zero dynamic stiffness axially, since one cannot rely on the friction to be sufficient to prevent cyclic sliding in the axial direction.

Under these assumptions of the standard assessment, the lowest frequency mode is in-line at 0.548 Hz. The cross-flow mode follows at 0.566 Hz. The maximum dynamic stress amplitudes at a vibration amplitude of $A=D$ are 206 MPa for in-line and 210 MPa for cross-flow.

The pure in-line fatigue assessment results in Figure 10 and Figure 11 clearly show the lower vibration amplitude and fatigue damage rate for ADM analysis as opposed to the standard one. At low vibration amplitudes, there is enough axial friction at the ZRB support to prevent slip there. This raises the in-line natural frequency delaying the onset of in-line VIV. Once axial slip at the ZRB sleeper starts, it introduces considerable damping with a strong effect on the response. Both of these effects are only accounted for in the ADM analysis and not in the standard one.

Indeed, most of the damping comes from axial slip at the ZRB support: Specifically, at $A/D = 0.077$, when the damping in Figure 6 reaches a maximum, the contributions to modal damping are: 12.6% for axial friction at the ZRB sleeper, 3.3% for the lateral friction on the seabed, and none for the other effects (i.e. ZRB lateral and seabed axial springs) to make for the total of 15.9% plotted in Figure 6.

For simplicity, to keep the assumptions in the ADM analysis and the time domain validation that follows the same, frictional slip is the only damping mechanism included in both cases.

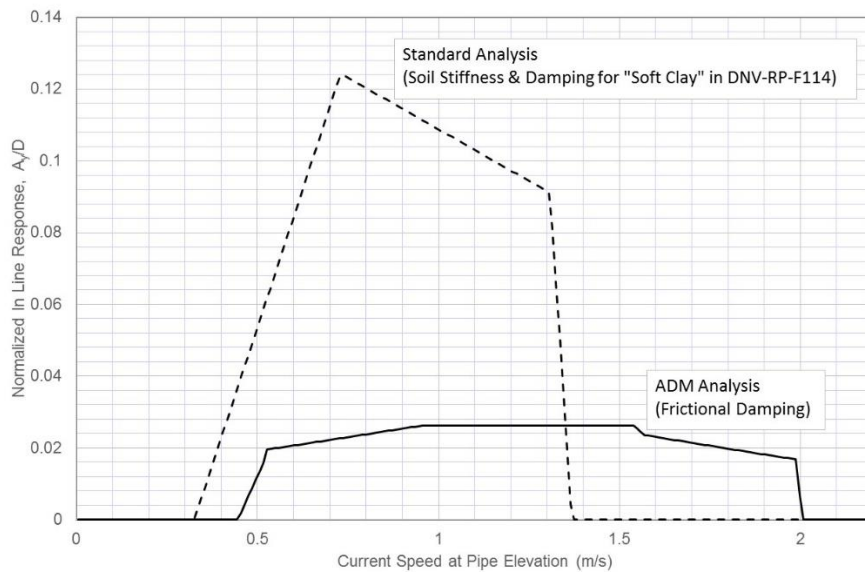


Figure 10 VIV response amplitudes from ADM analysis compared with standard analysis with soil stiffness and damping for “Soft Clay” from DNV-RP-F114 [2]

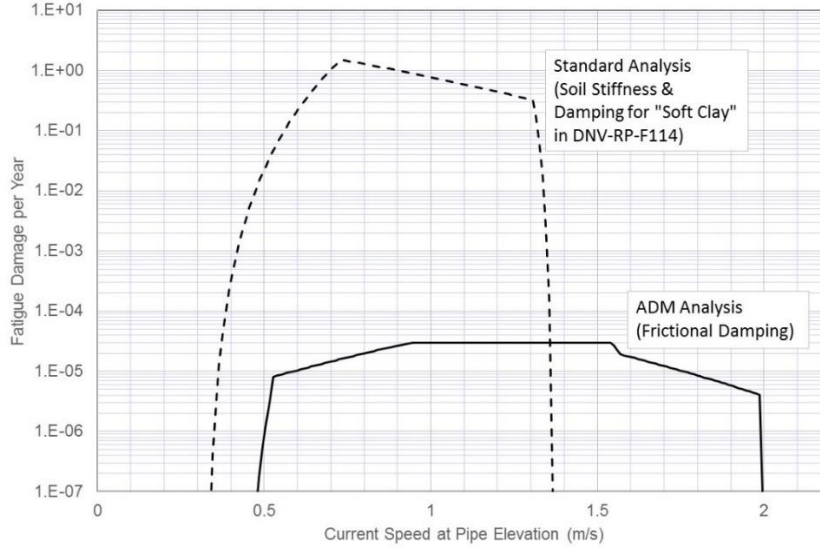


Figure 11 Fatigue damage rates

8 Time-Domain Validation

As in [58], a time-domain validation analysis is performed, by applying a harmonic load roughly representative of the hydrodynamic loads that could develop during VIV excitation, and checking that the response from the time-domain simulations matches that from the ADM analysis.

The equation of motion for the time domain analysis is:

$$\mathbf{s}(\mathbf{u}) + \mathbf{M} \ddot{\mathbf{u}} = H \mathbf{M} \boldsymbol{\psi} \sin(2\pi f_e t) \quad (11)$$

where H is a scalar real constant with units of acceleration, t denotes time, f_e the frequency of excitation, and the column vector $\boldsymbol{\psi}$ contains only the lateral (y) displacement components of the infinitesimal-amplitude mode $\boldsymbol{\phi}$ with all other components set to zero, so that the harmonic loading is only in the y direction; $\mathbf{s}(\mathbf{u})$ in this case represents the pipe, seabed, and ZRB support resistances to the displacements and rotations in $\mathbf{u}$, and depends not only on the value of $\mathbf{u}$ at time t , but also on the prior history of $\mathbf{u} = \mathbf{u}(t)$, as required to capture the frictional slip on the seabed and supports. The distributed harmonic, horizontal load represented by Eq. (11) has a maximum amplitude of H times the effective mass per unit length of the pipe, and varies proportionally with the lateral modal displacements associated with the infinitesimal-amplitude in-line mode.

Eq. (11) is solved using the Hilber, Hughes, and Taylor algorithm [64] with 128 time increments per excitation cycle (i.e., a time-discretization increment of $\Delta t = 1/(128f_e)$). The harmonic loading starts at $t = 0$ with the system at rest in the RSEC, and is continued for 100 excitation cycles (to $t = 100/f_e$), which in most cases is sufficient to get an approximately harmonic response. A typical result is shown in Figure 12. It shows the drift plus the dynamic response ($d(t) - d_{\text{rsec}}$) at the pivotal node, i.e. at $x = 20$ m, where the modal displacement $\boldsymbol{\phi}$ of the in-line mode is the largest in absolute value.

To extract a dynamic amplitude from the time-domain results, the last 8 excitation cycles of the 100 cycles of simulated response is approximated by

$$d(t) \cong d_{\text{rsec}} + d_{\text{drift}} + d_d \sin(2\pi f_e t - \delta) \quad (12)$$

where d could be a particular displacement component at a specified location or a stress component, $d(t)$ is the result of the time domain simulation, d_{rsec} is the value in the RSEC, d_{drift} represents the change from the RSEC due to drift or ratchetting of cyclic slip, and d_d the fitted dynamic value. By using discrete Fourier analysis, the values of the 3 fitting parameters d_{drift} , d_d , and phase shift δ are found to obtain a least-squares optimum fit. The largest absolute value of all the displacements d_d represents the

amplitude of vibration A from the time domain analysis. This is compared in Figure 13 with the ADM solution for the first in-line mode given in the Appendix.

The results show a good match in the dynamic amplitudes, except for one case, where the discrepancy is shown as a dashed double-headed arrow. This occurs at an excitation amplitude of $H = 0.2 \text{ m/s}^2$ at the resonant frequency ($f_e = 0.52 \text{ Hz}$). The expected vibration amplitude for this case is $A/D \cong 0.09$ from the ADM analysis, but processing the time-domain results by Eq. (12) leads to only $A/D = 0.055$. This is not surprising when one considers that at these amplitudes in-line and cross-flow amplitudes are very close, and can interact nonlinearly.

To investigate the discrepancy further, the time-domain displacement histories at the pivotal node are plotted in Figure 14. Despite periodic excitation, the response is not periodic. It is well known that, even for much simpler nonlinear systems, periodic excitation does not necessarily produce periodic response with the same period. The periodicity can bifurcate to a higher period, as has been found in [65], or the response can even become chaotic [66]. (The statement that the response shown in Figure 14 is “not periodic” does not mean that the solution could not converge to a periodic one if the simulation is extended sufficiently, though such periodic solution could have a higher period and several oscillations within each period.)

Figure 14 shows modulated vibration amplitudes, and the in-line amplitudes are relatively low for the last 8 cycles, which are used in the fitting procedure of Eq. (12). Hence the low amplitude of only $A/D = 0.055$ shown Figure 13. The maximum in-line amplitude in Figure 14 is about $A_y/D \cong 0.09$, in agreement with the ADM calculation. The maximum cross-flow amplitude is even higher, at $A_z/D \cong 0.12$. Somehow (e.g. by the dynamic axial force) energy is being transferred to the vertical mode, which has no damping at these amplitudes. The single mode ADM analysis also predicts some vertical vibration, based on the vertical component of the mode (see Figure 8), but that is only $A_z/D = 0.015$ when $A/D = 0.09$.

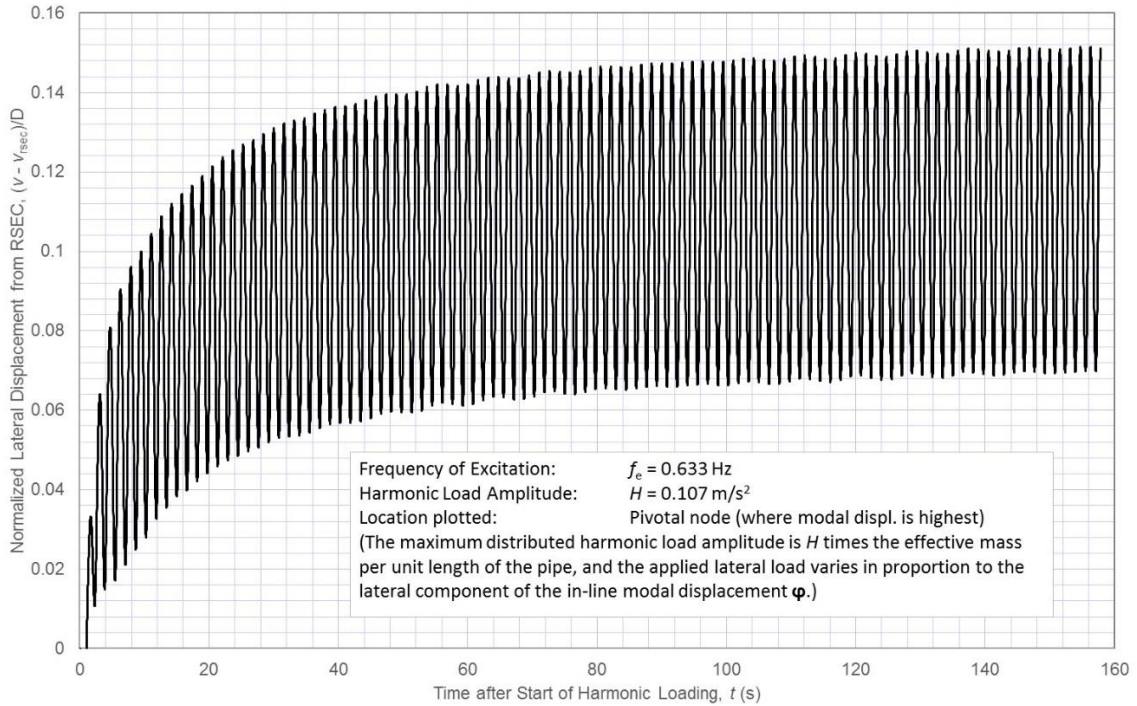


Figure 12 Typical displacement time history from time-domain analysis under harmonic loading

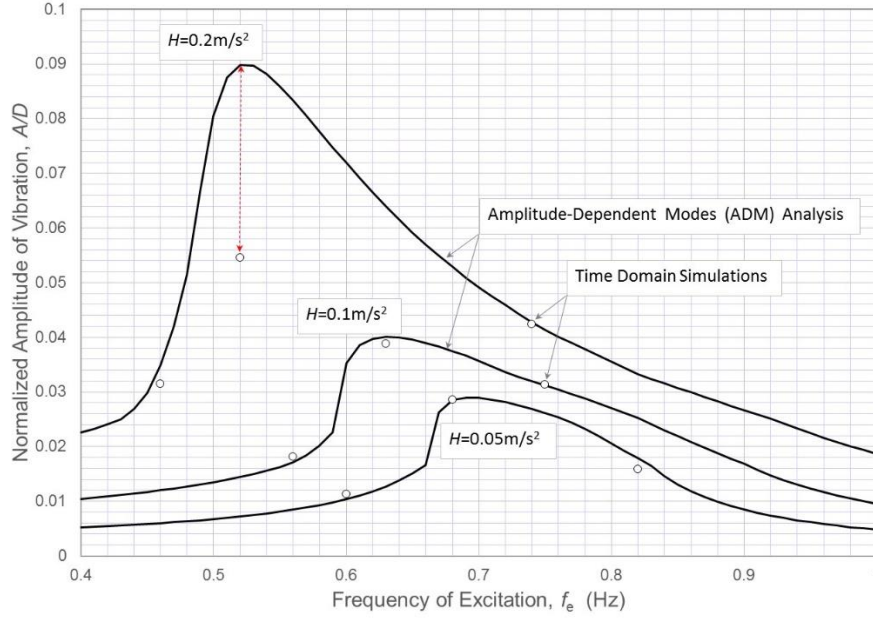


Figure 13 Comparison of results from the ADM analysis considering only the first in-line mode and the time domain simulations under harmonic loads defined by Eq. (11)

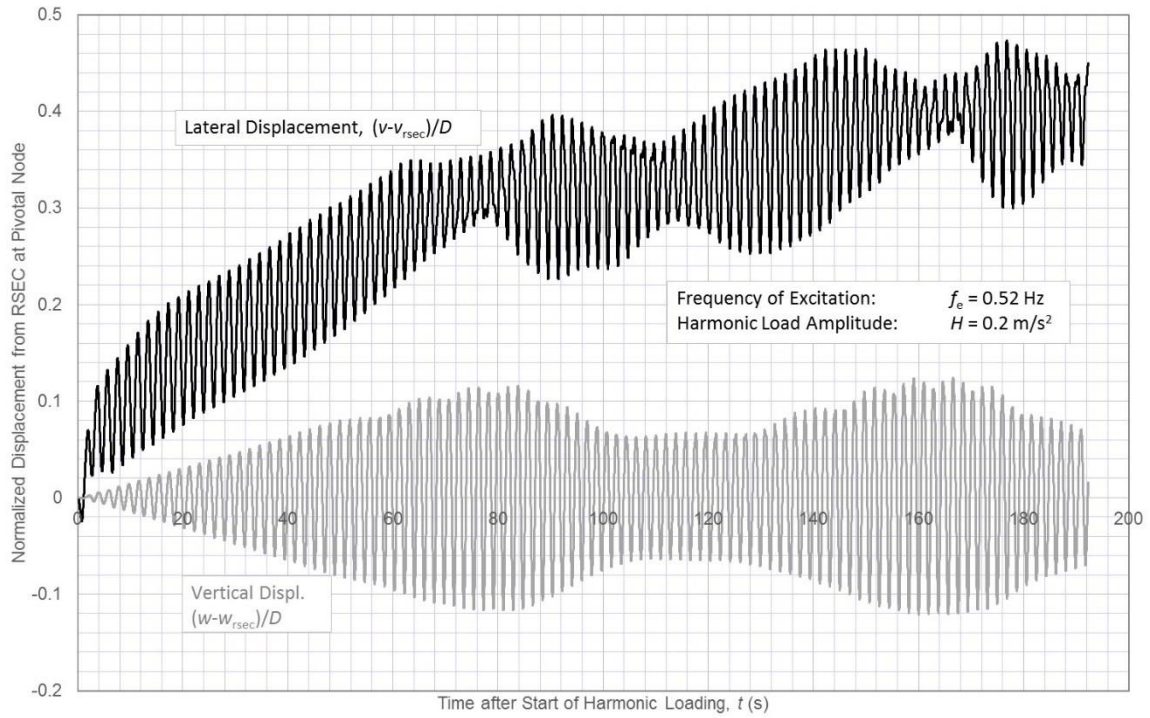


Figure 14 Time-domain response calculation for case where amplitudes in Figure 13 do not match

9 Conclusions

The in-line (horizontal) dynamic response of the spans created by a ZRB buckle trigger is affected by nonlinearity and inelasticity at the amplitudes of in-line VIV response. Axial slip at the ZRB support plays an important role in this context. The Amplitude-Dependent Modes (ADM) method described in this paper and the associated algorithm to solve the resulting nonlinear eigenvalue problem provides an efficient and effective method to address this problem within the framework of standard VIV assessment methods such as [55]. The example illustrates how properly accounting for the nonlinearity

and energy dissipation by inelastic slip leads to a large reduction in predicted VIV response and the associated fatigue damage rate.

The limitations of the ADM method are reached when either the nonlinearities become too strong, or there are several closely spaced modes that interact nonlinearly. Alternatives to the ADM method have been reviewed in the introduction but they are not as ready for application in the engineering practice, in view of high computational effort, and/or limited validation and industry acceptance of the methods.

10 Acknowledgements

This work arose via discussions with Phil Watson, Ali Karrech, Fraser Bransby, and others from the ARC Industrial Transformation Research Hub for Transforming Energy Infrastructure through Digital Engineering (TIDE) team at the University of Western Australia (UWA). The authors are grateful for the support received from TIDE, which is funded by the Australia Research Council, INPEX, Woodside Energy, Shell, Bureau Veritas, Lloyds Register, Fugro, Wood, RPS Group, the Australian Institute of Marine Science and the Bureau of Meteorology (Grant No. IH200100009).

11 Appendix – ADM Response Solution to Harmonic Load

The inputs to the analysis are the harmonic load multiplier H in Eq. (11), and, from Section 6, the ADM properties, as functions $\mathbf{u}_d = \mathbf{u}_d(A)$, $f = f(A)$, $\zeta = \zeta(A)$, for the amplitude-dependent mode shape, natural frequency, and damping ratio, respectively. (The mode could be normalized as $\mathbf{u}_d(A)/A$ so that it converges to ϕ as $A \rightarrow 0$, but it is simpler to use $\mathbf{u}_d(A)$ directly as the mode.) Here these functions are defined as piecewise linear functions by using linear interpolations between the amplitudes for which the bifurcated solution branches have been calculated and stored.

A steady-state dynamic part of the response of the form $\mathbf{u}_d \sin(2\pi f_e t - \delta)$ is sought. This leads to the complex but scalar modal response equation

$$4\pi^2 (f^2 + 2\zeta f f_e i - f_e^2) \mathbf{u}_d^T \mathbf{M} \mathbf{u}_d \exp(i\delta) = H \mathbf{u}_d^T \mathbf{M} \psi \quad (13)$$

where $i = \sqrt{-1}$. Taking the magnitude on both sides gives a single real equation that can be solved numerically for the unknown amplitude of vibration A . Then taking the argument on both sides (assuming H is positive) gives

$$\delta = \text{atan2}(f^2 - f_e^2, 2\zeta f f_e) \quad (14)$$

where $\text{atan2}(x, y)$ is the argument of the complex number $x + i y$, for any real x and y . This completes the analytical solution for the steady state response of the mode $\mathbf{u}_d$ to the harmonic loading of Eq. (11).

12 References

1. Bruschi, R. and Vitali, L. (1991) "Large-Amplitude Oscillations of Geometrically Nonlinear Elastic Beams Subjected to Hydrodynamic Excitation," *Journal of Offshore Mechanics and Arctic Engineering*, ASME, Vol. 113, May 1991, pp. 92-104.
2. DNVGL (2019) "Pipe-Soil Interaction for Submarine Pipelines," Det Norske Veritas, Recommended Practice, DNVGL-RP-F114, September 2019.
3. Templeton, J.S., Barrett, M.J., and Clukey, E.C. (2015) "Finite Element Analysis of Soil Damping for Pipeline Slugging," *Frontiers in Offshore Geotechnics III*, Meyer (Ed.), Taylor & Francis Group, London, ISBN 978-1-138-02848-7.
4. White, D.J., Clukey, E.C., Randolph, M.F., Boylan, N.P., Bransby, M.F., Zakeri, A. and Hill, A.J. (2017) "The State of Knowledge of Pipe-Soil Interaction for On-Bottom Pipeline," *Offshore Techn. Conf.*, Houston Texas, USA, 1-4 May 2017, Paper No. OTC-27623-MS.
5. Peek, R., Bernardo, C. A., Hong, H. M., O'Loughlin, C. D., White, D., & Ang, S. Y. (2020) "Pluck Tests on Operation Deepwater Pipeline Spans," *Engrxiv*, DOI: <https://doi.org/10.31224/osf.io/qypj7>.

6. Bernetti, R., Bruschi, R., Curti, G.B. and Carletti, A. (1989) "Sull'Utilizzo della Risposta Dinamica nella Identificazione dei Parametri Caratterizzanti L'Interazione Suolo-Tubo," XVII Convegno Nazionale della Associazione Italiana per L'Analisi delle Sollecitazioni, Ancona 12-16 September 1989.
7. Bernetti, R., Bruschi, R., Curti, G. and Simantiras, P. (1990) "Theoretical and Experimental Analysis of Pipe-Soil Interaction at Free Span Shoulders for Oscillating Pipelines," International Society of Offshore and Polar Engineers, The First ISOPE European Offshore Mechanics Symposium, 20-22 August, Trondheim, Paper No. ISOPE-E-90-044.
8. Papavasileiou, V., Bransby, F., Hou, Z., O'Neill, M., Watson, P., Low, H.E. Peek, R., and Rathbone, A. (2024) "Pipe-Soil Interaction for VIV of Free Spans in Carbonate Silty Sand," Proceedings of the ASME 2024 43rd International Conference on Ocean, Offshore and Arctic Engineering, OMAE2024 June 9-14, 2024, Singapore EXPO, Singapore, Paper No. OMAE2024-124740.
9. Randolph, M. and Quiggin, P. (2009) "Non-linear hysteretic seabed model for catenary pipeline contact," 28th International Conference on Ocean, Offshore and Arctic Engineering, ASME, May 31 - June 5, 2009, Honolulu, Hawaii, USA, Paper No. OMAE2009-79259.
10. Pontaza, J.P. and Menon, R.G. (2010), "On the numerical simulation of fluid-structure interaction to estimate the fatigue life of subsea pipeline spans: Effects of wall proximity," Proceedings of OMAE 2010, 29th International Conference on Offshore Mechanics and Arctic Engineering, June 6-11, 2010, Shanghai, China, Paper No. OMAE2010-20804.
11. Pontaza, J. P., Menon, R. G., Swanson, R. C., Jhingran, V., Hill, M., Kopp, F., & Hoffman, J. (2010). "Fluid-Structure interaction simulations of a pipeline span exposed to sea bottom currents," Offshore Technology Conference, 2010, ASME, Paper No. OTC 21070, January 2010.
12. Pontaza, J.P., and Menon, R.G., (2013), "Prediction of Vortex-Induced Vibration Response of a Pipeline Span by Coupling a Viscous Flow Solver and a Beam Finite Element Solver," Published in Journal of Offshore Mechanics and Arctic Engineering, August 2013, Vol. 135 Issue 3, p31702, 9 p.
13. Krwawzyk, P., Bijker, R., Chen, Z., and Boersma, H., (2013), "Managing Free Spans on Dynamic Seabed," OPT 2013.
14. Jiang, H., Ju, X., Zhao, M., Zhou, T., Draper, S., An, H., Watson, P., Lei, Z., Wang, L.; (2024), "Large-eddy simulation of vortex-induced vibration of a circular cylinder at Reynolds number 10 000," Physics of Fluids; 36 (8): 085113. <https://doi.org/10.1063/5.0219933>
15. Chen, L., Wang, Y., Sun, S., Wang, S., (2020), "The effect of boundary shear flow on hydrodynamic forces of a pipeline over a fully scoured seabed," Ocean Engineering, Volume 206, 107326, ISSN 0029-8018, <https://doi.org/10.1016/j.oceaneng.2020.107326>.
16. An, H., Cheng, L., Zhao, M., (2015), "Two-dimensional and three-dimensional simulations of oscillatory flow around a circular cylinder," Ocean Engineering, Volume 109, 2015, Pages 270-286, ISSN 0029-8018, <https://doi.org/10.1016/j.oceaneng.2015.09.013>.
17. Xiao, F. and Chen, H.-C., (2015), "CFD Simulation of Vortex-Induced Vibrations of Free Span Pipelines Including Pipe-Soil Interactions," Proceedings of the Twenty-fifth (2015) International Ocean and Polar Engineering Conference, Kona, Big Island, Hawaii, USA, June 21-26, 2015
18. Zhao, M., (2013), "Numerical investigation of two-degree-of-freedom vortex-induced vibration of a circular cylinder in oscillatory flow," Journal of Fluids and Structures, Volume 39, Pages 41-59, ISSN 0889-9746, <https://doi.org/10.1016/j.jfluidstructs.2013.02.003>.
19. Shittu, S., (2022), "Numerical Simulation of Vortex-Induced Vibration of Free Spanning Underwater Pipeline Close to the Seabed," PhD Thesis.
20. Birkhoff, G., and Zarantonello, E. H., 1957, Jets, Wakes, and Cavities, Academic Press Inc., New York.
21. Bishop, R. E. D., and Hassan, A. Y., 1964, "The Lift and Drag Forces on a Circular Cylinder in a Flowing Fluid," Proceedings of the Royal Society of London A, Royal Society, London, pp. 32–50. <https://doi.org/10.1098/rspa.1964.0004>.
22. Hartlen, Ro. T., and Currie, I. G., 1970, "Lift-Oscillator Model of Vortex-Induced Vibration," Journal of the Engineering Mechanics Division, 96(5), pp. 577–591. <https://doi.org/10.1061/JMCEA3.0001276>.

23. Skop, R. A., and Griffin, O. M., 1973, "A Model for the Vortex-Excited Resonant Response of Bluff Cylinders," *J. Sound Vib.*, 27(2), pp. 225–233. [Online]. Available: [https://doi.org/10.1016/0022-460X\(73\)90063-1](https://doi.org/10.1016/0022-460X(73)90063-1).
24. Griffin, O. M., Skop, R. A., and Koopmann, G. H., 1973, "The Vortex-Excited Resonant Vibrations of Circular Cylinders," *J. Sound Vib.*, 31(2), pp. 235–249. [https://doi.org/10.1016/S0022-460X\(73\)80377-3](https://doi.org/10.1016/S0022-460X(73)80377-3).
25. Iwan, W. D., and Blevins, R. D., 1974, "A Model for Vortex Induced Oscillation of Structures," *J Appl Mech*, pp. 581–586.
26. Iwan, W. D., 1981, "The Vortex-Induced Oscillation of Non-Uniform Structural Systems," *J. Sound Vib.*, 79(2), pp. 291–301.
27. Falco, M., Fossati, F. V., and Resta, F., 1999, "On the Vortex Shedding Vibration on Submarine Cables: Design Optimization of Wrapped Cables for Controlling Vibrations," *3rd International Symposium on Cable Dynamics*, Trondheim, Norway.
28. Facchinetti, M.L., Langre, E. and Biolley, F. (2004) "Coupling of structure and wake oscillators in vortex-induced vibrations," *Journal of Fluids and Structures* 19 (2004) 123–140.
29. Qu, Y., and Metrikine, A. V., 2020, "A Single van Der Pol Wake Oscillator Model for Coupled Cross-Flow and in-Line Vortex-Induced Vibrations," *Ocean Engineering*, 196. <https://doi.org/10.1016/j.oceaneng.2019.106732>.
30. Orcina, 2024, "OrcaFlex." Available at: <https://www.orcina.com/webhelp/OrcaFlex/>.
31. Srinil, N., and Zanganeh, H., 2012, "Modelling of Coupled Cross-Flow/in-Line Vortex-Induced Vibrations Using Double Duffing and van Der Pol Oscillators," *Ocean Engineering*, 53, pp. 83–97. <https://doi.org/10.1016/j.oceaneng.2012.06.025>.
32. Zanganeh, H., and Srinil, N., 2014, "Characterization of Variable Hydrodynamic Coefficients and Maximum Responses in Two-Dimensional Vortex-Induced Vibrations with Dual Resonances," *J. Vib. Acoust.*, 136(5), pp. 1–15. <https://doi.org/10.1115/1.4027805>.
33. Postnikov, A., Pavlovskaya, E., and Wiercigroch, M., 2017, "2DOF CFD Calibrated Wake Oscillator Model to Investigate Vortex-Induced Vibrations," *Int J Mech Sci*, 127, pp. 176–190. <https://doi.org/10.1016/j.ijmecsci.2016.05.019>.
34. Gabbai, R. D., and Benaroya, H., 2005, "An Overview of Modeling and Experiments of Vortex-Induced Vibration of Circular Cylinders," *J. Sound Vib.*, 282(3–5), pp. 575–616. <https://doi.org/10.1016/j.jsv.2004.04.017>.
35. Wu, Y., Cheng, Z., McConkey, R., Lien, F. S., and Yee, E., 2022, "Modelling of Flow-Induced Vibration of Bluff Bodies: A Comprehensive Survey and Future Prospects," *Energies (Basel)*, 15(22). <https://doi.org/10.3390/en15228719>.
36. Thorsen, M. J., Sævik, S., and Larsen, C. M., 2014, "A Simplified Method for Time Domain Simulation of Cross-Flow Vortex-Induced Vibrations," *J Fluids Struct*, 49, pp. 135–148. <https://doi.org/10.1016/j.jfluidstructs.2014.04.006>.
37. Thorsen, M. J., Sævik, S., and Larsen, C. M., 2015, "Fatigue Damage from Time Domain Simulation of Combined In-Line and Cross-Flow Vortex-Induced Vibrations," *Marine Structures*, 41, pp. 200–222. <https://doi.org/10.1016/j.marstruc.2015.02.005>.
38. Thorsen, M. J., 2016, "Time Domain Analysis of Vortex-Induced Vibrations," PhD Thesis, Norwegian University of Science and Technology.
39. Thorsen, M. J., Sævik, S., and Larsen, C. M., 2016, "Time Domain Simulation of Vortex-Induced Vibrations in Stationary and Oscillating Flows," *J Fluids Struct*, 61, pp. 1–19. <https://doi.org/10.1016/j.jfluidstructs.2015.11.006>.
40. Thorsen, M.J. and Sævik, S. (2017) "Simulating Riser VIV in Current and Waves using an Empirical Time Domain Model," *Proceedings of the ASME 2017 36th International Conference on Ocean, Offshore and Arctic Engineering*, OMAE2017, June 25-30, 2017, Trondheim, Norway, Paper No. OMAE2017-61217.
41. Thorsen, M. J., Sævik, S., and Larsen, C. M., 2017, "Non-Linear Time Domain Analysis of Cross-Flow Vortex-Induced Vibrations," *Marine Structures*, 51, pp. 134–151. <https://doi.org/10.1016/j.marstruc.2016.10.007>.
42. Ulveseter, J. V., Sævik, S., and Larsen, C. M., 2017, "Time Domain Model for Calculation of Pure In-Line Vortex-Induced Vibrations," *J Fluids Struct*, 68, pp. 158–173. <https://doi.org/10.1016/j.jfluidstructs.2016.10.013>.

43. Ulveseter, J. V., Thorsen, M. J., Sævik, S., and Larsen, C. M., 2018, "Time Domain Simulation of Riser VIV in Current and Irregular Waves," *Marine Structures*, **60**, pp. 241–260. <https://doi.org/10.1016/j.marstruc.2018.04.001>.
44. Wu, J., Jin, J., Yin, D., Lie, H., Passano, E., Sævik, S., Tognarelli, M.A., Grytøyr, G., Andersen, T., Karunakaran, D., & Igland, R., (2020), "Time Domain VIV Analysis Tool VIVANA-TD: Validations and Improvements." *Proceedings of the ASME 2020 39th International Conference on Ocean, Offshore and Arctic Engineering*. Volume 8: CFD and FSI. Virtual, Online. August 3–7, 2020. V008T08A031. ASME. <https://doi.org/10.1115/OMAE2020-18759>
45. Wu, J., Yin, D., Jin, J., Lie, H., Passano, E., Sævik, S., Tognarelli, M. A., Grytøyr, G., Andersen, T., Karunakaran, D., and Igland, R., (2022), "Time Domain Prediction of VIV Responses in Oscillatory Flow Conditions," *Proceedings of the ASME 2022 41st International Conference on Ocean, Offshore and Arctic Engineering*, Hamburg, Germany.
46. Wu, J., Jiang, F., Lie, H., Passano, E., Yin, D., Sævik, S., Grytøyr, G., Hogg, B. F., Resvanis, T., and Vandiver, K., (2023), "Evaluation of VIV Prediction Practice of Deep-Water Steel Lazy Wave Risers (SLWRs)," *Proceedings of the International Conference on Offshore Mechanics and Arctic Engineering - OMAE*, **7**. <https://doi.org/10.1115/OMAE2023-105583>.
47. Wu, J., Yin, D., Lie, H., Passano, E., Baarholm, G. S., and Sævik, S., (2024), "Time Domain Prediction of Combined CF and IL VIV Responses Under Constant Currents," *Proceedings of the ASME 2024 43rd International Conference on Ocean, Offshore and Arctic Engineering*, Singapore, Singapore.
48. Wu, J., Yin, D., Lie, H., Grytøyr, G., and Hogg, B. F., (2024), "Investigation of VIV Responses of Slender Structures Under Waves and Currents," *Proceedings of the ASME 2024 43rd International Conference on Ocean, Offshore and Arctic Engineering*, ASME, Singapore, Singapore.
49. Sarpkaya, T. and Shoaff, R.L. (1979) "A discrete-vortex analysis of flow about stationary and transversely oscillating circular cylinders," Report no. NPS-69SL79011, Naval Postgraduate School, Monterey, California.
50. Sarpkaya, T. and Schoaff, R.L. (1979) "Inviscid Model of Two-Dimensional Vortex Shedding by a Circular Cylinder," *AIAA Journal*, Vol. 17, No. 11, November 1979, Article No. 79-0281R.
51. Chaplin, J.R., Bearman, P.W., Cheng, Y., Fontaine, E., Graham, J.M.R., Herfjord, K., Huera Huarte, F.J., Isherwood, M., Lambrakos, K., Larsen, C.M., Meneghini, J.R., Moe, G., Pattenden, R.J., Triantafyllou, M.S., and Willden, R.H.J. (2005) "Blind predictions of laboratory measurements of vortex induced vibrations of a tension riser," *J. Fluids Struct.* **21**, 25-40.
52. Orcina (2025) "Sarpkaya & Shoaff 1979 Vortex Sheet model in Orcaflex," email reply to inquiry about this from David Heffernan, MD, Head of Software Development, Orcina Ltd. (Orcaflex developers.), 31 Mach 2025.
53. Cheng, Y. and Lambrakos, K.F. (2006) "Time Domain Computation of Riser VIV from Vessel Motions," *Proceedings of OMAE2006*, June 4-9, 2006, Hamburg, Germany, Paper No. OMAE2006-92432.
54. Cheng, Y. and Lambrakos, K.F. (2007) "Time Domain Riser VIV Predictions Compared to Field and Laboratory Test Data," *Proceedings of the 26th International Conference on Offshore Mechanics and Arctic Engineering*, OMAE2007, June 10-15, 2007, San Diego, California, USA, Paper No. OMAE2007-29548.
55. DNV (2017) "Free Spanning of Pipelines," Det Norske Veritas, Recommended Practice DNVGL-RP-F105, June 2017 Edition, Amended September 2021.
56. Riks, E. and Rankin, C.C., (1987), "Bordered Equations in Continuation Methods: An improved Solution Technique," NLR MP 87057U, National Aerospace Laboratory, The Netherlands.
57. Peek, R. (2022) "NPEX 22a Documentation Update," Documentation of updates in the Npex FE Analysis Code, available at www.peek.solutions.
58. Peek, R. (2020) "Nonlinear Span Assessment by Amplitude-Dependent Linearization," *Journal of Offshore Mechanics and Arctic Engineering (JOMAE)*, ASME, Vol. 42, October 2020.
59. Peek, R., Witz, M., and Vedeld, K. (2021) "Dynamics of a Pipeline Span on an Elastic Seabed," *Applied Ocean Research* **111** (2021) 102407.
60. Atkins (2014) "SAFEBUCK JIP - Feedback from Operating Pipelines," Atkins Report Number: 5087471.002.RP01.02, 10 November 2014.

61. Det Norske Veritas (2016) “Fatigue Design of Offshore Steel Structures,” Det Norske Veritas, Recommended Practice, DNVGL-RP-C203, April 2016.
62. Peek, R. and Kristiansen, N. Ø., (2009), “Zero-Radius Bend Method to Trigger Lateral Buckles,” *Journal of Transportation Engineering*, Vol. 135, No. 12, December 1, 2009.
63. Peek, R. (2024) “Dynamics of Pipeline Spans Created by ZRB Buckle Triggers: Analytical Approximation,” *Engrxiv*, DOI: <https://doi.org/10.31224/3440>.
64. Hilber, H.M. and Hughes, T.J.R., “Collocation, Dissipation and ‘Overshoot’ for Time Integration Schemes in Structural Dynamics,” *Earthquake Engineering and Structural Dynamics*, Vol. 6, pp. 99-117, 1978.
65. Karrech, A., et al. (2024) “Non-linear vibration of free spanning subsea pipelines with multi-dimensional mid-plane stretching,” *Engineering Structures* 301 (2024) 117265.
66. Strogatz, S.H. (2018) “Nonlinear Dynamics and Chaos,” 2nd Edition, CRC Press, Taylor & Francis Group, Boca Raton, Florida.