Lexicographic and Lexicographic Egalitarian Bi-Objective Linear Programming

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Abstract

We show that the set of solutions of a lexicographic bi-objective linear programming problem is equal to the set of solutions of a linear programming problem with the same constraints and whose objective function is a non-negative and non-zero weighted sum of the two objective functions. We also show that the set of solutions of a lexicographic egalitarian bi-objective linear programming problem is equal to the set of solutions of a linear programming problem with the same constraints and whose objective function is a non-negative and non-zero weighted sum of the minimum and maximum of the two objective functions.

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1. Introduction: This paper is concerned with bi-objective linear programming, which is a special case of the kind of multi-objective linear programming discussed in Luc (1989, 2016) , Jahn (2004) and Luan (2019), the last one being extensions of results on vector optimization to infinite dimensional spaces. The available literature on bi-objective linear programming is about its applications to specific problems (e.g. portfolio optimization problem as in Inzadi and Yaghoobi (2024)). To the best of our knowledge, the theory of bi-objective linear programming, has not been explored beyond what is applicable for it in the context of multi-objective linear programming. Further, the theory of multi-objective linear programming is concerned with general properties of "set of efficient points".

In this paper, we are concerned with two solutions for bi-objective linear programming problems, thereby leading to two categories of bi-objective linear programming problems. For the first category of problems, there is an exogenously

specified priority over the two objective functions and points satisfying the inequality constraints are ordered according to lexicographic preferences consistent with this priority over the pairs of values of the objective functions. We refer to such problems as lexicographic bi-objective linear programming problems. For the second category of problems, first priority is accorded to the lesser of the two values of the objective functions after which pairs of such values are compared lexicographically. We refer to the second category of problems as lexicographic egalitarian bi-objective linear programming problems. The second category of problems may be interpreted as the problems faced by a "egalitarian-minded decision-maker" whose choice rewards two groups of individuals and the decision-maker wants to maximize the rewards of the "poorer" group.

It turns out that if the constraint set is a bounded polyhedron and the zero vector is feasible but not chosen, then for each lexicographic bi-objective linear programming problem there exists a non-negative non-zero pair of weights that assigns a greater weight to the first priority, such that an alternative solves the problem if and only if it maximizes the weighted sum of the two objective functions subject to the same inequality constraints. Further, for each lexicographic egalitarian bi-objective linear programming problem there exists a non-negative non-zero pair of weights that assigns a greater weight to the first priority, such that an alternative solves the problem if and only if it maximizes the weighted sum of the maximum and minimum of the two objective functions subject to the same inequality constraints. The possibility of assigning a greater weight to the first priority, permits transforming the maximization of the weighted sum problem for lexicographic egalitarian bi-objective linear programming problem, into a linear programming problem.

Several points need to be clarified about our results. First, the non-negative, non-zero pair of weights defining the objective function of the linear programming problem in each context depends on the initial problem. This is no more demanding than the conclusion that the solution of a dual linear programming problem depends on the primal linear programming problem. Second, it is quite possible that for a lexicographic egalitarian linear programming problem, there are two solutions such that the minimum value for the former is obtained by one objective function and the minimum value for the latter is obtained by the other objective function. Thus, our main result for lexicographic egalitarian linear programming problems is not a corollary of our main result for lexicographic linear programming problems. Third,

while for both types of bi-objective linear programming problems that we consider here, the solutions are efficient points, for the first type the pair of optimal values of the objective functions is uniquely defined and for the second type the minimum value and the optimal maximum value in the optimal pair is uniquely defined.

To obtain the main result for each type of bi-objective linear programming problem that we are concerned with here, we make crucial use of the fact that the optimal value pairs are extreme points of the corresponding set of value pairs associated with alternatives that satisfy the constraints. The pair of values of the objective functions at an arbitrary efficient point, need not be an extreme point of the set of all possible pairs of values of the objective functions. This makes our results more context specific than the results for efficient points of multi-objective linear programming problems available in the existing literature on the subject.

2. Bi-Objective Linear Programming Problem: For positive integers m, n, let $A \epsilon \mathbb{R}^{m \times n}$, $b \epsilon \mathbb{R}^m$ and c, d $\epsilon \mathbb{R}^n$. Vectors (unless otherwise mentioned) are column vectors. Transpose of a vector or a matrix is denoted by the superscript "T".

Let $X(A,b) = \{x \epsilon \mathbb{R}^n_+ | \ Ax \leq b\}$.

Notation: We will write the zero vector as 0, when its dimension is apparent from the context. Otherwise the r-dimensional zero vector will be denoted by $0^{(r)}$. ▌

Let $X^+(A, b, c, d) = \{(x, u, v) | \ Ax \leq b, c^Tx - u = 0, d^Tx - v = 0, -I^{(n)}x \leq 0\}$ where $I^{(n)}$ is the n×n identity matrix.

Note 1: $X^+(A, b, c, d)$ is a polyhedron, since $c^Tx - u = 0$, $d^Tx - v$ can be expressed as $c^Tx - u \leq 0$, $d^Tx - v \leq 0$, $-c^Tx + u \leq 0$, $-d^Tx + v \leq 0$. (See Williamson (2014) or topic 6 in Lahiri (2023) for definition and properties of polyhedron).

Let $U(A, b, c, d) = \{(u, v) | (x, u, v) \epsilon X^+(A, b, c, d) \text{ for some } x\} = \{(u, v) | \ u = c^Tx, v = d^Tx, Ax \leq b, x \epsilon \mathbb{R}^n_+\}$.

Note 2: By applying Fourier-Motzkin elimination (see Frank (2019) or topic 6 in Lahiri (2023)) to $X^+(A, b, c, d)$ we get that U(A, b, c, d) is a polyhedron. ▌

We will refer to the ordered 4-tuple (A, b, c, d) as a **bi-objective linear programming problem (BO-LP)**.

Assumption 1: X(A, b) is a non-empty bounded subset of $\mathbb{R}^n_+$ containing 0. ▌

Note 3: An immediate consequence of assumption 1 is that $X^+(A, b, c, d)$ is a bounded polyhedron containing $(0, 0, 0) \epsilon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ and U(A, b, c, d) is a bounded polyhedron containing $0^{(2)}$. Thus, U(A, b, c, d) is a convex polytope (see topic 6 in

Lahiri (2023)), i.e., U(A, b, c, d) is the convex hull of a non-empty finite set of extreme points. ▌

An **efficient point** for the (BO-LP) (A, b, c, d) is a point $x^* \epsilon X(a, b)$ such that there *does not exist* any $x \epsilon X(A, b)$ satisfying $(c^T x, d^T x) > (c^T x^*, d^T x^*)$ (i.e., there *does not exist* any $x \epsilon X(A, b)$ satisfying $c^T x \geq c^T x^*$, $d^T x \geq d^T x^*$ with at least one strict inequality).

3. The Lexicographic Binary Relation on Ordered Pairs of Real Numbers: Let $\succcurlyeq$ be a binary relation on $\mathbb{R}^2$ such that for all $(\gamma_1, \eta_1), (\gamma_2, \eta_2) \epsilon \mathbb{R}^2$: $(\gamma_1, \eta_1) \succcurlyeq (\gamma_2, \eta_2)$ if and only if either $\gamma_1 > \gamma_2$ or $\gamma_1 = \gamma_2$ and $\eta_1 \geq \eta_2$.

Clearly $\succcurlyeq$ is a linear order on $\mathbb{R}^2$. It is generally known as the **lexicographic preference ordering (lexicographic preferences) on** $\mathbb{R}^2$ (see Varian (1978)).

Let $\succ$ denote the asymmetric part of $\succcurlyeq$.

4. The lexicographic linear programming problem: The lexicographic bi-objective linear programming **(L-BO-LP)** problem that we are concerned with is the following:

Maximize (with respect to $\succcurlyeq$) $(c^T x, d^T x)$, subject to $Ax \leq b$, $x \epsilon \mathbb{R}_+^n$.

which can also be expressed as:

Maximize (with respect to $\succcurlyeq$) (u, v), subject to $Ax \leq b$, $-c^T x + u \leq 0$, $-d^T x + v \leq 0$, $x \epsilon \mathbb{R}_+^n$, $u, v \epsilon \mathbb{R}$.

The above may be formulated as a two stage maximization problem: Maximize $d^T x$, subject to x being a solution of [Maximize $c^T y$, subject $Ay \leq b$, $y \epsilon \mathbb{R}_+^n$].

The above is equivalent to: Maximize v, subject to $-d^T x + v \leq 0$, $v \epsilon \mathbb{R}$ where (x, u^*) solves: Maximize u, subject to $-c^T y + u \leq 0$, $Ay \leq b$, $y \epsilon \mathbb{R}_+^n$, $u \epsilon \mathbb{R}$.

Let **LP(u)** denote the linear programming problem: Maximize $d^T x$, subject to $Ax \leq b$, $c^T x = u$, $x \epsilon \mathbb{R}_+^n$.

Let $\max_{x \epsilon X(A,b)} c^T x = u^*$.

Note 4: Since $0 \epsilon X(A, b)$, it must be the case that $u^* \geq 0$.

Then x^* solves L-BO-LP if and only if x^* solves the linear programming problem **(LP(u^*))**.

Let $v^* = \max_{x \epsilon \{z \epsilon X(A,b) | c^T z = u^*\}} d^T x$

Lemma 1: Suppose the n-dimensional 0-vector does not solve L-BO-LP (i.e., $0 \notin \operatorname{argmax}_{x \epsilon \{z \epsilon X(A,b) | c^T z = u^*\}} d^T x$). Then, there exists a pair of real numbers $(\alpha, \beta) \neq 0$ such that x^*solves L-BO-LP if and only if $x^* \epsilon \operatorname{argmax}_{x \epsilon X(A,b)} (\alpha c^T x + \beta d^T x)$.

Proof: (u^*, v^*) is an extreme point of U(A, b, c, d). For if there exists $(u^{(1)}, v^{(1)})$, $(u^{(2)}, v^{(2)})$ and $t\epsilon(0, 1)$ such that $(u^{(1)}, v^{(1)}) \neq (u^{(2)}, v^{(2)})$ and $(u^*, v^*) = t(u^{(1)}, v^{(1)}) + (1-t)(u^{(2)}, v^{(2)})$, then it must be the case that $u^{(1)} = u^{(2)} = u^*$ and $v^{(1)} \neq v^{(2)}$ and $v^* = tv^{(1)} + (1-t)v^{(2)}$. Thus, $\max\{v^{(1)}, v^{(2)}\} > v^* > \min\{v^{(1)}, v^{(2)}\}$ and $(u^*, \max\{v^{(1)}, v^{(2)}\}) \epsilon U(A, b, c, d)$ thereby contradicting the definition of v^*.

Thus there does not exist any array $<t(u,v)|$ (u, v) is an extreme point of U(A, b, c, d), $(u, v) \neq (u^*, v^*)>$ in [0, 1] satisfying

$\sum_{(u,v)\} \text{ is an extreme point of } U(A,b,c,d),\ (u,v)\neq(u^*,v^*)} t(u, v) = 1$, such that

$\sum_{(u,v)\} \text{ is an extreme point of } U(A,b,c,d),\ (u,v)\neq(u^*,v^*)} t(u, v)(u,v) = (u^*, v^*)$.

Thus, by Farka's lemma there exists real numbers α, β, γ such that $\alpha u + \beta v + \gamma \leq 0 < \alpha u^* + \beta v^* + \gamma$ for every extreme point $(u, v) \neq (u^*, v^*)$.

Thus, $\alpha u + \beta v < \alpha u^* + \beta v^*$ for every extreme point of U(A, b, c, d)\{(u^*, v^*)}.

Now x^* solves L-BO-LP if and only if $x^* \epsilon$ X(A, b) and $(c^T x^*, d^T x^*) = (u^*, v^*)$

Thus, x^* solves L-BO-LP if and only if $x^* \epsilon \underset{x\epsilon X(A,b)}{\operatorname{argmax}} (\alpha c^T x + \beta d^T x)$.

Since $0^{(n)} \notin \underset{x\epsilon\{z\epsilon X(A,b)|c^T z = u^*\}}{\operatorname{argmax}} d^T x$, $(u^*, v^*) \neq (0, 0)$.

Thus, $\alpha u^* + \beta v^* > 0$.

Hence, $(\alpha, \beta) \neq (0, 0)$. Q.E.D.

The straightforward proof of the following lemma is being omitted.

Lemma 2: If for all $y\epsilon \underset{x\epsilon X(A,b)}{\operatorname{argmax}} c^T x$ it is the case that $d^T y = v^*$, then x^* solves L-BO-LP <u>if and only if</u> x^* solves [Maximize $c^T x$ subject to $x\epsilon X(A, b)$].

Lemma 3: If there exists $y\epsilon \underset{x\epsilon X(A,b)}{\operatorname{argmax}} c^T x$ satisfying $d^T y < v^*$, then $\beta > 0$ whenever $\underset{x\epsilon X(A,b)}{\operatorname{argmax}} (\alpha c^T x + \beta d^T x) = \underset{x\epsilon\{z\epsilon X(A,b)|c^T z = u^*\}}{\operatorname{argmax}} d^T x$. Further, α may be chosen to satisfy $\alpha > \beta$.

Proof: If there exists $y\epsilon \underset{x\epsilon X(A,b)}{\operatorname{argmax}} c^T x$ satisfying $d^T y < v^*$, then $c^T y = u^*$ and $y \notin \underset{x\epsilon\{z\epsilon X(A,b)|c^T z = u^*\}}{\operatorname{argmax}} d^T x$.

Thus, for (α, β) as in the statement of this lemma, $\alpha u^* + \beta v^* > \alpha u^* + \beta d^T y$.

Since, $d^T y < v^*$, $\beta v^* > \beta d^T y$ if and only if $\beta > 0$.

Now $u^* \geq u$ for all u satisfying $(u, v) \epsilon W(A, b, c, d)$ for some v. and $\alpha u^* + \beta v^* > \alpha u + \beta v$ for all $(u, v)\epsilon U(A, b, c, d)\backslash\{(u^*, v^*)\}$ implies $(\alpha+1)u^* + \beta v^* > (\alpha + 1)u + \beta v$ for all $(u, v)\epsilon U(A, b, c, d)\backslash\{(u^*, v^*)\}$.

Thus, we may without loss of generality assume that (α, β) satisfies α > β > 0. Q.E.D.

In view of lemmas 1, 2 and 3, we have the following proposition.

Proposition 1: Suppose the n-dimensional 0-vector does not solve L-BO-LP (i.e., $0 \notin \underset{x \epsilon \{z \epsilon X(A,b) | c^T z = u^*\}}{\operatorname{argmax}} d^T x$). Then, there exists a pair of real numbers (α, β) satisfying α > β ≥ 0 such that x^*solves L-BO-LP if and only if $x^* \epsilon \underset{x \epsilon X(A,b)}{\operatorname{argmax}} (\alpha c^T x + \beta d^T x)$.

Proof: By lemma 1, there exists a pair of real numbers (α, β) ≠ 0 such that x^*solves L-BO-LP if and only if $x^* \epsilon \underset{x \epsilon X(A,b)}{\operatorname{argmax}} (\alpha c^T x + \beta d^T x)$.

Case 1: There exists $y \epsilon \underset{x \epsilon X(A,b)}{\operatorname{argmax}} c^T x$ satisfying $d^T y < v^*$.

By lemma 3, β > 0 and by lemma 3 once again α may be chosen to satisfy α > β.

Case 2: For all $y \epsilon \underset{x \epsilon X(A,b)}{\operatorname{argmax}} c^T x$ it is the case that $d^T y = v^*$.

In this case, from lemma 2 it follows that $\underset{x \epsilon \{z \epsilon X(A,b) | c^T z = u^*\}}{\operatorname{argmax}} d^T x = \underset{x \epsilon X(A,b)}{\operatorname{argmax}} (c^T x + d^T x)$ for α = 1, β = 0.

Thus, under all circumstances, there exists a pair of real numbers (α, β) satisfying α > β ≥ 0 such that $\underset{x \epsilon \{z \epsilon X(A,b) | c^T z = u^*\}}{\operatorname{argmax}} d^T x = \underset{x \epsilon X(A,b)}{\operatorname{argmax}} (c^T x + d^T x)$. Q.E.D.

5. Lexicographic Egalitarian Linear Programming: A somewhat more egalitarian solution for the bi-objective linear programming problem which we refer to as the **"lexicographic egalitarian bi-objective linear programming" problem (LE-BO-LP)** is the following.

Maximize (with respect to ≽) $(\min\{c^T x, d^T x\}, \max\{c^T x, d^T x\})$ subject to $Ax \leq b$, $x \epsilon \mathbb{R}_+^n$.

In a more familiar two-stage optimization form, the above problem is equivalent to the following:

Maximize $\max\{d^T x, c^T x\}$ subject to x being a solution of [Maximize $\min\{d^T y, c^T y\}$ subject to $Ay \leq b$, $y \epsilon \mathbb{R}_+^n$].

Note 5: Under our Assumption 1, there exist real numbers $\overline{u}$, $\overline{v}$ with $\overline{u} \leq 0 \leq \overline{v}$ such that for all $x \epsilon X(A, b)$, $\overline{u} \leq \min\{c^T x, d^T x\} \leq \max\{c^T x, d^T x\} \leq \overline{v}$.

Thus the maximization problem above can be reformulated as:

Maximize $d^T x + c^T x - u^{**}$ subject to (x, u^{**}) being a solution of [Maximize u subject to $u - c^T y \leq 0$, $u - d^T y \leq 0$, $Ay \leq b$, $y \epsilon \mathbb{R}_+^n$, $u \epsilon [\overline{u}, \overline{v}]$.

If (x, u^{**}) solves [Maximize u subject to u - $c^T y \leq 0$, u - $d^T y \leq 0$, $Ay \leq b$, $y \in \mathbb{R}^n_+$], then $v^{**} = d^T x + c^T x - u^{**} = \max\{d^T x, c^T x\} \in [\overline{u}, \overline{v}]$.

Thus, $v^{**} = \max_{\{x \in X(A,b) | min\{c^T x, d^T x\} = u^{**}\}} \max\{c^T x, d^T x\} = \max_{x \in \{z \in X(A,b) | min\{c^T z, d^T z\} = u^{**}\}} c^T x + t^T x - u^{**})$.

Let $X^{e+}(A, b, c, d) = \{(x, u, v) | Ax \leq b, u - c^T y \leq 0, u - d^T y \leq 0, v - d^T x - c^T x + u = 0, u \leq \overline{v}, v \leq \overline{v}, -I^{(n)} x \leq 0, -u \leq -\overline{u}, , -v \leq -\overline{u}\}$.

Clearly $X^{e+}(A, b, c, d)$ is a non-empty bounded polyhedron.

Let $W(A, b, c, d) = \{(u, v) | (x, u, v) \in X^{e+}(A, b, c, d) \text{ for some } x\}$.

Note 6: It follows from the definitions of u^{**}, v^{**} that if $(u, v) \in W(A, b, c, d)$, then $u \leq u^{**}$ and if $(u^{**}, v) \in W(A, b, c, d)$, then $v \leq v^{**}$.

By applying Fourier-Motzkin elimination we may conclude that W(A, b, c, d) is a non-empty bounded polyhedron and hence a convex polytope.

Lemma 4: Suppose the n-dimensional 0-vector does not solve LE-BO-LP (i.e., $0 \notin \operatorname{argmax}_{x \in \{z \in X(A,b) | min\{c^T z, d^T z\} = u^{**}\}} (c^T x + t^T x - u^{**})$. Then, there exists a pair of real numbers $(\alpha, \beta) \neq 0$ such that x^* solves LE-BO-LP if and only if $x^* \in \operatorname{argmax}_{x \in X(A,b)} (\alpha \min\{c^T x, d^T x\} + \beta \max\{c^T x, d^T x\})$.

Proof: (u^{**}, v^{**}) is an extreme point of W(A, b, c, d). For if there exists $(u^{(1)}, v^{(1)})$, $(u^{(2)}, v^{(2)})$ and $t \in (0, 1)$ such that $(u^{(1)}, v^{(1)}) \neq (u^{(2)}, v^{(2)})$ and $(u^{**}, v^{**}) = t(u^{(1)}, v^{(1)}) + (1-t)(u^{(2)}, v^{(2)})$, then since $u^{(1)} \leq u^{**}$ and $u^{(2)} \leq u^{**}$ it must be the case that $u^{(1)} = u^{(2)} = u^{**}$ and $v^{(1)} \neq v^{(2)}$ and $v^{**} = tv^{(1)} + (1-t)v^{(2)}$.

Thus, $\max\{v^{(1)}, v^{(2)}\} > v^{**} > \min\{v^{(1)}, v^{(2)}\}$ and $(u^{**}, \max\{v^{(1)}, v^{(2)}\}) \in W(A, b, c, d)$ thereby contradicting the definition of v^{**}.

Thus there does not exist any array $<t(u,v) | (u, v)$ is an extreme point of W(A, b, c, d), $(u, v) \neq (u^{**}, v^{**})>$ in [0, 1] satisfying $\sum_{(u,v) \text{ is an extreme point of } W(A,b,c,d), (u,v) \neq (u^*, v^*)} t(u, v) = 1$, such that $\sum_{(u,v) \text{ is an extreme point of } W(A,b,c,d), (u,v) \neq (u^*, v^*)} t(u, v)(u,v) = (u^{**}, v^{**})$.

Thus, by Farka's lemma there exist real numbers α, β, γ such that $\alpha u + \beta v + \gamma \leq 0 < \alpha u^{**} + \beta v^{**} + \gamma$ for every extreme point $(u, v) \neq (u^{**}, v^{**})$.

Thus, $\alpha u + \beta v < \alpha u^{**} + \beta v^{**}$ for every extreme point of W(A, b, c, d)$\backslash\{(u^{**}, v^{**})\}$.

Now x^* solves LE-BO-LP if and only if $x^* \in X(A, b)$ and $(\min\{c^T x^*, d^T x^*\}, \max\{c^T x^*, d^T x^*\}) = (u^{**}, v^{**})$

Thus, x^* solves LE-BO-LP if and only if $x^* \epsilon \underset{x \epsilon X(A,b)}{\operatorname{argmax}} (\alpha \min\{c^T x, d^T x\} + \beta \max\{c^T x, d^T x\})$.

Since $0^{(n)} \notin \underset{x \epsilon \{z \epsilon X(A,b) | min\{c^T z, d^T z\} = u^{**}\}}{\operatorname{argmax}} (c^T x + t^T x - u^{**})$, and $0^{(2)} \epsilon$ W(A, b, c, d), it must be the case that $\alpha u^* + \beta v^* > 0$.

Hence, $(\alpha, \beta) \neq (0, 0)$. Q.E.D.

Note 7: Having established that (u^{**}, v^{**}) is an extreme point of W(A, b, c, d), the proof of lemma 4 differs from the proof of lemma 1 at one point, which we have highlighted in *italics*. ▌

The following lemma which is analogous to lemma 2 has an equally simple proof that we are omitting.

Lemma 5: If for all $y \epsilon X(A, b)$ satisfying $\min\{c^T y, d^T y\} = u^{**}$ it is the case that $\max\{c^T y, d^T y\} = v^{**}$, then x^* solves LE-BO-LP <u>if and only if</u> x^* solves [Maximize $\min\{c^T x, d^T x\}$ subject to $x \epsilon X(A, b)$].

Lemma 6: If there exists $y \epsilon \underset{x \epsilon X(A,b)}{\operatorname{argmax}} \min\{c^T x, d^T x\}$ satisfying $\max\{c^T y, d^T y\} < v^{**}$, then $\beta > 0$ whenever $\underset{x \epsilon X(A,b)}{\operatorname{argmax}} (\alpha \min\{c^T x, d^T x\} + \beta \max\{c^T x, d^T x\})$.

$= \underset{x \epsilon \{z \epsilon X(A,b) | min\{c^T z, d^T z\} = u^{**}\}}{\operatorname{argmax}} (c^T x + t^T x - u^{**})$. Further, α may be chosen to satisfy $\alpha > \beta$.

Proof: If there exists $y \epsilon \underset{x \epsilon X(A,b)}{\operatorname{argmax}} \min\{c^T x, d^T x\}$ satisfying $\max\{c^T y, d^T y\} < v^{**}$, then $\min\{c^T y, d^T y\} = u^{**}$ and $y \notin \underset{x \epsilon \{z \epsilon X(A,b) | min\{c^T z, d^T z\} = u^{**}\}}{\operatorname{argmax}} (c^T x + t^T x - u^{**})$.

Thus, for (α, β) as in the statement of this lemma, $\alpha u^{**} + \beta v^{**} > \alpha u^{**} + \beta \max\{c^T y, d^T y\}$.

Since, $\max\{c^T y, d^T y\} < v^{**}$, $\beta v^{**} > \beta \max\{c^T y, d^T y\}$ if and only if $\beta > 0$.

Now $u^{**} \geq u$ for all u satisfying $(u, v) \epsilon$W(A, b, c, d) for some v and $\alpha u^{**} + \beta v^{**} > \alpha u + \beta v$ for all $(u, v) \epsilon$W(A, b, c, d)\{(u^{**}, v^{**})} implies $(\alpha+1)u^{**} + \beta v^{**} > (\alpha + 1)u + \beta v$ for all $(u, v) \epsilon$W(A, b, c, d)\{(u^{**}, v^{**})}.

Thus, we may without loss of generality assume that (α, β) satisfies $\alpha > \beta > 0$. Q.E.D.

In view of lemmas 4, 5 and 6, we have the following proposition.

Proposition 2: Suppose the n-dimensional 0-vector does not solve LE-BO-LP (i.e., $0 \notin \underset{x \epsilon \{z \epsilon X(A,b) | min\{c^T z, d^T z\} = u^{**}\}}{\operatorname{argmax}} \max\{c^T x, d^T x\}$. Then, there exists a pair of real numbers

$(\alpha, \beta) > 0$ satisfying $\alpha > \beta \geq 0$ such that x^*solves LE-BO-LP <u>if and only if</u> $x^* \epsilon$ $\underset{x \epsilon X(A,b)}{\operatorname{argmax}} (\alpha \min\{c^T x, d^T x\} + \beta \max\{c^T x, d^T x\})$.

Proof: By lemma 4, there exists a pair of real numbers $(\alpha, \beta) \neq 0$ such that x^* solves LE-BO-LP if and only if $x^* \epsilon \underset{x \epsilon X(A,b)}{\operatorname{argmax}} (\alpha \min\{c^T x, d^T x\} + \beta \max\{c^T x, d^T x\})$.

Case 1: There exists $y \epsilon \underset{x \epsilon X(A,b)}{\operatorname{argmax}} \min\{c^T x, d^T x\}$ satisfying $\max\{c^T y, d^T y\} < v^{**}$.

By lemma 6, $\beta > 0$ and by lemma 6 once again, we may assume $\alpha > \beta$.

Case 2: For all $y \epsilon \underset{x \epsilon X(A,b)}{\operatorname{argmax}} \min\{c^T x, d^T x\}$ it is the case that $\max\{c^T y, d^T y\} = v^{**}$..

In this case, from lemma 5 it follows that $\underset{x \epsilon \{z \epsilon X(A,b) | min\{c^T z, d^T z\} = u^{**}\}}{\operatorname{argmax}} \max\{c^T x, d^T x\} = \underset{x \epsilon X(A,b)}{\operatorname{argmax}} (\alpha \min\{c^T x, d^T x\} + \beta \max\{c^T x, d^T x\})$ for $\alpha = 1, \beta = 0$.

Thus, under all circumstances, there exists a pair of real numbers (α, β) satisfying $\alpha > \beta \geq 0$ such that $\underset{x \epsilon \{z \epsilon X(A,b) | min\{c^T z, d^T z\} = u^{**}\}}{\operatorname{argmax}} \max\{c^T x, d^T x\} = \underset{x \epsilon X(A,b)}{\operatorname{argmax}} (\alpha \min\{c^T x, d^T x\} + \beta \max\{c^T x, d^T x\})$. Q.E.D.

Note 8: The requirement $\alpha > \beta$ in the statement of proposition 2 allows us to formulate the maximization problem $\max_{x \epsilon X(A,b)} (\alpha \min\{c^T x, d^T x\} + \beta \max\{c^T x, d^T x\})$ as the *linear programming maximization problem* $\max_{x \epsilon X(A,b), \xi \leq c^T x, \xi \leq d^T x, \xi \epsilon \mathbb{R}} ((\alpha - \beta)\xi + \beta (c^T x + d^T x))$.

Note 9: Since proposition 2 continues to remain valid if we replace α by $\frac{\alpha}{\alpha+\beta}$ and β by $\frac{\beta}{\alpha+\beta}$, proposition 2 may be rewarded as follows:

Suppose the n-dimensional 0-vector does not solve LE-BO-LP (i.e., $0 \notin \underset{x \epsilon \{z \epsilon X(A,b) | min\{c^T z, d^T z\} = u^{**}\}}{\operatorname{argmax}} \max\{c^T x, d^T x\}$. Then, the set $\underset{x \epsilon \{z \epsilon X(A,b) | min\{c^T z, d^T z\} = u^{**}\}}{\operatorname{argmax}} \max\{c^T x, d^T x\}$ satisfies the **Hurwicz (1951) criterion**, i.e., there exists non-negative real numbers α, β satisfying $\alpha + \beta = 1$, such that x^*solves LE-BO-LP <u>if and only if</u> $x^* \epsilon \underset{x \epsilon X(A,b)}{\operatorname{argmax}} (\alpha \min\{c^T x, d^T x\} + \beta \max\{c^T x, d^T x\})$. ▌

Note 10: It quite possible that $x^{(1)}, x^{(2)} \epsilon \underset{x \epsilon \{z \epsilon X(A,b) | min\{c^T z, d^T z\} = u^{**}\}}{\operatorname{argmax}} \max\{c^T x, d^T x\}$, with $d^T x^{(1)} = c^T x^{(2)} = u^{**} < v^{**} = c^T x^{(1)} = d^T x^{(2)}$. Thus, proposition 2 is not a corollary of proposition 1.▌

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