

Capped Repulsion Limits the Size of Safely Separated Swarms in Potential-Game Flocking

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Abstract—This article introduces *Flockwork*, a distributed flocking law for drone swarms, and analyzes its energy and safety properties. Each drone follows the saturated negative gradient of a local cost combining a goal term and a coupling term to nearby neighbors. The pair repulsion is capped at a finite value, and the attraction between them switches off once they exceed a fixed sensing range. We first show that the individual costs form an exact potential game, so a single function decreases along every trajectory. We then prove that the sensing-range cutoff is harmless: for almost every starting condition the solution is unique and spends zero time on the switching surface. Next, we give a condition on the step size under which the discrete-time implementation still decreases the potential, and show that once the step size exceeds this bound, equilibria go unstable through period doubling. Finally, capping the repulsion at a finite value limits the size of a safely separated swarm: once the swarm exceeds a threshold that grows as the square of the cap-to-navigation-speed ratio, every equilibrium has a pair pushed into the saturated zone, where repulsion can no longer maintain the safety gap. This order is sharp: we build safe equilibria of the same order from concentric rings of drones. Numerical experiments on a literature parameter set and a deployed controller confirm the results.

Index Terms—Multiagent systems, flocking, potential games, discontinuous systems, sampled-data control, collision avoidance.

I. INTRODUCTION

COLLECTIONS of drones operating without centralized coordination are now routinely deployed for terrain mapping, search-and-rescue, and aerial displays. Each vehicle executes identical local rules and responds only to neighboring vehicles, making the collective both scalable and tolerant of individual failures. The canonical local rules repel neighbors at close range, attract them at intermediate range, and steer the group toward an objective [1], [2]. Translating these rules into controllers with provable guarantees has occupied the multiagent control community for nearly two decades.

In the following, we survey the literature on flocking controllers admitting formal analysis. We begin with potential-based designs, then address safety, sensing-range discontinuities, and sampled-data implementation.

A large body of flocking controllers is founded on pairwise interaction potentials. Leonard and Fiorelli [3] introduced artificial potentials and virtual leaders to orchestrate group motion, while Gazi and Passino [4] established cohesion for swarms governed by attraction–repulsion functions. Olfati-Saber [5] developed a complete framework employing smooth pair potentials—bump functions that taper interactions to zero

at the sensing boundary—together with a navigational feedback term. Alignment and consensus properties were analyzed in [6]–[8]. Outdoor flight experiments [9], [10] demonstrated that such rules are viable in practice, while also revealing that real implementations saturate every command: speeds, forces, and accelerations are all subject to limits. When each vehicle minimizes its own cost, the group constitutes a game; symmetric pairwise costs make it an exact potential game [11], [12], yielding a single Lyapunov-like certificate for the entire group.

Collision avoidance in these frameworks typically relies on a repulsive term that diverges as inter-vehicle separation vanishes, rendering the potential infinite at contact [5], [13]. An alternative approach enforces safety via control barrier functions, resolved at each timestep through a quadratic program [14], [15]. Their guarantee assumes that this program is always feasible, which can fail when the commands are bounded. In a deployed controller the repulsion manifests as a velocity command and is clipped to a finite value C like every other output. Far less is understood about what this saturation does to the group’s safety guarantees.

A second concern is the sensing range. If pair attraction terminates abruptly at the sensing radius R , the closed-loop system is discontinuous. Solutions must then be interpreted in the Filippov sense [16]–[18], and the discontinuity can in principle trap a trajectory on the switching surface, where it slides or chatters. Classical results require the field on one side of the discontinuity surface to point across it [16], [17], and results for irregular fields recover uniqueness for almost every initial condition when the field is Sobolev or of bounded variation with controlled divergence [19]–[21]. A discontinuous cutoff satisfies neither condition (Remark 3). Smooth bump functions [5] sidestep this issue, yet many deployed systems use a discontinuous cutoff. Finally, every controller operates at a fixed sample period h , and gradient-like flows are preserved by their Euler discretization only for sufficiently small steps [22].

The foregoing discussion identifies three open questions for flocking laws with caps, discontinuous cutoffs, and sampling: whether the discontinuous cutoff is well posed, how large the sample period may be made, and whether a finite cap still prevents collisions when many vehicles share a common goal. The last question has a well-known analogue in crowd dynamics, where pressure propagates through a dense group until individuals near the center are crushed [23]. In a swarm directed to a single point, every vehicle pushes inward with the navigation speed v_{nav} , while each pair can resist with at most C . In swarming models with bounded soft-core repulsion,

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large groups can collapse when the pair potential is not H-stable [24], [25]. Here collapse is driven by a shared goal and a capped command rather than by the potential, and we are not aware of results that quantify this trade-off.

This work makes two main contributions and one supporting one. First, we prove that the sensing-range cutoff of the Flockwork law is well posed without additional assumptions: for almost every initial condition the Filippov solution is unique and crosses the switching surfaces transversally, spending zero time on them. The argument rests on a sign property of the cutoff, which we show expands volume in state space and never compresses it. Unlike classical results, it needs neither transversality at the switching surface nor an absolutely continuous divergence. As a supporting result, we derive a sample period below which the sampled-data law continues to decrease the potential, and show that above $2/\lambda_{\max}$ of the Hessian, equilibria destabilize through period doubling. Second, we prove that a finite cap bounds the size of a swarm with a common goal: beyond a threshold of order $(C/v_{\text{nav}})^2$, every equilibrium contains at least one pair in the saturated region. We show that this order is sharp by building safe equilibria of the same order from concentric rings of drones. Numerical experiments on parameter sets drawn from the literature [5], [9] and on a deployed controller confirm each result and show that the $(C/v_{\text{nav}})^2$ scaling is visible in simulation.

The remainder of this article is organized as follows. Section II states the Flockwork law and necessary preliminaries. Section III establishes its potential-game structure. Section IV proves well-posedness at the cutoff. Section V analyzes the sampled-data implementation. Section VI derives the swarm-size limit under a common goal. Section VII presents numerical examples, and Section VIII concludes. Longer proofs are collected in the appendices.

Notations: For a set S , $|S|$ denotes its Lebesgue measure, $S^{(\delta)}$ its open δ -neighborhood, and $\text{co } S$ its convex hull. Let $|x|$ be the Euclidean norm of a vector x and $\|X\|$ the induced norm of a matrix X . B_ρ is the open ball of radius ρ centered at the origin. We write $A \preceq B$ when $B - A$ is positive semidefinite, and $\lambda_{\max}(A)$ for the largest eigenvalue of a symmetric matrix A . For a function ϕ , $\nabla\phi$ denotes its gradient and $\nabla^2\phi$ its Hessian; ∇_{x_i} is the gradient with respect to the block x_i .

II. PROBLEM STATEMENT AND PRELIMINARIES

A. The Flockwork Law

We place N drones in the plane with positions $x_i \in \mathbb{R}^2$, $i \in \{1, \dots, N\}$, collecting their states as $x = (x_1, \dots, x_N) \in \mathbb{R}^{2N}$. Each drone is modeled as a velocity-controlled point satisfying $\dot{x}_i = u_i$, which reflects the presence of a fast inner position loop. We write $r_{ij} = |x_i - x_j|$ and $\hat{e}_{ij} = (x_i - x_j)/r_{ij}$.

Assumption 1 (Parameters): The lengths satisfy $0 < d_m < d_{\text{in}} < d_{\text{out}} < R$, and the gains $b, k, C, k_n, v_{\text{nav}}, \bar{v}$ are positive.

The parameter d_m sets the minimum safe separation, $[d_{\text{in}}, d_{\text{out}}]$ is a dead zone in which a pair generates no force, R is the sensing range, b and k are the barrier and tether gains, C caps the pair repulsion, k_n and v_{nav} are the navigation gain

and speed, and $\bar{v}$ is the speed limit. From these we form the derived quantities

$$\begin{aligned} a &= d_{\text{in}} - d_m, & p &= k(R - d_{\text{out}}), \\ r_c &= d_m + \left(\frac{C}{b} + \frac{1}{a}\right)^{-1}, & \sigma^* &= \max\left\{\frac{b}{(r_c - d_m)^2}, k\right\}. \end{aligned} \quad (1)$$

Definition 1 (Pair force): Taking $f > 0$ to denote repulsion,

$$f(r) = \begin{cases} C, & r \leq d_m, \\ \min\left\{b\left(\frac{1}{r - d_m} - \frac{1}{a}\right), C\right\}, & d_m < r < d_{\text{in}}, \\ 0, & d_{\text{in}} \leq r \leq d_{\text{out}}, \\ -k(r - d_{\text{out}}), & d_{\text{out}} < r \leq R, \\ 0, & r > R. \end{cases} \quad (2)$$

As two drones close in on d_m the barrier grows, though it saturates at the cap C once r falls to $r_c > d_m$; for all separations below r_c the repulsion holds constant at C . Beyond d_{out} the tether draws drones together and cuts off at the sensing range, so f jumps from $f(R^-) = -p$ to $f(R^+) = 0$. We define the pair potential $w(r) = \int_r^\infty f(s) ds$, so that $w' = -f$ and w vanishes beyond R .

Each drone i is assigned a goal $g_i \in \mathbb{R}^2$. Its navigation command $\text{sat}(k_n(g_i - x_i), v_{\text{nav}})$ arises as the negative gradient of the Huber potential

$$V_i(x_i) = \begin{cases} \frac{k_n}{2}|x_i - g_i|^2, & |x_i - g_i| \leq v_{\text{nav}}/k_n, \\ v_{\text{nav}}|x_i - g_i| - \frac{v_{\text{nav}}^2}{2k_n}, & \text{otherwise,} \end{cases} \quad (3)$$

where $\text{sat}(v, \bar{v}) = v \min\{1, \bar{v}/|v|\}$ projects onto the closed ball of radius $\bar{v}$. The function V_i has a Lipschitz gradient and satisfies $0 \preceq \nabla^2 V_i \preceq k_n I$ wherever the Hessian exists.

Definition 2 (Flockwork law): Each drone carries the cost

$$J_i(x) = V_i(x_i) + \sum_{j \neq i} w(r_{ij}), \quad (4)$$

and its velocity is

$$u_i = D_\varepsilon(\text{sat}(-\nabla_{x_i} J_i, \bar{v})), \quad D_\varepsilon(v) = \begin{cases} 0, & |v| < \varepsilon, \\ v, & \text{otherwise,} \end{cases} \quad (5)$$

where $\varepsilon \geq 0$ is a dead band.

Expanding the gradient gives

$$-\nabla_{x_i} J_i = -\nabla V_i(x_i) + \sum_{j \neq i} f(r_{ij}) \hat{e}_{ij}.$$

Each drone sums its navigation command and the pairwise forces from its neighbors, saturates the result, and suppresses it when the magnitude falls below ε . Only the relative positions of drones within range R are required. The pairwise sum carries no $1/N$ normalization, consistent with deployed controllers.

Remark 1: The law (5) forms the gradient core of an open-source ROS 2 controller, referred to below as the implementation. That controller additionally incorporates velocity

alignment, acceleration feed-forward, formation slots, obstacle avoidance, and a command rate limiter. None of these extra terms are gradients, so they are omitted here and may be regarded as perturbations.

Lemma 1: The pair potential w is bounded and Lipschitz with $|w'| \leq C$. On each interval where f is smooth, the potential $W(z) = w(|z|)$, $z \in \mathbb{R}^2$, satisfies $\nabla^2 W \preceq \sigma^* I$. At $r = R$ the function w has a concave kink, and near $z = 0$ it takes the form $W(z) = w(0) - C|z|$, a concave cone.

Proof: The bound $|f| \leq C$ gives the first claim. On smooth pieces

$$\nabla^2 W = w''(r) \hat{e} \hat{e}^\top + \frac{w'(r)}{r} (I - \hat{e} \hat{e}^\top).$$

The quantity $w'' = -f'$ vanishes below r_c , equals $b/(r - d_m)^2 \leq b/(r_c - d_m)^2$ on (r_c, d_m) , vanishes in the dead zone, equals k on (d_{out}, R) , and vanishes beyond R . Meanwhile $w'/r = -f/r$ is positive only on (d_{out}, R) , where it does not exceed $p/R < k$. At $r = R$ the derivative w' drops from p to 0, producing the concave kink, and below r_c we have $w' = -C$. $\square$

B. Solutions

The right-hand side of (5) is discontinuous wherever some $r_{ij} = R$, so solutions are understood in the Filippov sense [16], [17]. Informally, wherever the vector field jumps, a Filippov solution is permitted to travel with any convex combination of the nearby field values. Away from these jump surfaces and from the collision set $r_{ij} = 0$, the field is locally Lipschitz and Filippov solutions coincide with classical ones. The bound $|u_i| \leq \bar{v}$ ensures solutions exist for all $t \geq 0$.

III. POTENTIAL-GAME STRUCTURE

Define the potential

$$\Phi(x) = \sum_i V_i(x_i) + \frac{1}{2} \sum_i \sum_{j \neq i} w(r_{ij}). \quad (6)$$

Lemma 2: At every x for which $r_{ij} \notin \{0, R\}$ for all $i \neq j$, one has $\nabla_{x_i} \Phi = \nabla_{x_i} J_i$ for every i , so the costs (4) constitute an exact potential game [11] with potential Φ .

Proof: The terms of Φ depending on x_i are $V_i(x_i)$ and, for each $j \neq i$, the two identical contributions $w(|x_i - x_j|)/2$ and $w(|x_j - x_i|)/2$. Differentiating gives

$$\nabla_{x_i} \Phi = \nabla V_i(x_i) + \sum_{j \neq i} w'(r_{ij}) \hat{e}_{ij} = \nabla_{x_i} J_i. \quad \square$$

Lemma 3: For $q \in \mathbb{R}^2$, $s = D_\varepsilon(\text{sat}(q, \bar{v}))$ satisfies $q \cdot s \geq |s|^2$. The map sat is also monotone in the sense that $(\text{sat}(q_1) - \text{sat}(q_2)) \cdot (q_1 - q_2) \geq 0$.

Proof: If $s = 0$ the bound is trivial. Otherwise $s = \theta q$ for some $\theta \in (0, 1]$, so $q \cdot s = \theta |q|^2 \geq \theta^2 |q|^2 = |s|^2$. The monotonicity property follows from the fact that projection onto a closed convex set is firmly nonexpansive. $\square$

Proposition 1: Along solutions of (5), on every time interval where no r_{ij} lies in $\{0, R\}$,

$$\dot{\Phi} = \sum_i \nabla_{x_i} \Phi \cdot u_i \leq - \sum_i |u_i|^2 \leq 0, \quad (7)$$

with equality precisely when $u = 0$. The rest points are those at which $|\text{sat}(-\nabla_{x_i} \Phi, \bar{v})| < \varepsilon$ holds for every i .

Proof: Lemma 2 gives $u_i = D_\varepsilon(\text{sat}(-\nabla_{x_i} \Phi, \bar{v}))$. The inequality then follows from Lemma 3 applied with $q = -\nabla_{x_i} \Phi$. $\square$

Proposition 1 explains why we view the law as a game. Although each drone descends only its own cost, the entire group jointly decreases the single function Φ . The condition that no r_{ij} equals R is the gap that Section IV fills.

IV. WELL-POSEDNESS AT THE CUTOFF

Throughout this section we set $\varepsilon = 0$. Let $S_{ij} = \{x : r_{ij} = R\}$ denote the surface on which the pair (i, j) leaves the sensing range. Fix a point of S_{ij} that belongs to no other surface S_{kl} , and let a_i, a_j denote the inputs $-\nabla_{x_i} J_i, -\nabla_{x_j} J_j$ computed without the (i, j) tether contribution. Approaching from inside the range ($r < R$, tether active) and from outside ($r > R$, tether inactive), the separation rate takes the values

$$\begin{aligned} g^- &= \hat{e}_{ij} \cdot [\text{sat}(a_i - p \hat{e}_{ij}) - \text{sat}(a_j + p \hat{e}_{ij})], \\ g^+ &= \hat{e}_{ij} \cdot [\text{sat}(a_i) - \text{sat}(a_j)]. \end{aligned} \quad (8)$$

Lemma 4: At every such point, $g^- \leq g^+$.

Proof: Applying the monotonicity from Lemma 3 with $q_1 - q_2 = -p \hat{e}_{ij}$ gives $\hat{e}_{ij} \cdot (\text{sat}(a_i - p \hat{e}_{ij}) - \text{sat}(a_i)) \leq 0$. Taking instead $q_1 - q_2 = p \hat{e}_{ij}$ gives $\hat{e}_{ij} \cdot (\text{sat}(a_j + p \hat{e}_{ij}) - \text{sat}(a_j)) \geq 0$. Subtracting yields the result. $\square$

Lemma 4 states that deactivating the tether can only accelerate the separation of the pair. The reason is that the tether is attractive and the saturation operator is monotone. This eliminates the standard mechanism by which a solution becomes trapped on a jump surface.

Corollary 1: On S_{ij} , away from the other surfaces, Filippov solutions satisfy $\dot{r}_{ij} \in [g^-, g^+]$. Three consequences follow.

- 1) No sliding mode exists on S_{ij} , since sliding would require $g^- > 0 > g^+$.
- 2) A solution that reaches S_{ij} from inside with $g^- > 0$ has $g^+ > 0$ and crosses outward, and a solution that reaches it from outside with $g^+ < 0$ crosses inward.
- 3) A solution can stay on S_{ij} for positive time only where $g^- \leq 0 \leq g^+$.

Item 3 leaves one case open. A solution may linger on S_{ij} where the pair feels no net push in either direction, for instance when no other force acts on it. Such points can occupy an open portion of S_{ij} ; consider for example $V_i \equiv 0$ with a pair sitting at distance R while every other drone is far away. We cannot therefore dismiss these points as rare. The next result shows instead that almost no initial condition leads to them.

Theorem 1: Let $\varepsilon = 0$. For almost every initial condition $x(0) \in \mathbb{R}^{2N}$, the Filippov solution of (5) is unique on $[0, \infty)$ and intersects $\bigcup S_{ij}$ only at transversal crossings, with at most finitely many occurring in any bounded time interval. In particular, the set of times at which $x(t) \in \bigcup S_{ij}$ has measure zero, and (7) holds for almost every t .

The proof is given in Appendix A. We sketch the main idea here. A solution can only enter and stay on S_{ij} through the boundary of the set where $g^+ = 0$ or $g^- = 0$. Since the law is built from polynomials, square roots, and minima, this

boundary has dimension at most $2N - 2$, two lower than the ambient state space, so its δ -neighborhood has volume of order δ^2 . Meanwhile the flow compresses volume at most as fast as e^{-Mt} for some constant M . Away from the jump surfaces the divergence of u is bounded below, and across them the normal velocity jump is $g^+ - g^- \geq 0$ by Lemma 4, which expands rather than compresses volume. No set of initial conditions with positive volume can be mapped onto a set of measure zero in finite time, so almost no solution reaches those points.

Remark 2: The direction of the inequality in Lemma 4 is what drives the argument. A repulsive force with a discontinuous cutoff would produce the opposite sign, compress volume, and could admit sliding. This means that an attractive tether with a discontinuous cutoff can safely be implemented without a bump function.

Remark 3 (Relation to known uniqueness results): Classical uniqueness for Filippov solutions of piecewise smooth fields requires that at every point of the switching surface the field on one side points into the other [16], [17]. Here this means $g^- > 0$ or $g^+ < 0$ on S_{ij} . The condition fails where $g^- \leq 0 \leq g^+$, and this set can fill an open part of S_{ij} , as noted after Corollary 1, so these results do not apply. A second line of work establishes uniqueness of the flow for almost every initial condition from regularity of the field: DiPerna–Lions [19] for Sobolev fields with bounded divergence, Ambrosio [20] for fields of bounded variation with absolutely continuous bounded divergence, and Bianchini–Bonicatto [21] for nearly incompressible fields of bounded variation, a condition that bounds the density transported by the flow from above and below. The Flockwork field is of bounded variation, but its divergence has a singular part on S_{ij} equal to the jump $g^+ - g^-$, so it is not absolutely continuous, and Lemma 12 controls it only from below. Theorem 1 replaces the missing two-sided bound by two properties of the law: the sign condition $g^- \leq g^+$ of Lemma 4, which makes the singular part nonnegative, and the semialgebraic structure of the law, which makes the set on S_{ij} where a solution can stop thin. Semialgebraic geometry is standard in nonsmooth optimization [26].

V. SAMPLED-DATA IMPLEMENTATION

The controller advances with sample period h according to

$$x^{k+1} = x^k + h u(x^k), \quad \varepsilon \geq 0. \quad (9)$$

Lemma 5: Let $\mathcal{G}$ be the graph whose edges are those pairs with $r_{ij} < R$ at some point along the segment $[x, y]$, and let Δ denote its maximum degree. One then has

$$\Phi(y) \leq \Phi(x) + G(x) \cdot (y - x) + \frac{L}{2} |y - x|^2, \quad L = k_n + 2\sigma^* \Delta, \quad (10)$$

where $G(x)$ is the gradient used by the controller.

Proof: On smooth pieces, Lemma 1 gives the pair term Hessian as $\preceq \sigma^*(L_{\mathcal{G}} \otimes I_2)$, where $L_{\mathcal{G}}$ is the Laplacian of $\mathcal{G}$ and $\lambda_{\max}(L_{\mathcal{G}}) \leq 2\Delta$. Combined with $\nabla^2 V_i \preceq k_n I$ this yields the curvature constant L . At the remaining nonsmooth points, which are either C^1 junctions or concave kinks by Lemma 1, the potential is a minimum over smooth branches; a concave kink only pulls the function below each branch, so the upper bound (10) remains valid across them. $\square$

Proposition 2: If $h < 2/L$ with L as in Lemma 5 along the step, then

$$\Phi(x^{k+1}) \leq \Phi(x^k) - h(1 - \frac{hL}{2})|u(x^k)|^2. \quad (11)$$

Proof: Apply Lemma 5 with $y - x = hu$ and $G = \nabla \Phi$ from Lemma 2, then use $\nabla_{x_i} \Phi \cdot u_i \leq -|u_i|^2$ from Lemma 3. $\square$

The bound $2/L$ is pessimistic, since L reflects the worst-case curvature of the barrier. The following result shows that the threshold becomes tight at an equilibrium when L is replaced by the actual curvature at that point.

Proposition 3: Let $\varepsilon = 0$ and let x^* be a local minimum of Φ near which Φ is twice continuously differentiable. When $h \lambda_{\max}(\nabla^2 \Phi(x^*)) > 2$, the point x^* is an unstable fixed point of (9), with an associated eigenvalue $1 - h \lambda_{\max} < -1$.

Proof: At x^* we have $u(x^*) = 0$, so the saturation is inactive in a neighborhood of x^* and (9) reduces to the map $T(x) = x - h \nabla \Phi(x)$ with Jacobian $DT(x^*) = I - h \nabla^2 \Phi(x^*)$, whose eigenvalues are $1 - h \lambda$. An eigenvalue whose modulus exceeds one renders the fixed point unstable. $\square$

An eigenvalue below -1 indicates that the instability takes the form of a period-two oscillation where drones in a tight pack jump back and forth across their equilibrium at every sample. For $\varepsilon > 0$ the dead band fills a neighborhood of x^* with fixed points, though only within a region of size of order $\varepsilon/\lambda_{\max}$.

Corollary 2: When no pair at x^* lies closer than d_{in} , we have $\lambda_{\max}(\nabla^2 \Phi(x^*)) \leq k_n + 2k \Delta_t$, where

$$\Delta_t \leq \frac{(R + d_{\text{in}}/2)^2 - (d_{\text{out}} - d_{\text{in}}/2)^2}{(d_{\text{in}}/2)^2} \quad (12)$$

gives an upper bound on the number of neighbors a drone can have in the tether band (d_{out}, R) .

Proof: Pairs in the dead zone or beyond R contribute nothing to the Hessian, while pairs in (d_{out}, R) satisfy $\nabla^2 W \preceq kI$ by Lemma 1. Neighbors of x_i in the tether band are separated by at least d_{in} , so disjoint discs of radius $d_{\text{in}}/2$ centered at them all fit within the annulus $d_{\text{out}} - d_{\text{in}}/2 < |z - x_i| < R + d_{\text{in}}/2$. The bound follows by comparing areas. $\square$

Corollary 2 shows that an equilibrium in which all drones sit within their dead zones can tolerate a sample period governed only by the soft tether gain k . A compressed equilibrium, where pairs are pressed against the stiff barrier, requires a considerably shorter period. Section VI shows that a shared goal produces exactly this kind of compressed configuration.

VI. CRUSHING UNDER A COMMON GOAL

Throughout this section we set $\varepsilon = 0$ and assign all drones a single shared goal at the origin. Write $\rho_i = |x_i|$ and $\nu(\rho) = \min\{k_n \rho, v_{\text{nav}}\}$, so the navigation force on drone i is $-\nu(\rho_i) x_i / \rho_i$. An *equilibrium* is a configuration with no $r_{ij} = 0$ and $-\nabla_{x_i} J_i = 0$ for all i , where a pair sitting at $r_{ij} = R$ may carry any force in $[-p, 0]$. By Proposition 1, every rest point is an equilibrium.

Definition 3: A pair is *capped* if $r_{ij} \leq r_c$.

When a pair is capped, its repulsive force is C regardless of the exact separation. The equilibrium conditions therefore

place no constraint on the length of such a pair within $(0, r_c]$, and in particular they do not prevent it from falling below d_m . A capped pair is one whose safety separation is no longer enforced by the controller.

Lemma 6: At every equilibrium,

$$\sum_i \nu(\rho_i) \rho_i = \sum_{i < j} f(r_{ij}) r_{ij}. \quad (13)$$

Proof: At an equilibrium,

$$0 = \sum_i x_i \cdot (-\nabla_{x_i} J_i) = - \sum_i \nu(\rho_i) \rho_i + \sum_i \sum_{j \neq i} f(r_{ij}) \hat{e}_{ij} \cdot x_i.$$

Combining the (i, j) and (j, i) terms yields $f(r_{ij}) \hat{e}_{ij} \cdot (x_i - x_j) = f(r_{ij}) r_{ij}$. $\square$

Identity (13) is a pressure balance: the left side is the inward navigation pull, weighted by distance to the goal, and the right side is the outward pairwise push, weighted by separation. In a pack of N drones the radius grows like $\sqrt{N}$, so the left side scales as $N^{3/2}$. Each drone has only a bounded number of close neighbors and each pair pushes with at most C , so the right side grows no faster than N . For large enough N the repulsion cannot keep pace.

Lemma 7: Suppose all $r_{ij} > r_c$ and let $t = d_{in}/r_c$.

- 1) A closed ball of radius s contains at most $(1 + 2s/r_c)^2$ drones. Ordering $\rho_{(1)} \leq \dots \leq \rho_{(N)}$, this implies $\rho_{(m)} \geq (r_c/2)(\sqrt{m} - 1)$.
- 2) Each drone has at most $K := \lfloor 4t(t+1) \rfloor$ neighbors with $r_{ij} < d_{in}$.

Proof: Discs of radius $r_c/2$ centered at the drones are pairwise disjoint. For claim 1, the discs belonging to drones inside the ball fit within the concentric ball of radius $s + r_c/2$; comparing areas gives the count, and the ordering bound follows by applying it to the m drones with $\rho \leq \rho_{(m)}$. For claim 2, the discs of close neighbors of x_i lie in the annulus $r_c/2 < |z - x_i| < d_{in} + r_c/2$, whose area is $\pi(r_c/2)^2 \cdot 4t(t+1)$. $\square$

Theorem 2: If an equilibrium has no capped pair, then

$$\sqrt{N} \leq \frac{3K}{2} \frac{C}{v_{nav}} + \frac{3}{2} + \frac{3v_{nav}}{k_n r_c}. \quad (14)$$

Equivalently, whenever N exceeds the square of the right side, every equilibrium must contain a capped pair.

Proof: *Right side of (13).* For $r_c < r < d_{in}$,

$$f(r) r = b \left(\frac{r}{r - d_m} - \frac{r}{a} \right),$$

$$\frac{d}{dr} (f(r) r) = - \frac{bd_m}{(r - d_m)^2} - \frac{b}{a} < 0,$$

so $f(r) r < f(r_c) r_c = Cr_c$. Pairs with $r \geq d_{in}$ contribute $f(r) r \leq 0$. By Lemma 7 there are at most $KN/2$ pairs with $r < d_{in}$, bounding the right side by $KNCr_c/2$.

Left side. The inequality $\nu(\rho)\rho \geq v_{nav}(\rho - v_{nav}/k_n)$ holds for all ρ : for $\rho \geq v_{nav}/k_n$ it is immediate, and below that value the discriminant of $k_n \rho^2 - v_{nav} \rho + v_{nav}^2/k_n$ is negative. Applying Lemma 7 and $\sum_{m \leq N} \sqrt{m} \geq (2/3)N^{3/2}$ gives

$$\sum_i \nu(\rho_i) \rho_i \geq v_{nav} \left[\frac{r_c}{3} N^{3/2} - \left(\frac{r_c}{2} + \frac{v_{nav}}{k_n} \right) N \right].$$

Combining the two bounds and dividing by $v_{nav} r_c N/3$ yields (14). $\square$

Theorem 2 states that the largest safe swarm with a common goal scales as $(C/v_{nav})^2$: doubling the cap allows four times as many drones. The constant K depends only on the barrier geometry through the ratio d_{in}/r_c , and the bound applies to every equilibrium configuration, regardless of the shape of the pack.

Remark 4: For $\varepsilon \in (0, \bar{v}]$ the rest points satisfy $|\nabla_{x_i} J_i| < \varepsilon$, giving $\sum_i x_i \cdot (-\nabla_{x_i} J_i) \geq -\varepsilon \sum_i \rho_i$. Theorem 2 then holds with v_{nav} replaced by $v_{nav} - \varepsilon$.

Remark 5: The navigation pull equals $k_n \rho$ inside the radius v_{nav}/k_n and v_{nav} beyond it. A pack with spacing close to r_c has radius near $(r_c/2)\sqrt{N}$, placing it inside this radius when $N \lesssim N_\times := (2v_{nav}/(k_n r_c))^2$. In that inner regime the left side of (13) is approximately $k_n \sum \rho_i^2$, which grows like $k_n r_c^2 N^2$. Balancing against NCr_c gives a critical size proportional to $C/(k_n r_c)$, linear in C . In the outer regime the left side grows as $v_{nav} r_c N^{3/2}$ and the critical size is proportional to $(C/v_{nav})^2$, as in Theorem 2. Small swarms with a gentle navigation gain may therefore exhibit a linear scaling before the quadratic law takes over.

Theorem 2 gives a necessary condition for safety. We show that the order $(C/v_{nav})^2$ is tight, by constructing safe equilibria of that size. The idea is simple. A ring of drones under inward pressure acts like an arch: the repulsion between adjacent drones has an outward component that counteracts the navigation pull. Rings with radii differing by more than R do not interact, so each ring can be balanced independently.

A *ring* of n drones with radius ρ is the arrangement $x_i = \rho(\cos(2\pi i/n + \phi), \sin(2\pi i/n + \phi))$, $i = 1, \dots, n$, for some phase ϕ . The separation between drones j steps apart is $c_j = 2\rho \sin(\pi j/n)$, and $s = c_1$ is the spacing.

Lemma 8: For a ring centered at the goal, the net force on every drone is radial, and the ring is an equilibrium if and only if

$$h(\rho, n) := \sum_{j=1}^{n-1} f\left(2\rho \sin \frac{\pi j}{n}\right) \sin \frac{\pi j}{n} - \nu(\rho) = 0. \quad (15)$$

Proof: By symmetry, it is enough to consider one drone. Its neighbors j and $n - j$ steps away are reflections across the line through the drone and the goal, so their tangential force components cancel. The radial component of the unit vector from drone j to the drone is $\rho(1 - \cos(2\pi j/n))/c_j = \sin(\pi j/n)$, so the outward force equals the sum in (15), and the navigation pull is $\nu(\rho)$ inward. $\square$

Assumption 2 (Range): Let J be the largest integer with $Jd_m < R$. Then $Jd_{in} < R$.

Equivalently, no integer falls in $[R/d_{in}, R/d_m)$. The assumption guarantees that, in a ring with spacing between r_c and d_{in} , the chord c_j lies below R for $j \leq J$ and above R for larger j , so no chord equals the cutoff. Both parameter sets of Section VII satisfy it, with $J = 1$ (reference) and $J = 2$ (implementation). We use the following constants.

- $n_0 \geq 3(J + 1)$ is an integer with $(J + 1)r_c(1 - \pi^2(J + 1)^2/(6n_0^2)) > R$. Such a value exists because $(J + 1)d_m \geq R$ and $r_c > d_m$.

- J_b is the largest $j \leq J$ with $j r_c (1 - \pi^2 j^2 / (6n_0^2)) < d_{\text{in}}$. Set $S_b = \sum_{j \leq J_b} j^2$ and $S_t = \sum_{2 \leq j \leq J} j^2$.
- The growth rate and step are

$$\delta = \frac{d_m \cos(\pi/n_0)}{2\pi (S_b + ka^2 S_t/b)}, \quad \Delta = \lfloor R/\delta \rfloor + 1.$$

- $n_{\text{lo}} = \max\{n_0, \lceil 2\pi v_{\text{nav}} / (k_n r_c) \rceil\}$, and n_{hi} is the largest integer below

$$\frac{\pi}{v_{\text{nav}}} \left(2C \cos \frac{\pi}{n_0} - 2p \sum_{j=2}^J j \right).$$

J_b counts the nearby neighbors on the barrier and S_t those in the tether band. The quantity δ bounds below how much the ring radius grows per added drone, and Δ is the increment in drone count that separates two rings by more than R .

Theorem 3: Let Assumption 2 hold and $n_{\text{lo}} \leq n_{\text{hi}}$. For each real $n \in [n_{\text{lo}}, n_{\text{hi}}]$, (15) has a unique root $\rho(n)$ with spacing in (r_c, d_{in}) , and $\rho(n') - \rho(n) \geq \delta(n' - n)$ for $n \leq n'$. Thus the rings with $n_m = n_{\text{lo}} + m\Delta$ drones, $m = 0, \dots, \lfloor (n_{\text{hi}} - n_{\text{lo}})/\Delta \rfloor$, and radii $\rho(n_m)$, at any phase, form an equilibrium with no capped pair. Its size is

$$N = \sum_m n_m \geq \frac{(n_{\text{hi}} - n_{\text{lo}})(n_{\text{hi}} + n_{\text{lo}} - \Delta)}{2\Delta}. \quad (16)$$

The proof appears in Appendix B. Since $n_{\text{hi}} \approx 2\pi C/v_{\text{nav}}$, the right side of (16) is approximately $(2\pi^2/\Delta)(C/v_{\text{nav}})^2$. Combined with Theorem 2, this confines the largest safe swarm between two multiples of $(C/v_{\text{nav}})^2$. Let $N_{\text{safe}}(C)$ be the largest N for which an equilibrium with no capped pair exists.

Corollary 3: Let Assumption 2 hold and fix all parameters except C . Let J_b^∞ be the largest $j \leq J$ with $j d_m \leq d_{\text{in}}$, and define

$$S_b^\infty = \sum_{j=1}^{J_b^\infty} j^2, \quad \delta_\infty = \frac{d_m}{2\pi (S_b^\infty + ka^2 S_t/b)},$$

$$\Delta_\infty = \lfloor R/\delta_\infty \rfloor + 1, \quad K_\infty = \lfloor 4t(t+1) \rfloor, \quad t = d_{\text{in}}/d_m.$$

Then

$$\begin{aligned} \frac{2\pi^2}{\Delta_\infty} &\leq \liminf_{C \rightarrow \infty} \frac{N_{\text{safe}}(C) v_{\text{nav}}^2}{C^2} \\ &\leq \limsup_{C \rightarrow \infty} \frac{N_{\text{safe}}(C) v_{\text{nav}}^2}{C^2} \leq \left(\frac{3K_\infty}{2} \right)^2. \end{aligned} \quad (17)$$

Proof: As C increases, $r_c - d_m = (C/b + 1/a)^{-1}$ decays like b/C . Since $(J+1)d_m \geq R$, the constraint on n_0 is met once $\pi^2(J+1)^2/(6n_0^2) < (r_c - d_m)/r_c$, so the smallest admissible n_0 scales like $C^{1/2}$. All conditions on n_0 are preserved under increase, so we may choose $n_0(C) = \max\{n_0^{\text{min}}(C), \lceil C^{1/3} \rceil\}$. Then $n_0(C) \rightarrow \infty$ and $n_0(C)/C \rightarrow 0$, giving $\cos(\pi/n_0) \rightarrow 1$, $n_{\text{lo}}/C \rightarrow 0$ and $n_{\text{hi}} v_{\text{nav}}/C \rightarrow 2\pi$. Moreover $J_b \leq J_b^\infty$ for large C , so $\delta \geq \delta_\infty \cos(\pi/n_0)$ and $\Delta \leq \Delta_\infty$ for large C . Substituting into (16) yields the lower bound. For the upper bound, $K \leq K_\infty$ since $r_c > d_m$, and the remaining terms in (14) stay bounded, so Theorem 2 gives $\sqrt{N_{\text{safe}}} \leq (3K_\infty/2)(C/v_{\text{nav}}) + O(1)$. $\square$

TABLE I
PARAMETER SETS. LENGTHS IN M, TIME IN S.

symbol	meaning	reference	source	implem.
R	sensing range	8.4	[5], $R/d = 1.2$	2.5
d_m	safe distance	4.2	[5], $0.6d$	0.9
$d_{\text{in}}, d_{\text{out}}$	dead zone	6.3, 7.7	design, $d \pm 0.1d$	1.2, 1.8
k	tether gain	1	[9]	0.5
b	barrier gain	4.41	design, $b = ka^2$	3
C	force cap	4	[9]	20
k_n	navigation gain	1	[9], $v_{\text{nav}}/2\text{m}$	0.6
v_{nav}	navigation speed	2	[9]	2
$\bar{v}$	speed limit	4	[9]	2
r_c, σ^*	from (1)	4.923, 8.438		1.0, 300

Corollary 3 settles the scaling question: the largest safe swarm with a common goal is exactly of order $(C/v_{\text{nav}})^2$. The two constants are widely separated, because rings are a simple, loose construction and the proof of Theorem 2 charges every close pair the full repulsion. The constant observed in simulation falls between them (Section VII). The condition $n \geq 2\pi v_{\text{nav}}/(k_n r_c)$ in n_{lo} places every ring outside the radius v_{nav}/k_n , which is the quadratic regime of Remark 5; note that $2\pi v_{\text{nav}}/(k_n r_c) = \pi\sqrt{N_\times}$. Theorem 3 establishes that safe equilibria exist. It does not identify which equilibrium the dynamics reach.

Remark 6 (Higher dimensions): Lemma 6 and the proof of Theorem 2 are dimension-independent. For drones in $\mathbb{R}^d$, replacing areas by volumes in Lemma 7 gives $\rho_{(m)} \geq (r_c/2)(m^{1/d} - 1)$ and at most $K_d = \lfloor (2t+1)^d - 1 \rfloor$ close neighbors, and $\sum_{m \leq N} m^{1/d} \geq \frac{d}{d+1} N^{1+1/d}$ then yields

$$N^{1/d} \leq \frac{d+1}{d} \left(K_d \frac{C}{v_{\text{nav}}} + 1 + \frac{2v_{\text{nav}}}{k_n r_c} \right), \quad (18)$$

which reduces to (14) for $d = 2$. In three dimensions the safe swarm size is therefore at most of order $(C/v_{\text{nav}})^3$. The rings of Theorem 3, placed in a plane through the goal, remain an equilibrium in $\mathbb{R}^d$ since every force lies in that plane, so N_{safe} is at least of order $(C/v_{\text{nav}})^2$ in every dimension. For $d = 3$ the simulations of Section VII follow the $(C/v_{\text{nav}})^3$ scaling, while a matching safe construction for $d \geq 3$ remains open.

VII. NUMERICAL EXAMPLES

A. Setup

Parameter sets. Two parameter sets are used, listed in Table I. The *reference* set is constructed so that results can be compared with the literature: every value with a counterpart in a published flocking design is drawn from it. The ratio $R/d = 1.2$ and scale $d = 7\text{m}$ follow the simulations in [5], and $d_m/d = 0.6$ is the obstacle-clearance ratio they adopt. The tether gain is the repulsion gain of the outdoor drone experiments in [9], the navigation speed is their preferred speed, and both the cap and the speed limit equal their maximum speed. The navigation gain $k_n = v_{\text{nav}}/(2\text{m})$ follows the 2 m decay length over which their target-tracking command fades near the goal. Two values have no published counterpart. The design in [5] has a single preferred distance d , which we broaden into the dead zone $d \pm 0.1d$, and

$b = ka^2$ sets the barrier slope at d_{in} equal to the tether gain, so the pair is equally stiff on both sides of the dead zone. The *implementation* set contains the defaults of the deployed controller of Remark 1, which flies simulated quadrotors in Gazebo. There $d_{\text{in}} = 0.9\text{ m}$ exceeds the 0.71 m at which the rotors of two vehicles touch. The two sets differ in the two ratios relevant to Section VI: C/v_{nav} is 2 in the reference set and 10 in the implementation, and the radius v_{nav}/k_n at which the navigation command saturates is $0.4 r_c$ and $3.3 r_c$, placing them in the two regimes of Remark 5.

Simulation method. Each run integrates the law (5) via its sampled-data form (9), namely the forward Euler step the controller executes. Outside Section VII-C, where the sample period is the quantity under study, the step is small enough that runs track the continuous-time flow: 10^{-3} s and $5 \cdot 10^{-3}\text{ s}$ for the cutoff experiment, and $h = \min\{5\text{ ms}, 1/(2(k_n + 12\sigma^*))\}$ for the common-goal experiment, giving $1.8\text{--}5.0\text{ ms}$ (reference) and $0.39\text{--}0.87\text{ ms}$ (implementation) as C varies. We verified that this step satisfies Proposition 2: on four packs, including the largest that crushed in each set, the interaction graph had maximum degree at most 10 (reference) and 18 (implementation), so $h < 2/L$, and Φ decreased at every step. Every run is deterministic given the random seed of the initial condition. Initial condition lengths are expressed in units of $\ell = d_{\text{in}}/1.2$, equal to 1 m in the implementation and 5.25 m in the reference set, so the two sets start from identical configurations up to scale.

Experiment design. Each experiment targets one result, in the regime where it is most demanding.

- 1) *Cutoff (Section VII-B).* Testing Lemma 4 and Theorem 1 requires many crossings of $r_{ij} = R$ in both directions. Twelve drones are initialized uniformly in a square of side 6ℓ centered at $(5\ell, 2\ell)$ and fly toward a common goal at the origin, so the pack assembles in motion and pairs repeatedly enter and exit each other's range. The thresholds τ halve three times from 1.6% of R (implementation) and 2.4% of R (reference), sufficient to distinguish linear decay in τ from a positive limit.
- 2) *Sampled data (Section VII-C).* A common goal yields the stiffest equilibria (Section VI), making it the worst case for the sample period. Twelve drones are placed uniformly in a square of side 3.6ℓ centered at $(6\ell, 0)$ and simulated for 30 s (20 seeds, implementation) or 120 s (10 seeds, reference). The sample periods straddle $2/\lambda_{\text{max}}$ of the settled packs and include the periods used in [5] and the sensor period of [9]. A step registers as an increase of Φ if Φ rises by more than 10^{-9} , and the late steps are the final quarter of each run.
- 3) *Common goal (Section VII-D).* Theorems 2 and 3 predict how the critical size scales with C/v_{nav} , so we vary C and hold all other values of each set fixed. The range $C/v_{\text{nav}} \in [1, 4]$ includes the reference value 2 and covers a factor of 16 in $(C/v_{\text{nav}})^2$, enough to fit a power law, and the swarm sizes form a roughly geometric grid with ratio about 1.25. Each initial pack is sampled uniformly in a square of side $2\sqrt{N} d_{\text{in}}$ and resampled point by point until every gap is at least d_{in} , ensuring no pair begins on the barrier and any capped pair is

created by the navigation pressure. Each pack runs until $\max_i |u_i| < 10^{-3}\text{ m/s}$, for at least 100 s (reference) or 40 s (implementation); every pack reached this within 200 s . Four packs are simulated per (C, N) , and N^* is the smallest N for which at least half have a capped pair.

Code. The code and data for all examples are available at [27].

B. The Cutoff

We simulated $N = 12$ drones from 10 random initial conditions for each parameter set: 20 s with step 10^{-3} s for the implementation, and 60 s with step $5 \cdot 10^{-3}\text{ s}$ for the reference. We recorded 285 and 213 cutoff crossings respectively, and the inequality $g^- \leq g^+$ of Lemma 4 held at every one. To verify Theorem 1, we measured the fraction of time with some $|r_{ij} - R| < \tau$. Transversal crossings with zero time on the cutoff would cause this fraction to vanish linearly in τ . Its ratio to τ was 2.40, 2.78, 2.97, 3.05 for $\tau = 0.04, 0.02, 0.01, 0.005\text{ m}$ in the implementation, and 1.67, 1.49, 1.65, 1.69 for $\tau = 0.2, 0.1, 0.05, 0.025\text{ m}$ in the reference, confirming the linear vanishing in both cases.

C. Sampled Data

We sent $N = 12$ drones to a common goal from 10 to 20 random starting configurations, producing compressed packs. In the implementation the settled packs had $\lambda_{\text{max}}(\nabla^2\Phi) \in [205, 211]\text{ s}^{-1}$, so Proposition 3 predicts instability for $h > 2/\lambda_{\text{max}} \approx 9.5\text{ ms}$. Consistent with this, Φ never increased for $h \in \{2, 5\}\text{ ms}$, while for $h \geq 10\text{ ms}$ it increased on exactly half of the late steps, indicating a period-two oscillation. In the reference set $\lambda_{\text{max}} \in [14.2, 21.9]\text{ s}^{-1}$ and $2/\lambda_{\text{max}} \in [0.091, 0.141]\text{ s}$; the potential never increased for $h \leq 0.05\text{ s}$, which covers the periods $0.01\text{--}0.03\text{ s}$ used in [5]. At the sensor period $h = 0.2\text{ s}$ of [9] it increased on 49.5% of late steps, again with period two. For equilibria in the dead zone, Corollary 2 gives $\lambda_{\text{max}} \leq 23$ (reference) and $\lambda_{\text{max}} \leq 22.6$ (implementation), so no flip instability arises below about 87 ms in either set. The compressed pack of the implementation requires a sample period roughly ten times shorter than that of its uncompressed formation.

D. Common Goal

To measure the critical size, we varied C in both sets while holding all other parameters fixed, spanning $C/v_{\text{nav}} \in \{1, 1.5, 2, 3, 4\}$. For each value and each $N \in \{4, 6, 8, 10, 12, 15, 19, 24, 30, 38, 48, 60, 75, 95, 120\}$, we ran 4 random initial packs with every gap above d_{in} , ensuring that any capped pair arises from navigation pressure alone. Packs were run until $\max_i |u_i| < 10^{-3}\text{ m/s}$ with a step below $2/L$ from Proposition 2 (Section VII-A). We define N^* as the smallest N for which at least half of the settled packs contain a capped pair.

Fig. 2 shows the outcome. In the reference set, $N^* = 6, 12, 19, 60, 75$, with a log-log slope of 1.93 and the fit $N^* \approx 5.4 (C/v_{\text{nav}})^2$, in agreement with the quadratic law of

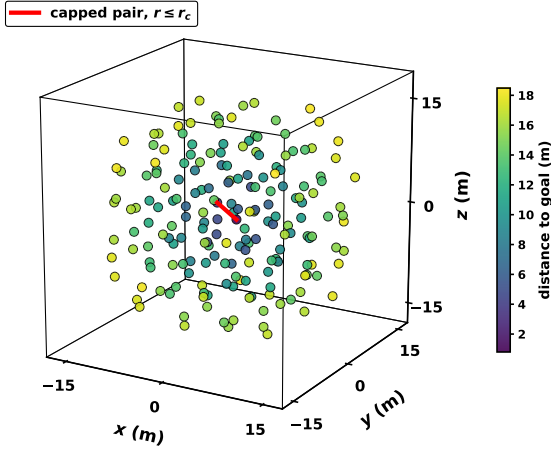


Fig. 1. Settled pack of $N = 192$ drones in $\mathbb{R}^3$, reference set, $C/v_{\text{nav}} = 3$. Color: distance to goal. Red: capped pair, $r \leq r_c$.

Theorem 2. In the implementation set, $N^* = 15, 24, 38, 48$ for $C/v_{\text{nav}} \leq 3$, with slope 1.09, and $N^* > 120$ at $C/v_{\text{nav}} = 4$. Here $N_\times \approx 37$ from Remark 5, placing the first four points in the linear regime; the jump at $C/v_{\text{nav}} = 4$ marks the transition into the quadratic one. In the reference set $N_\times < 1$, so only the quadratic regime appears. At N^* , the smallest recorded gap was 1.0–3.3 m (reference) and 0.06–0.76 m (implementation), well below d_m in both cases.

The bound (14) yields $N \leq 1275$ for the reference set ($K = 11$) and $N \leq 26082$ for the implementation ($K = 10$), roughly 60 times larger than the measured values. The proof attributes the full push Cr_c to every close pair, whereas in a settled pack only the pairs nearest the goal approach the cap.

The ring construction of Theorem 3 produces safe equilibria with $N = 7, 32, 130, 621$ drones at $C/v_{\text{nav}} = 2, 4, 10, 20$ in the reference set, and $N = 20, 62, 310$ at $C/v_{\text{nav}} = 4, 10, 20$ in the implementation, where $n_{\text{lo}} > n_{\text{hi}}$ for $C/v_{\text{nav}} \leq 3$. Each was verified directly: the largest residual $|\nabla_{x_i} J_i|$ fell below 10^{-10} , and the smallest gap exceeded r_c . The ring radii grew by at least 0.67 (reference) and 0.145 (implementation) per added drone, exceeding the values $\delta = 0.60$ – 0.66 and 0.125 – 0.130 of Theorem 3. Corollary 3 gives $2\pi^2/\Delta_\infty = 1.52$ and 1.04 for the two sets, so the measured $N^* \approx 5.4 (C/v_{\text{nav}})^2$ of the reference set falls between the two constants of (17). The rings are unstable in the manner of an arch under load: after a perturbation of 10^{-6} m the drones drifted 4–14 m away from the ring configuration. In every case, however, the packs they ultimately settled into still contained no capped pair.

The same sweep in three dimensions with the reference set gives $N^* = 16, 24, 64, 192, 512$ at $C/v_{\text{nav}} = 1, 1.5, 2, 3, 4$, with a log–log slope of 3.06 for $C/v_{\text{nav}} \geq 1.5$, consistent with the $(C/v_{\text{nav}})^3$ scaling of (18). Fig. 1 shows a settled pack at N^* for $C/v_{\text{nav}} = 3$, where the capped pair forms near the goal.

VIII. CONCLUSION

This article analyzed the Flockwork law, a distributed flocking controller in which each drone descends its own cost subject to a capped pair repulsion and a discontinuous

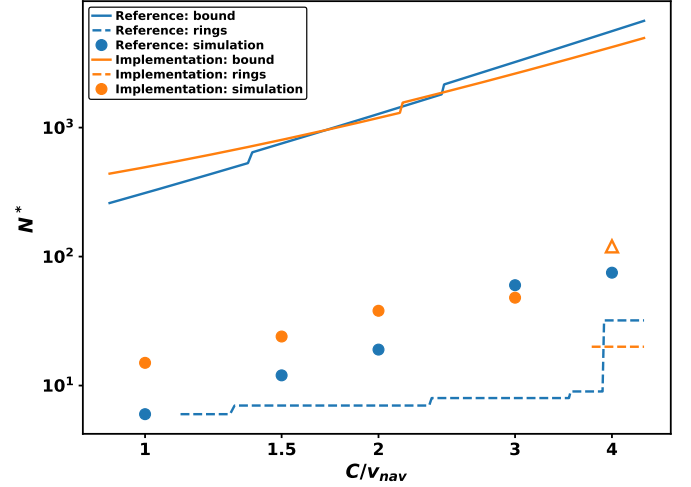


Fig. 2. Critical swarm size N^* against C/v_{nav} . Circles: simulation, reference $N^* = 6, 12, 19, 60, 75$ and implementation $N^* = 15, 24, 38, 48$ at $C/v_{\text{nav}} = 1, 1.5, 2, 3, 4$. Open triangle: implementation at $C/v_{\text{nav}} = 4$, $N^* > 120$. Solid lines: the bound of Theorem 2 for each set, with steps where K changes as r_c moves with C . Dashed lines: size of the safe ring equilibrium of Theorem 3.

sensing-range cutoff. We established that the individual costs form an exact potential game, that the cutoff is well posed for almost every initial condition with no sliding and zero time spent on the switching surfaces, and that the sampled-data implementation continues to decrease the potential below an explicit step-size threshold while losing stability through period doubling above $2/\lambda_{\text{max}}$. The main result is a bound on swarm size under a common goal: once the number of drones exceeds a threshold of order $(C/v_{\text{nav}})^2$, every equilibrium contains at least one pair pressed into the capped region. The practical implication is that a capped controller should not direct a large swarm to a single point; the remedy is either distinct goals, formation slots, or a cap that grows with the swarm size. We further showed that this order is tight by constructing safe equilibria of size of order $(C/v_{\text{nav}})^2$ from concentric rings. Except for Remark 6 and the three-dimensional sweep of Section VII, the analysis is planar and models each drone as a velocity-controlled point; second-order and delayed dynamics and hardware experiments are left for future work. Several directions remain open, among them the sharp constant, which equilibrium the dynamics select, behavior on the dead-band surface, and guarantees for the non-gradient terms of Remark 1. The ground states of the law in the absence of navigation, and their geometry in the large- N limit, are taken up in a companion paper.

APPENDIX A PROOF OF THEOREM 1

Throughout, $\varepsilon = 0$.

A. Setup

Let Σ be the union of all pairwise intersections $S_{ij} \cap S_{kl}$, $(ij) \neq (kl)$, together with the collision sets $\{r_{ij} = 0\}$. Let $D = \bigcup S_{ij} \cup \Sigma$ and $S_{ij}^\circ = S_{ij} \setminus \Sigma$. Each S_{ij}° is a smooth

hypersurface ($|\nabla r_{ij}| = \sqrt{2}$), these hypersurfaces are pairwise disjoint, and u is locally Lipschitz off D . Near $y \in S_{ij}^\circ$, let u^- (u^+) be the field with the (i, j) pair force set to $-k(r_{ij} - d_{\text{out}})$ (to 0). Both are Lipschitz on a ball around y , u equals u^- inside and u^+ outside, and $g^\pm = \nabla r_{ij} \cdot u^\pm$ on S_{ij}° are the rates (8), continuous on S_{ij}° . On S_{ij}° the Filippov set is $\text{co}\{u^-, u^+\}$ [16], [17]. Set

$$Z_{ij}^\pm = \{g^\pm = 0\}, \quad C_{ij} = \{g^- < 0 < g^+\},$$

$$\Gamma = \Sigma \cup \bigcup_{ij} (Z_{ij}^+ \cup Z_{ij}^- \cup C_{ij}),$$

all as subsets of S_{ij}° . By Lemma 4, every point of S_{ij}° outside Γ has $g^- > 0$ or $g^+ < 0$, so solutions cross there. Γ is closed, so $W := \mathbb{R}^{2N} \setminus \Gamma$ is open. Finally, let

$$\mathcal{B} = \Sigma \cup \bigcup_{ij} [(Z_{ij}^+ \setminus \text{int } Z_{ij}^+) \cup (Z_{ij}^- \setminus \text{int } Z_{ij}^-)],$$

with interiors relative to S_{ij}° . This is the boundary of the set where solutions could stall.

B. The Arrival Set Is Thin

A set is *semialgebraic* if finitely many polynomial equations and inequalities describe it. We use three facts.

- (F1) [28, Thm. 2.2.1, Props. 2.2.2, 2.2.4, 2.2.6] Unions, intersections, complements, closures, interiors and projections of semialgebraic sets are semialgebraic, and so are sums, products, quotients, compositions, minima, maxima and square roots of semialgebraic functions.
- (F2) [28, Prop. 2.8.2, Cor. 2.8.9, Prop. 2.9.10] A semialgebraic set A has a dimension $\dim A$, equal to that of the smallest algebraic set containing it. A is a finite disjoint union of smooth pieces, each diffeomorphic to an open cube $(0, 1)^m$ with $m \leq \dim A$, and at least one with $m = \dim A$.
- (F3) [29, Main Theorem] If $X \subset \mathbb{R}^n$ is an algebraic set of codimension m , then for every $\rho > 0$ there is c with $|X^{(\delta)} \cap B_\rho| \leq c\delta^m$ for $\delta \leq 1$.

Lemma 9: $\dim \mathcal{B} \leq 2N - 2$. Hence, for every $\rho > 0$ there is c with $|\mathcal{B}^{(\delta)} \cap B_\rho| \leq c\delta^2$ for $\delta \leq 1$.

Proof: The distance r_{ij} is the square root of a polynomial, $\hat{e}_{ij}$ is a quotient, f is piecewise rational with a minimum, and sat is a product with a minimum. By (F1), $u^\pm$, $g^\pm$, $Z_{ij}^\pm$, their interiors and $\mathcal{B}$ are semialgebraic. Σ is a finite union of sets $\{r_{ij}^2 = R^2, r_{kl}^2 = R^2\}$ and subspaces $\{x_i = x_j\}$. The gradients of r_{ij} and r_{kl} are independent where both equal R , since they differ in the x_j or x_l block, so each piece has dimension $2N - 2$. Next, let $Z = Z_{ij}^\pm$, closed in S_{ij}° . The set $Z \setminus \text{int } Z$ has empty interior in S_{ij}° , since a relatively open subset would lie in $\text{int } Z$. Were its dimension $2N - 1$, by (F2) it would contain a piece of dimension $2N - 1$, open in the $(2N - 1)$ -manifold S_{ij}° by invariance of domain, a contradiction. So $\mathcal{B}$ has dimension at most $2N - 2$, and by (F2) it lies in an algebraic set of the same dimension. Apply (F3) with $n = 2N$ and $m = 2$. $\square$

Lemma 10: For $x_0 \in W$, let $\tau(x_0) \in (0, \infty]$ be the exit time from W . The Filippov solution from x_0 is unique on $[0, \tau)$, crosses $\bigcup S_{ij}$ only transversally and finitely often in each $[0, t]$ with $t < \tau$, and $x_0 \mapsto x(t; x_0)$ is continuous on the open set $\{\tau > t\}$.

Proof: Off D solutions are classical and unique. Take $y \in S_{ij}^\circ \cap W$ with $g^-(y) > 0$; the case $g^+(y) < 0$ is symmetric. By Lemma 4, $g^+(y) > 0$. By continuity, there are a ball $O \ni y$ and $\delta > 0$ with $\nabla r_{ij} \cdot v \geq \delta$ for every Filippov velocity v in O . So along any solution in O , r_{ij} grows at rate at least δ and equals R at exactly one instant s . Two solutions from the same point in O with $r_{ij} < R$ both follow $\dot{x} = u^-$ until s and $\dot{x} = u^+$ after, so they agree. Hence each point of $W \cap D$ has a time neighborhood with at most one crossing, and a compact arc in W has finitely many. Continuity follows because the set of Filippov solutions depends upper semicontinuously on the initial point [16] and is a single solution here. If $\tau(x_0) > t$, the arc $x([0, t]; x_0)$ has positive distance from the closed set Γ , so nearby initial points also stay in W on $[0, t]$. $\square$

Lemma 11: If $\tau(x_0) < \infty$, then $x(\tau; x_0) \in \mathcal{B}$.

Proof: Since $|u| \leq \sqrt{N}\bar{v}$, $y := x(\tau)$ exists, and $y \in \Gamma$. Suppose $y \notin \mathcal{B}$. Then $y \in S_{ij}^\circ$ for one pair, and there is a ball $O \ni y$ with $O \cap D \subset S_{ij}$. By the definition of $\mathcal{B}$, Lemma 4 and continuity of $g^\pm$, after shrinking O we have, on all of $S_{ij} \cap O$, either $g^+ \equiv 0$ or $g^+ > 0$, and either $g^- \equiv 0$ or $g^- < 0$. In every case $S_{ij} \cap O \subset \Gamma$, so the last segment $x([t_1, \tau]) \subset O$ avoids S_{ij} and stays on one side, say outside. If $g^+ > 0$, then $\nabla r_{ij} \cdot u^+ > 0$ on O after shrinking, so r_{ij} increases and $r_{ij}(\tau) > R$. If $g^+ \equiv 0$, then $\varphi = \nabla r_{ij} \cdot u^+$ is Lipschitz on O with some constant L and vanishes on $S_{ij} \cap O$. For x near y with $r_{ij} > R$, moving x_i and x_j toward each other by $(r_{ij} - R)/2$ each gives a point of $S_{ij} \cap O$ at distance $(r_{ij} - R)/\sqrt{2}$. So $\dot{r}_{ij} = \varphi(x) \geq -L'(r_{ij} - R)$ with $L' = L/\sqrt{2}$, and by Gronwall's inequality

$$r_{ij}(t) - R \geq (r_{ij}(t_1) - R)e^{-L'(t-t_1)} > 0,$$

so again $r_{ij}(\tau) > R$. The inside case is the same with u^- . Each case contradicts $y \in S_{ij}$. $\square$

C. The Flow Does Not Compress Volume Too Fast

Lemma 12: Let $M = 2N(k_n + (N-1)(\sigma^* + p/d_{\text{out}}))$. Then $\text{div } u \geq -M$ in the sense of distributions on $\mathbb{R}^{2N}$, that is, $-\int u \cdot \nabla \phi \geq -M \int \phi$ for every smooth $\phi \geq 0$ with compact support.

Proof: Smooth part. Let $F_i = -\nabla_{x_i} J_i$. Off D and off a finite union of null sets where f or ∇V_i changes formula, F_i is C^1 and

$$\begin{aligned} D_{x_i} F_i &= -\nabla^2 V_i + \sum_{k \neq i} \left[f'(r_{ik}) \hat{e} \hat{e}^\top + \frac{f(r_{ik})}{r_{ik}} (I - \hat{e} \hat{e}^\top) \right] \\ &= P_i + Q_i, \end{aligned}$$

where $P_i \succeq 0$ collects the nonnegative coefficients and Q_i the negative ones. Since $f' \geq -\sigma^*$, $f/r \geq -p/d_{\text{out}}$ and $0 \preceq \nabla^2 V_i \preceq k_n I$, $|Q_i| \leq k_n + (N-1)(\sigma^* + p/d_{\text{out}})$. For almost every such x , $D_{x_i} u_i = A_i D_{x_i} F_i$ with A_i symmetric and $0 \preceq A_i \preceq I$. Where $|F_i| \neq \bar{v}$, A_i is the derivative of sat , which is I or $(\bar{v}/|F_i|)(I - \hat{F}_i \hat{F}_i^\top)$. On $E = \{|F_i| = \bar{v}\}$, $u_i = F_i$, and two Lipschitz functions that agree on E have equal derivatives almost everywhere on E [30]; since $|F_i|^2$ is constant on E , this gives $A_i = I - \hat{F}_i \hat{F}_i^\top$ there. As $\text{tr}(AP) \geq 0$ and $\text{tr}(AQ) \geq -2|Q|$ for $0 \preceq A \preceq I$ and $P \succeq 0$ in $\mathbb{R}^{2 \times 2}$, we get $\text{div } u = \sum_i \text{tr}(A_i P_i + A_i Q_i) \geq -M$ almost everywhere.

Jump part. Let $\psi \geq 0$ be smooth with support off Σ . There, D is the disjoint union of the hypersurfaces S_{ij}° , and u is Lipschitz up to each of them from both sides. The divergence theorem on both sides, with unit normal $\nabla r_{ij}/\sqrt{2}$, gives

$$\begin{aligned} -\int u \cdot \nabla \psi &= \int (\operatorname{div} u) \psi + \sum_{ij} \int_{S_{ij}^\circ} \psi \frac{g^+ - g^-}{\sqrt{2}} dA \\ &\geq -M \int \psi, \end{aligned}$$

since $g^+ - g^- \geq 0$ by Lemma 4.

Removing Σ . Let $\chi_\eta = \min\{1, \operatorname{dist}(\cdot, \Sigma)/\eta - 1\}^+$, so $\chi_\eta = 0$ on $\Sigma^{(\eta)}$ and $|\nabla \chi_\eta| \leq 1/\eta$ on $\Sigma^{(2\eta)}$. By Lemma 9, $|\Sigma^{(2\eta)} \cap \operatorname{supp} \phi| = O(\eta^2)$, so $\int \phi |\nabla \chi_\eta| = O(\eta)$. Apply the jump part to a mollified $\chi_\eta \phi$ and let $\eta \rightarrow 0$, using $|u| \leq \sqrt{N} \bar{v}$. $\square$

Lemma 13: Let $q \geq 0$, U open, and $E = \{x_0 \in W : \tau(x_0) > q, x(q; x_0) \in U\}$. Then $|E| \leq e^{Mq}|U|$.

Proof: E is open by Lemma 10. Smooth u by convolution, $u_\delta = u * \rho_\delta$. Then $|u_\delta| \leq \sqrt{N} \bar{v}$ and $\operatorname{div} u_\delta \geq -M$ by Lemma 12. The flow $\mathcal{F}_q^\delta$ of u_δ is a diffeomorphism with Jacobian determinant $\exp \int_0^q \operatorname{div} u_\delta \geq e^{-Mq}$ by Liouville's formula. Moreover, $u_\delta(x)$ lies in the closed convex hull of the values of u near x , so by the convergence theorem for differential inclusions [31, Ch. 1, §4], every limit of $\mathcal{F}^\delta(x_0)$ as $\delta \rightarrow 0$ is a Filippov solution. For $x_0 \in E$ this solution is unique, so $\mathcal{F}_q^\delta(x_0) \rightarrow x(q; x_0) \in U$. For a compact $F \subset E$, let $F_\delta = \{x_0 \in F : \mathcal{F}_q^\delta(x_0) \in U\}$. Then $|F_\delta| \rightarrow |F|$ and $|U| \geq |\mathcal{F}_q^\delta(F_\delta)| \geq e^{-Mq}|F_\delta|$. Let $\delta \rightarrow 0$, then let F grow to E . $\square$

D. Conclusion of the Proof

By Lemma 10 it suffices to show $\tau = \infty$ for almost every x_0 ; the complement of W lies in D , which has measure zero. Fix $T, \rho > 0$, let $A = \{x_0 \in W \cap B_\rho : \tau(x_0) \leq T\}$, and let $c_0 = \sqrt{N} \bar{v}$. Solutions from B_ρ stay in $B_{\rho'}$, $\rho' = \rho + c_0 T$, on $[0, T]$. For $\eta \in (0, 1]$, let $q_k = k\eta$ and $U = \mathcal{B}^{(2c_0\eta)} \cap B_{\rho'+2c_0}$. If $\tau(x_0) \in (q_k, q_{k+1}]$, then $x(\tau) \in \mathcal{B}$ by Lemma 11 and $|x(q_k) - x(\tau)| \leq c_0\eta$, so $x_0 \in E_k := \{\tau > q_k, x(q_k) \in U\}$. By Lemmas 13 and 9,

$$|A| \leq \sum_{k=0}^{\lceil T/\eta \rceil} |E_k| \leq \left(\frac{T}{\eta} + 2\right) e^{MT} c \eta^2 \xrightarrow{\eta \rightarrow 0} 0.$$

Taking the union over integer T and ρ completes the proof. $\square$

APPENDIX B PROOF OF THEOREM 3

Throughout, $u_j = \pi j/n$ for real $n \geq n_0$, and Ω denotes the set of pairs (ρ, n) with $n \geq n_0$ and spacing $s = 2\rho \sin(\pi/n) \in [r_c, d_{\text{in}}]$.

Step 1: distances in the ring. On Ω , c_j is increasing in j for $1 \leq j \leq n/2$, and because $\sin(jx) \leq j \sin x$ and $\sin y \geq y(1 - y^2/6)$,

$$js \left(1 - \frac{\pi^2 j^2}{6n^2}\right) \leq c_j \leq js. \quad (19)$$

For $j \leq J$, $c_j \leq J d_{\text{in}} < R$ by Assumption 2. For $J < j \leq n/2$, $c_j \geq c_{J+1} \geq (J+1)r_c(1 - \pi^2(J+1)^2/(6n_0^2)) > R$

by the selection of n_0 . The function $j(1 - \pi^2 j^2/(6n_0^2))$ is increasing for $j \leq J+1$, since $n_0 \geq 3(J+1)$, so $c_j \geq d_{\text{in}}$ for $J_b < j \leq J$. Applying the symmetry $j \leftrightarrow n-j$ and $n_0 > 2J$,

$$h(\rho, n) = 2 \sum_{j=1}^J f(c_j) \sin u_j - \nu(\rho), \quad (20)$$

where $c_1 = s \in [r_c, d_{\text{in}}]$, the chords with $j \leq J$ lie in $[s, R)$, those with $J_b < j \leq J$ lie in $[d_{\text{in}}, R)$, and no chord equals R . Because f is continuous on $(0, R)$, h is continuous on Ω , and each term of (20) admits one-sided derivatives in every direction.

Step 2: a unique root. Fix $n \in [n_{\text{lo}}, n_{\text{hi}}]$. The function f is nonincreasing on $(0, R)$ and strictly decreasing on (r_c, d_{in}) , every c_j grows with ρ , and ν is nondecreasing, so $h(\cdot, n)$ is strictly decreasing on Ω . At $s = d_{\text{in}}$, $f(c_1) = 0$ and $f(c_j) \leq 0$ for $j \geq 2$, giving $h < 0$. At $s = r_c$, $f(c_1) = C$ and $f(c_j) \geq -p$ for $j \geq 2$. Using $\sin(\pi/n) \geq (\pi/n) \cos(\pi/n)$ and $\sin u_j \leq \pi j/n$,

$$h \geq \frac{\pi}{n} \left(2C \cos \frac{\pi}{n_0} - 2p \sum_{j=2}^J j \right) - v_{\text{nav}} > 0,$$

since $n \leq n_{\text{hi}}$. Thus $h(\cdot, n)$ has a unique root $\rho(n)$, its spacing lies in (r_c, d_{in}) , and $\rho(n)$ is continuous in n . Because $\sin x \leq x$, $\rho(n) > nr_c/(2\pi) \geq v_{\text{nav}}/k_n$, so $\nu = v_{\text{nav}}$ near the root.

Step 3: the radius grows with n . Let (ρ, n) be a root with $n < n_{\text{hi}}$. Differentiating (20) with ν held constant,

$$\frac{\partial h}{\partial \rho} = 4 \sum_{j=1}^J f'(c_j) \sin^2 u_j,$$

$$\frac{\partial h}{\partial n} = -\frac{2}{n} \sum_{j=1}^J u_j \cos u_j (fc)'(c_j),$$

where at a break point of f the one-sided value of f' is taken. On (r_c, R) , $f' \leq 0$ and $(fc)' \leq 0$: on the barrier $(fc)' = -bd_m/(c - d_m)^2 - b/a$, in the dead zone it vanishes, and in the tether band it equals $-k(2c - d_{\text{out}})$. For $j = 1$, $|(fc)'(s)| = |f'(s)|(d_m + (s - d_m)^2/a) \geq d_m|f'(s)|$. Also $|f'(c_j)| \leq |f'(s)|$ on the barrier, since $|f'|$ is decreasing there and $c_j \geq s$, and $|f'| = k$ in the tether band. Retaining only $j = 1$ in $\partial h/\partial n$, and applying $\sin u_j \leq u_j$,

$$\frac{\partial h}{\partial n} \geq \frac{2\pi}{n^2} \cos \frac{\pi}{n_0} d_m |f'(s)|,$$

$$\left| \frac{\partial h}{\partial \rho} \right| \leq \frac{4\pi^2}{n^2} (S_b |f'(s)| + k S_t).$$

Since $|f'(s)| \geq b/a^2$ on (r_c, d_{in}) , the ratio of the two is at least δ . These bounds hold for both one-sided values of f' . Hence for $\delta' < \delta$, the one-sided derivative of h in the direction $(\delta', 1)$ is at least $(\delta - \delta')|\partial h/\partial \rho| > 0$, so $h(\rho + \delta'\tau, n + \tau) > 0$ for small $\tau > 0$, and $\rho(n + \tau) > \rho(n) + \delta'\tau$ because $h(\cdot, n + \tau)$ is decreasing. Thus $g(n) = \rho(n) - \delta'n$ is continuous with $g(n + \tau) > g(n)$ for small $\tau > 0$, at every n . Such a g is nondecreasing: if $g(n') < g(n)$ for some $n < n'$, the largest $t \in [n, n']$ with $g(t) \geq g(n)$ satisfies $t < n'$, and $g < g(t)$ just to its right, a contradiction. Taking $\delta' \rightarrow \delta$ yields $\rho(n') - \rho(n) \geq \delta(n' - n)$.

Step 4: assembly. Adjacent rings satisfy $\rho(n_{m+1}) - \rho(n_m) \geq \delta\Delta > R$. Two drones on distinct rings are separated by at least the radius difference, so they do not interact, and each drone responds only to its own ring and the goal. By Lemma 8 and Step 2, every drone is balanced. Intra-ring distances are at least $s > r_c$, and inter-ring distances exceed R , so no pair is capped and no pair lies at R . For the count, set $M = \lfloor (n_{hi} - n_{lo})/\Delta \rfloor$. Then $M + 1 \geq (n_{hi} - n_{lo})/\Delta$, $n_M > n_{hi} - \Delta$, and $\sum_m n_m = (M + 1)(n_{lo} + n_M)/2$, giving (16). $\square$

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