

Motion mem-converters: A paradigm shift in mechanical element modeling[★]

From converters and mem-converters to an array of mechanical two-port elements

Xiao-Liang Zhang^{a,*}, Jia-Mei Nie^b

^a*Automotive Engineering Research Institute, Jiangsu University, 301 Xuefu Road, Zhenjiang, 212013, China*

^b*School of Automotive and Traffic Engineering, Jiangsu University, 301 Xuefu Road, Zhenjiang, 212013, China*

ARTICLE INFO

Keywords:

Memory element
Motion converter
Mem-converter
Diamond-shaped mechanism
Nonlinear modeling
Mem-inerter

ABSTRACT

Since the introduction of memristors, memory element theories have advanced significantly, yet their application in mechanical systems remains limited due to the dominance of force-based linear modeling thinking. This paper proposes a mem-nonlinear modeling paradigm by identifying the diamond-shaped mechanism as a motion mem-converter, a purely mechanical two-port memory element. It is demonstrated that displacement, as the across variable, should dominate the modeling of geometrically nonlinear mechanical elements, challenging the traditional force-dominated approach. Through a systematic port-variable selection method within the framework of the periodic table of mechanical elements, correct constitutive relations for diamond-shaped memory elements are derived. The validity of this paradigm is confirmed via self-consistency analysis, including conservation of co-evolution, co-action, and energy, as well as experimental validation on a diamond-shaped mem-inerter. Furthermore, the framework successfully identifies the relativistic mass as a nonlinear converter with memory properties via decomposition. Finally, converters and mem-converters are generalized to a two-port mechanical-element array, enabling the prediction of higher-order elements. This work establishes a fundamental shift from force-dominated linear modeling to an across-variable-dominated mem-nonlinear paradigm, providing a robust theoretical foundation for future mechanical memory systems.

1. Introduction

As a key branch of nonlinear science, mem-element and system theories have advanced significantly since Chua's introduction of the memristor concept in 1971 [1, 2, 3, 4, 5, 6]. The memristor is a two-terminal element characterized by the flux–charge constitutive relationship (CR) and exhibiting the pinched hysteresis loop (PHL) in the voltage-current plane [7]. Chua and Kang [8] generalized the memristor concept to a class of nonlinear dynamical systems, establishing a mathematical theory for memristive systems. Di Ventra et al. [9] further extended this theory to memcapacitors and meminductors, along with their corresponding memcapacitive and meminductive systems. On the basis of these theories, Chua [10] proposed a periodic table of circuit elements, capable of predicting new ones just as Mendeleev's table predicts unknown chemical elements. Such studies have laid a solid foundation for the theories of memory elements and systems [11]. The discovery of memristor devices sparked unprecedented worldwide interest from both industry and academia, leading to an exponential growth in related publications and consequently propelling the further development of relevant theories [12, 13]. These theories have attracted considerable attention for applications in non-volatile data storage, neuromorphic computing mimicking brain synapses, ultra-low-power artificial intelligence, and in-memory logic circuits [14, 15, 16, 17, 18].

However, transferring these theories from the electrical to the mechanical domain is far from straightforward. The difficulty stems from the fact that mechanical engineering introduces many unfamiliar concepts, such as momentement, yank, tug, and evolution¹ [19, 20, 21, 22, 23], which are not always physically intuitive. Moreover, the theories are mathematically abstract, posing a considerable challenge in translating their terminology and mathematical notation into the mechanical domain. Even developing a simple mechanical mem-model usable for practical data analysis and

[★] This work was supported by the National Natural Science Foundation of China (Grant Nos. U24A6008 and 51805223).

*Corresponding author

✉ zxl1979@ujs.edu.cn (X. Zhang)

ORCID(s): 0000-0002-5368-3664 (X. Zhang)

¹ Absement is the time integral of displacement, momentement is the second time integral of force, yank and tug are the first and second time derivatives of force, and evolution is the second time integral of energy.

Table 1

Mechanical elements and devices with memory published in literature.

Elements	Devices
Mem-inerter or Mem-mass (−1, −2)	Displacement-dependent fluid inerter [24] Crank-connecting-rod inerter [25] Diamond-shaped or X-shaped inerter [26] cable-reel [27, 28] and yoyo-toy [29, 11]
Mem-spring (−2, −1)	Superelasticity of shape-memory alloys [30, 31, 32]
Mem-damper (−1, −1)	Tapered dashpot [33] Diamond-shaped or X-shaped damper [26] Displacement-dependent fluid damper [34]
Nonlinear mass with memory (0, −1)	Relativistic mass [10, 35, 21]
Nonlinear spring with memory (−1, 0)	Diamond-shaped or X-shaped spring [26, 36]

modeling requires evaluating many functional forms through extensive experimentation. Due to the aforementioned difficulties, there has been little published research on mechanical memory elements and their associated mem-models. This is why these memory elements are so “new,” even though the physical and mathematical foundations of memristors were first introduced many years ago. Table 1 lists these elements with memory and marks their positions in the periodic table of mechanical two-terminal (α, β) elements (see Fig. 6 below), which is the mechanical version of Chua’s periodic table of fundamental circuit elements.

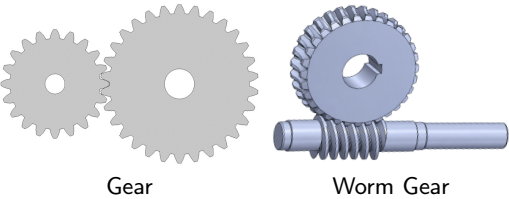
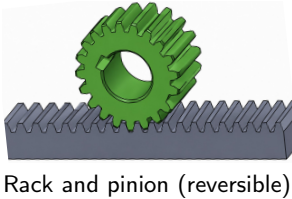
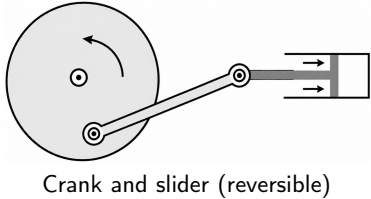
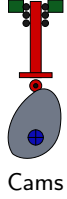
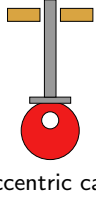
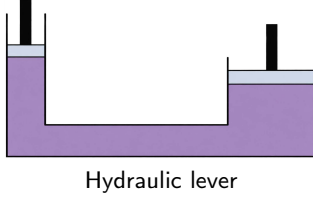
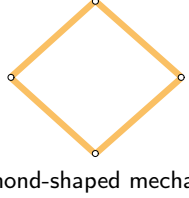
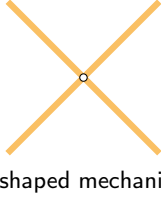
Chua’s table of fundamental elements introduces the so-called two-terminal (α, β) elements, or higher-order elements (HOEs), which are useful as both model and physical building blocks. HOE models take the form of unambiguous constitutive relations, independent of the element’s interaction with its surroundings or its initial state. Chua emphasized that HOEs are mutually independent and cannot be synthesized from any combination of existing elements from his table, and thus serve as fundamental building blocks that make up a circuit model [37]. However, research and practice in the mechanical domain have revealed that HOEs in the mechanical version of Chua’s table are not the most basic elements, as they can be further decomposed. For instance, the diamond-shaped memory element [26], a geometrically nonlinear element, can be decomposed into a diamond-shaped mechanism and a linear element, where the mechanism is in fact a motion converter (see Tab. 2 below). In other words, motion converters and linear elements are more fundamental than HOEs. Accordingly, investigating the mechanical motion converter family and its memory properties is one of the main motivations for this work.

Table 2 presents common mechanical motion conversion mechanisms and their types. As is well known, with the mass and rotational inertia of their components neglected, these mechanisms may be modeled as ideal converter elements. It is worth noting that ideal converters require only across variables (such as displacement and velocity) to describe their input-output relationships, without involving through variables like force or momentum. Linear converters like gears, worm gears, rack and pinion gears, ball screws, hydraulic levers shown in the last column of Tab. 2 have fixed transmission ratios, the ratios of output to input for displacement² and all its successive time derivatives (velocity, acceleration, jerk, etc.) are therefore all equal to one another and to the transmission ratio. However, the case is very different for the nonlinear converters in the last column of Tab. 2, with the diamond-shaped mechanism as a representative example. This study finds that the diamond-shaped mechanism is a two-port element whose CR is expressed in terms of input and output port displacements, whose velocity ratio is displacement-dependent rather than constant, and which therefore exhibits PHLs in the plane of port velocities — the fingerprint of a memory element (see Figs. 2 and 3 below). Accordingly, what were previously referred to as nonlinear converters in Tab. 2 should now be formally termed mem-converters.

²Including displacement for translational motion and angular displacement for rotational motion.

Table 2

Common mechanical motion conversion mechanisms and their types.

Motion conversion	Linear converters (Fixed transmission ratio)	Nonlinear converters (Variable transmission ratio)
Rotary to Rotary	 <p>Gear Worm Gear</p>	 <p>Elliptical gear</p>
Linear to Rotary	 <p>Rack and pinion (reversible)</p>	 <p>Crank and slider (reversible)</p>
Rotary to Linear	 <p>Ball screw (reversible)</p>	 <p>Cams</p>  <p>Eccentric cam</p>
Linear to Linear	 <p>Hydraulic lever</p>	 <p>Diamond-shaped mechanism</p>  <p>X-shaped mechanism</p>

The true significance of this work, however, lies not in the terminology of mem-converters per se, but in the paradigm shift it triggers in the modeling of mechanical elements. As stated above, a memory element with geometrical nonlinearity is composed of a motion mem-converter combined with a linear element, wherein the mem-converter provides only the across variable for the nonlinear element, while the through variable is naturally supplied by the linear element. In the linear case, the memory element and the mem-converter discussed here degenerate into their linear counterparts. For example, the rack-and-pinion and ball-screw inerters both consist of a linear motion converter and a flywheel. Under the conventional linear modeling paradigm, force is taken for granted as the dominant quantity, with the across variable from the converter playing only a subsidiary role to the through variable, i.e., force. For linear elements like the rack-and-pinion and ball-screw inerters, such a modeling paradigm poses no problem. This is because the converters that provide the across variables are linear, and the output-to-input ratios for their port displacements and all successive time derivatives are always identical. Thus, one may select any across variables from displacement, velocity, acceleration, jerk, or higher-order derivatives to form complementary variable pairs with force for modeling these linear elements. For linear inerters, dampers, and springs, the variable pairs are invariably chosen as {acceleration, force}, {velocity, force}, and {displacement, force}, respectively, with their corresponding positions in the periodic table of mechanical elements being (1, 0), (0, 0), and (−1, 0). This is a typical manifestation of the linear modeling paradigm.

However, applying this paradigm to memory elements will inevitably lead to errors. The reason lies in the fact that, in memory elements, the mem-converter is defined by the constitutive relationship between input and output

displacements. Consequently, the displacement across variable dominates, whereas the through variable from the linear element must be identified via the periodic table. This is entirely different from the modeling philosophy for linear elements. To date, almost all geometrically nonlinear inertial and damping devices, more precisely displacement-dependent ones, have been modeled under the constraints of the linear modeling paradigm. The models of such devices, especially diamond-shaped inerters and dampers [38, 39, 40], X-shaped inerters and dampers [41, 42], and crank-connecting-rod inerters [43] that are frequently reported in the literature, are all in need of reevaluation. Thus, another motivation of this work is to propose a mem-nonlinear modeling paradigm and to verify its validity.

The main contributions of this work are threefold. First, this study reveals that the diamond-shaped mechanism is a purely mechanical two-port memory element, termed the motion mem-converter. It is precisely from the perspective of the mem-converter that the limitations of the linear modeling paradigm come to light. Second, the present work proposes a systematic approach for port-variable selection and through-variable analysis, derives the correct CRs and port equations of diamond-shaped memory elements (DSMEs), establishes the mem-nonlinear modeling paradigm, and validates it through self-consistency analysis and experimental studies on DSMEs. More importantly, by applying this modeling paradigm, it identifies the relativistic converter — a velocity-dependent nonlinear converter — via decomposition of the relativistic mass into the relativistic converter and the rest mass. Third, this work generalizes converters and mem-converters to an array of mechanical two-port elements and predicts higher-order two-port elements.

The rest of this paper is organized as follows. Section 2 reveals that the diamond-shaped mechanism is a motion mem-converter, which is a purely mechanical two-port element with memory. Section 3 presents the mem-nonlinear modeling paradigm, whose validity is evaluated through self-consistency analysis and an experimental study on DSMEs. Section 4 establishes the relativistic converter model by decomposing the relativistic mass into the relativistic converter and the rest mass under the mem-nonlinear modeling paradigm. In Sect. 5, converters and mem-converters are generalized to the array of mechanical two-port elements, from which higher-order two-port elements can be predicted. Finally, Section 6 concludes the paper.

2. Mem-converter

2.1. Diamond-shaped mechanism

2.1.1. Geometrically nonlinear CR

The diamond-shaped mechanism consists of four equal-length rods, as shown in Fig. 1. The rods are interconnected by four joints to form a diamond-shaped configuration, which is supported by a horizontal base through joint 2. Moreover, the line connecting joints 1 and 2 is vertical, and consequently, the line connecting joints 3 and 4 is horizontal. Assuming that all rods are rigid and massless and that all joints are smooth and frictionless, the diamond-shaped mechanism can be regarded as an ideal mechanism.

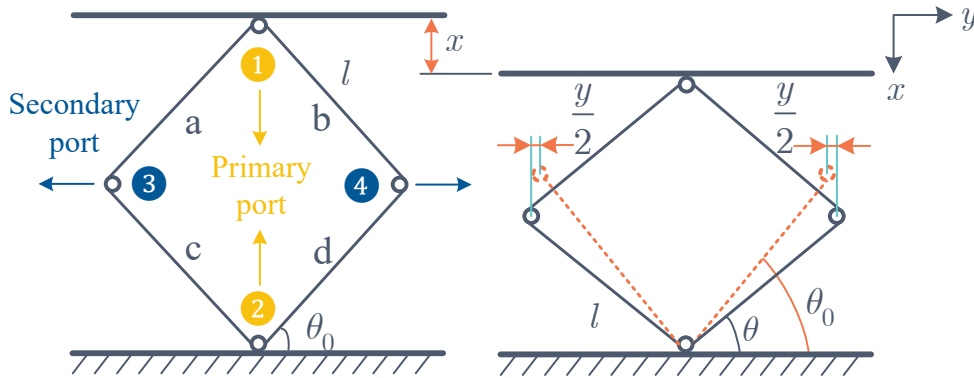


Figure 1: Schematic of the diamond-shaped motion converter identified as a mem-converter in this work.

Let joints 1 and 2 be the terminals of the primary side, and joints 3 and 4 be the terminals of the secondary side. Thus, the four terminals form two ports, namely, the primary and secondary ports. Since joint 2 is fixed to the base, the displacement x of the primary port is equal to the displacement of joint 1, while the displacement y of the secondary

port is equal to the sum of the displacements of joints 3 and 4. According to the geometric relationship shown in Fig. 1, the port displacements x and y can be expressed as

$$x = 2l \sin \theta_0 - 2l \sin \theta, \quad (1)$$

and

$$y = 2l \cos \theta - 2l \cos \theta_0, \quad (2)$$

where l is the rod length, which is set to 0.2 m in this work; θ_0 and θ denote the initial and instantaneous angles, respectively, between rod d and the base, corresponding to the initial and instantaneous configurations of the rod, respectively. Accordingly, $\cos \theta$ can be written as

$$\cos \theta = \sqrt{1 - \left(\sin \theta_0 - \frac{x}{2l} \right)^2}. \quad (3)$$

According to Eqs. (1) and (2), the geometric relationship among the three variables x , y and θ can be derived as

$$\tan \theta = \frac{2l \sin \theta_0 - x}{2l \cos \theta_0 + y}. \quad (4)$$

With the help of Eqs. (1), (2) and (3), the motion relationship between the primary-port displacement x and the secondary-port displacement y can be expressed as

$$y = \hat{y}(x) = \sqrt{4l^2 - (2l \sin \theta_0 - x)^2} - 2l \cos \theta_0, \quad (5)$$

Equation (5) clearly demonstrates that there is a nonlinear algebraic relationship between the variable pair $\{x, y\}$.

Substituting Eq. (5) into Eq. (4) and taking the inverse tangent, one can obtain

$$\theta = \hat{\theta}(x) = \arctan \left(\frac{2l \sin \theta_0 - x}{\sqrt{4l^2 - (2l \sin \theta_0 - x)^2}} \right). \quad (6)$$

Differentiating both sides of Eq. (5) with respect to time yields

$$v_y = \hat{v}_y(x, v_x) = \frac{2l \sin \theta_0 - x}{\sqrt{4l^2 - (2l \sin \theta_0 - x)^2}} v_x = \tan \hat{\theta}(x) v_x, \quad (7)$$

where $v_x = dx/dt$ and $v_y = dy/dt$ denote the velocities of the primary and secondary ports, respectively.

Obviously, Eq. (7) contains three variables $\{v_x, x, v_y\}$, and is therefore a dynamic equation. In contrast to this dynamic relationship, the algebraic relationship represented by Eq. (5) intrinsically reflects the geometrically nonlinear kinematic relationship between the primary and secondary ports. Therefore, x and y are the appropriate choice of variables for characterizing the CR of the diamond-shaped mechanism.

2.1.2. Three memory fingerprints

Consider a cosine velocity input defined by

$$\begin{cases} v_x(t) = A \cos \Omega t, & t \geq 0 \\ = 0, & t < 0 \end{cases} \quad (8)$$

across the primary port, as shown in magenta in Fig. 2d for $A = 0.5$ m/s and $\Omega = 2\pi$ rad/s. To determine the velocity response $v_y(t)$ of the secondary port from the CR (5), the displacement x of the primary port (shown in green in Fig.

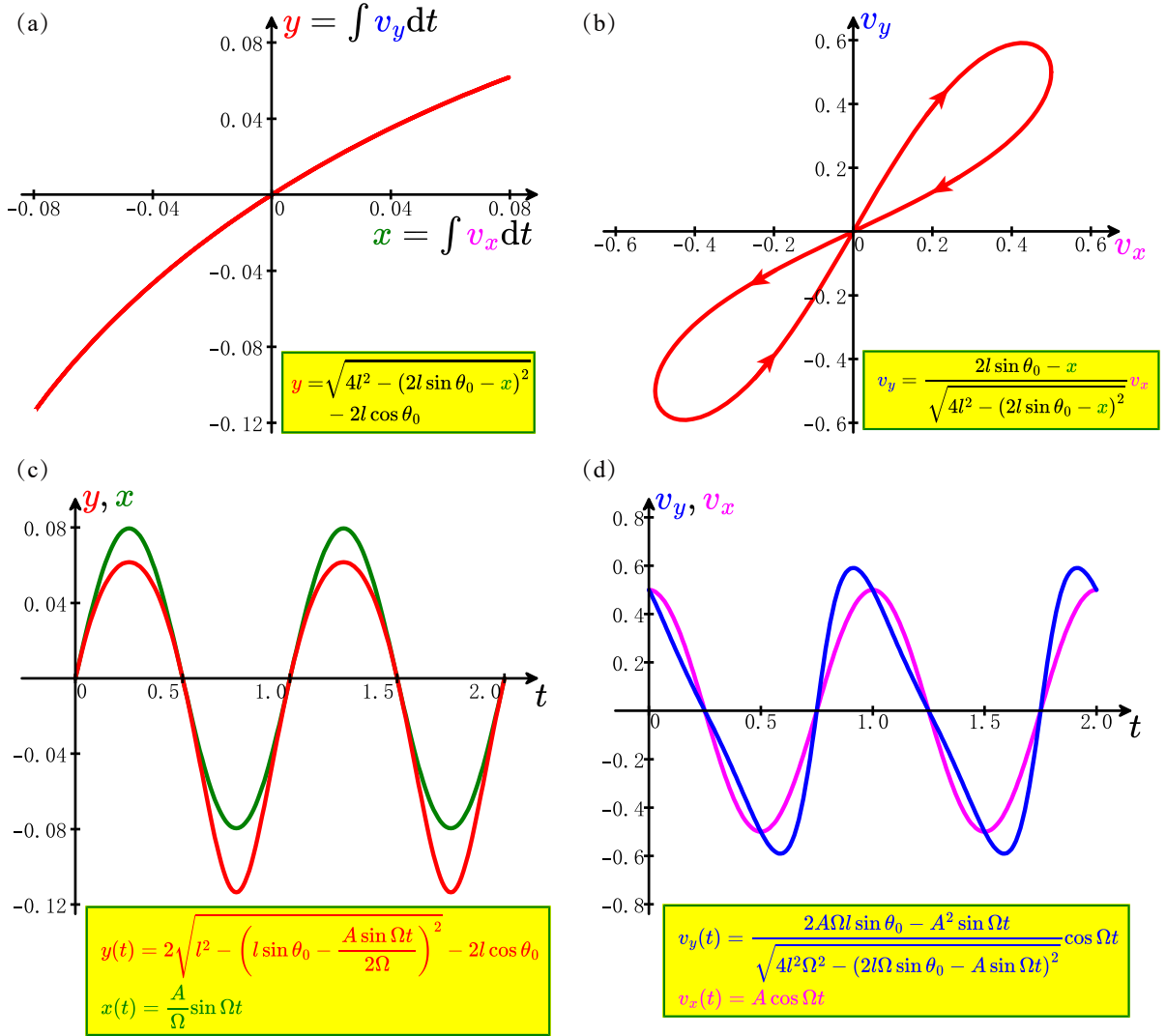


Figure 2: CR and PHL of the diamond-shaped mechanism, together with the associated waveforms, for $A = 0.5$ m/s and $\Omega = 2\pi$ rad/s. (a) CR, $y = \hat{y}(x)$. (b) PHL in the $v_x(t) - v_y(t)$ plane. (c) Periodic waveforms of $y(t) = \int_0^t v_y(\tau) d\tau$ and $x(t) = \int_0^t v_x(\tau) d\tau$. (d) Periodic waveforms of $v_y(t)$ and $v_x(t)$.

2c) must be calculated first. Assuming an initial angle $\theta_0 = \pi/4$ and an initial displacement $x_0 = x(0) = 0$, integration of Eq. (8) yields the following expression for the displacement $x(t)$ of the primary port:

$$x(t) = \int_0^t A\Omega \cos(\Omega\tau) d\tau = \frac{A}{\Omega} \sin \Omega t, \quad t \geq 0. \quad (9)$$

Substituting Eq. (9) into Eq. (5) yields the corresponding displacement $y(t)$ of the secondary port, as shown in green in Fig. 2c, given by

$$y(t) = 2\sqrt{l^2 - \left(l \sin \theta_0 - \frac{A \sin \Omega t}{2\Omega}\right)^2} - 2l \cos \theta_0. \quad (10)$$

Differentiating Eq. (10) with respect to t leads to

$$v_y(t) = \frac{2A\Omega l \sin \theta_0 - A^2 \sin \Omega t}{\sqrt{4l^2\Omega^2 - (2l\Omega \sin \theta_0 - A \sin \Omega t)^2}} \cos \Omega t. \quad (11)$$

PHL. Plotting the loci of $(v_x(t), v_y(t))$ in the $v_y - v_x$ plane using Eqs. (8) and (11) yields the PHL shown in Fig. 2b for $A = 0.5$ m/s and $\Omega = 2\pi$ rad/s, a characteristic fingerprint of memory elements in both the electrical and mechanical domains.

Coincident Zero-Crossing signature. Observing Fig. 2d, one can find that the waveform of the secondary port velocity $v_y(t)$ must cross the time axis at the same instants of time whenever the waveform of the primary port velocity $v_x(t)$ crosses the time axis. This property is referred to as the Coincident Zero-Crossing signature by Chua in [44]. More general than the pinched hysteresis loop, this signature provides another important fingerprint for identifying memory elements.

High-frequency degradation phenomenon of pinched hysteresis loops. Figure 3 shows three pinched hysteresis loops corresponding to three frequencies, $\Omega = 2\pi$ rad/s, 5π rad/s, and ∞ rad/s, for $v_x(t) = 0.4\Omega \cos(\Omega t)$ and initial state $x(0) = 0$. As the frequency Ω increases, the pinched loop gradually shrinks and eventually collapses to a straight line at very high frequencies.

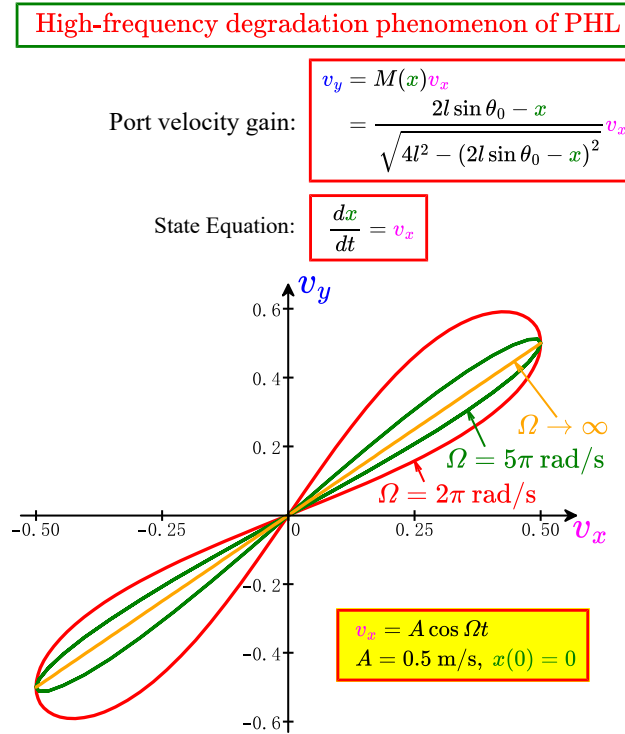


Figure 3: High-frequency degradation phenomenon of pinched hysteresis loops. The areas of the two lobes of the PHL of the mem-converter decrease as the frequency increases, and the PHL tends to a single-valued linear function at very high frequencies.

Both the pinched hysteresis loop fingerprint and the Coincident Zero-Crossing signature observed in Fig. 2 reveal that the diamond-shaped mechanism in Fig. 1 should be identified as a motion converter with memory, whose CR is described by the curve in the x - y plane shown in Fig. 2a. It can be observed from Fig. 2a that this CR establishes a one-to-one relationship between the primary-port displacement $x(t)$ and the secondary-port displacement $y(t)$, as shown in Fig. 2c. Moreover, the degradation of the pinched hysteresis loop into a straight line as the frequency tends to infinity, as shown in Fig. 3, further indicates the diamond-shaped mechanism can be regarded as an ideal motion converter with memory.

2.2. Defining an ideal motion mem-converter

It should be noted that, unlike a conventional two-port element, the diamond-shaped mechanism is a pure motion two-port without inertia, stiffness or damping characteristics. In this sense, a mem-converter is a two-port element characterized by a CR between two variables y and x representing the time integral of the two port velocities v_y and v_x ; namely,

$$y(t) \triangleq \int_{-\infty}^t v_y(\tau) d\tau, \quad (12)$$

$$x(t) \triangleq \int_{-\infty}^t v_x(\tau) d\tau. \quad (13)$$

If the primary-port displacement is controlled by the secondary-port displacement of the mem-converter, its CR can be expressed by

$$y = \hat{y}(x). \quad (14)$$

Differentiating Eq. (14) with respect to time yields

$$v_y = \frac{dy}{dt} = \frac{d\hat{y}(x)}{dx} \frac{dx}{dt} = M(x)v_x, \quad (15)$$

where

$$M(x) = \frac{d\hat{y}(x)}{dx} \quad (16)$$

is called the port velocity gain with memory from the primary to the secondary side, except that the gain $M(x)$ at any time $t = t_0$ depends on the entire past history of $v_x(t)$ from $t = -\infty$ to $t = t_0$. If the gain $M(x)$ is a constant, the mem-converter degenerates into a two-port linear converter.

For the diamond-shaped mechanism shown in Fig. 1, Eq. (5) gives the expression of its CR corresponding to Eq. (14), while Eq. (7) gives the expression of its dynamic relation corresponding to Eq. (15). Furthermore, substituting Eq. (15) into Eq. (7) yields $M(x) = \tan \hat{\theta}(x)$.

2.3. Mem-converter two-port

A mem-converter is a pure motion two-port element with memory, as shown in Fig. 4. As a motion two-port, its port through variables cannot be determined until an external element is connected to one of its ports. In this sense, such a two-port element has well-defined port across variables, i.e., displacements, while its port through variables remain undetermined.

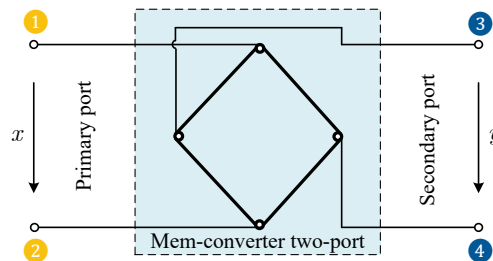


Figure 4: Mem-converter: a pure motion two-port model with memory characterized by a CR between port displacement variables.

As a pure motion two-port, it has the following properties.

Passivity. Since this two-port contains no internal power source, it cannot supply energy to the external system and is therefore passive.

Port dimension consistency. A mem-converter two-port is said to be dimensionally consistent if its two ports have the same dimensions when an external element is connected to one of them. For example, for a mem-converter two-port with one port connected to a linear element, the primary and secondary ports should have the same physical dimension, which can be $J \cdot s^2$, $J \cdot s$, or J .

Memory property and fundamental building block. Owing to its inherent memory property, the mem-converter two-port has the capability of transforming the memoryless linear elements into the memory elements. Specifically, when a linear element is connected to the secondary port of the mem-converter, the resulting one-port element, as viewed from the primary port, becomes a DSME, as shown in Fig. 5. In other words, the mem-converter is the underlying source of memory, while the resulting DSME inherits its memory property from the mem-converter. This also indicates that the mem-converter is more fundamental than the memory elements in the periodic table of mechanical elements and can serve as both a modeling building block and a physical building block.

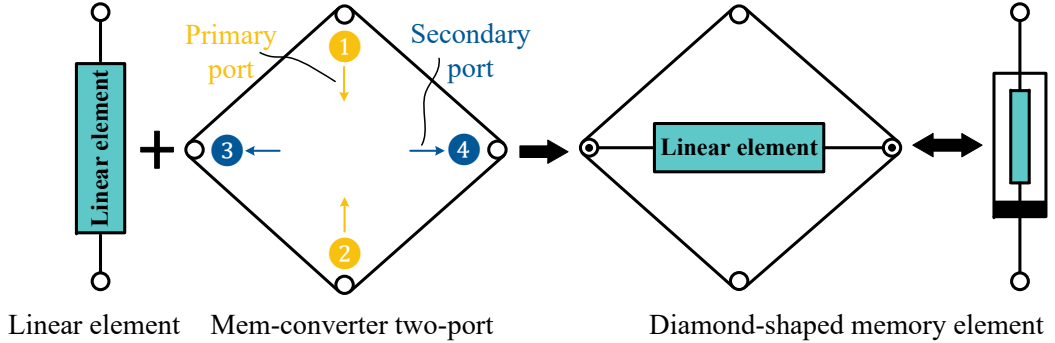


Figure 5: Motion mem-converter two-port: a building block for transforming linear into memory elements.

CR's decisive role. It is well known that CR is the most fundamental relationship of an element and, for a nonlinear element, is uniquely determined. Likewise, for the mem-converter, the CR between its two port displacements is also sufficiently fundamental to play a decisive role in determining the across variable of both a linear element connected to either port and the memory element formed by the mem-converter. Specifically, the CR of the mem-converter requires that the across variable of the connected linear element to be displacement, rather than velocity or acceleration. Such CR-based mem-nonlinear modeling paradigm is counterintuitive and fundamentally different from the conventional linear modeling paradigm, which, due to its strong cognitive inertia, dictates that the across variable of a spring, damper, or inerter must be displacement, velocity, or acceleration, respectively, regardless of the specific modeling context.

3. Mem-nonlinear modeling paradigm: a case study of DSMEs

3.1. Correct location in the periodic table and variable choices for DSMEs

The periodic table of mechanical elements. As shown in Fig. 6, a mechanical counterpart to Chua's periodic table of circuit elements can be established via the force-current analogy. Here, integers α and β index the orders of time-domain derivatives (for positive integers) or integrals (for negative integers) of the port velocity v and force f , with zero representing the variables themselves. Under this convention, momentum $p = f^{(-1)}$, displacement $x = v^{(-1)}$, momentement $\delta = f^{(-2)}$, and absement $\rho = v^{(-2)}$. Thus, the damper (C) with its $v - f$ constitutive relation resides at coordinate $(0, 0)$. Likewise, the inerter (B), spring (K), mem-damper (MC), mem-inerter (MB), and mem-spring (MK) occupy positions $(0, -1)$, $(-1, 0)$, $(-1, -1)$, $(-1, -2)$, and $(-2, -1)$, respectively.

In this table, elements along the axis $\alpha = n$ or $\beta = n$ ($n = 0, \pm 1, \pm 2, \dots, \pm N$) share the same across variable or through variable, respectively. For example, the K, MC, and MB elements located on $\alpha = -1$ all possess displacement as their across variable, while the MK, MC, and B elements on $\beta = -1$ all have momentum as their through variable.

From a small-signal perspective, all elements located along the same Δ -diagonal exhibit an identical characteristic, classified into damping, stiffness, or inertance types. Specifically, the elements along each Δ -diagonal can be characterized as frequency-dependent positive or negative dampers (^+C or ^-C), inerters (B), or springs (K) of the corresponding order Δ , as shown in the periodic table. The four corresponding types of Δ -diagonals are colored orange, gold, turquoise, and blue, respectively, in Fig. 6.

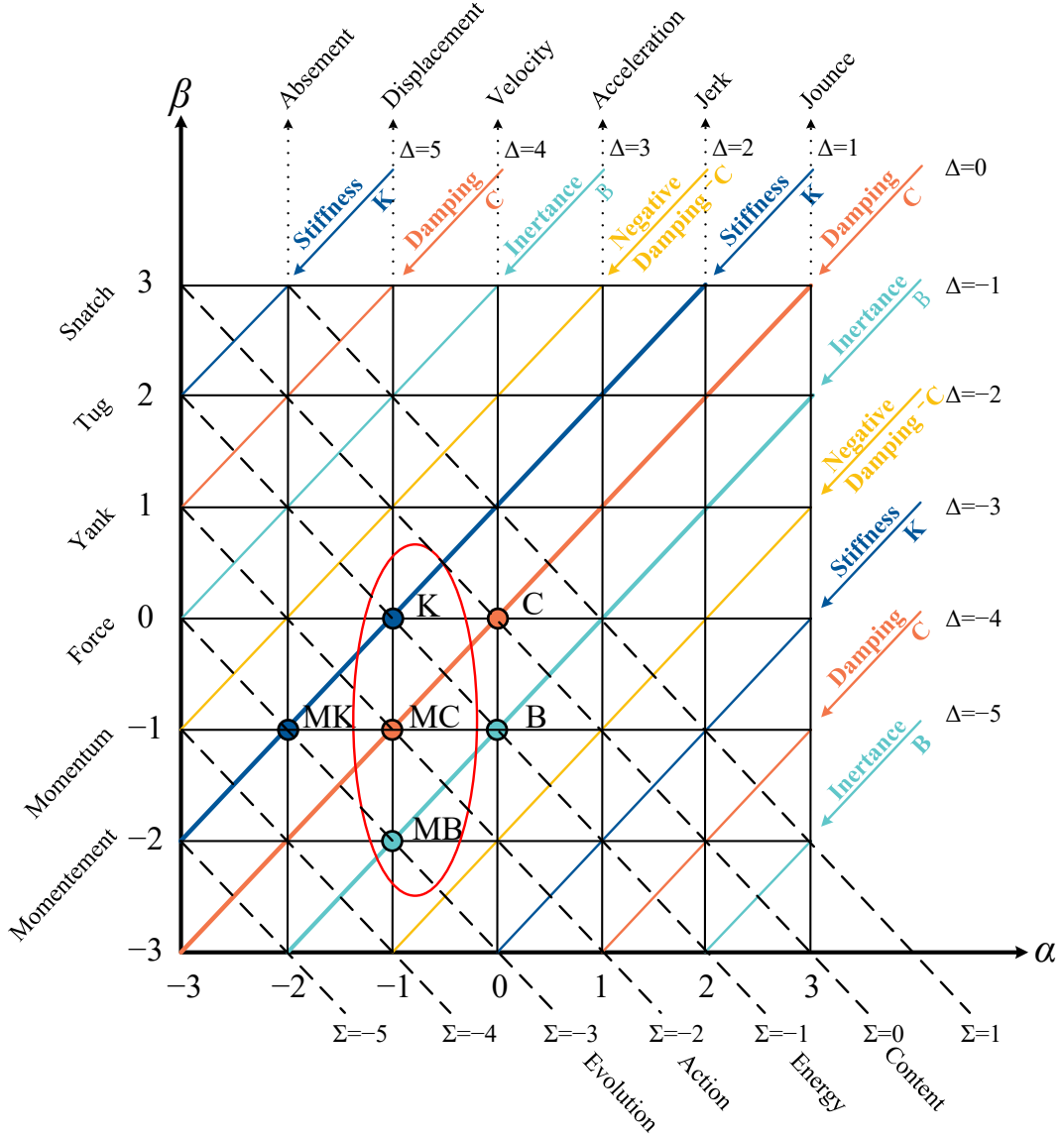


Figure 6: Correct location in the periodic table of mechanical elements for memory elements consisting of a mem-converter two-port and a linear element. K, C, B denote the spring, damper, inerter, respectively, while MK, MC, MB denote the mem-spring, mem-damper, mem-inerter, respectively. ($\Delta = \beta - \alpha$, $\Sigma = \beta + \alpha$).

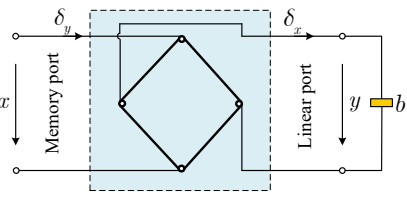
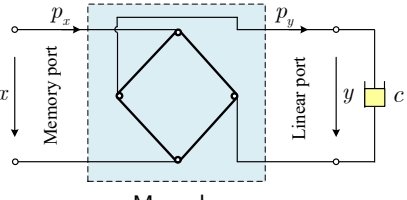
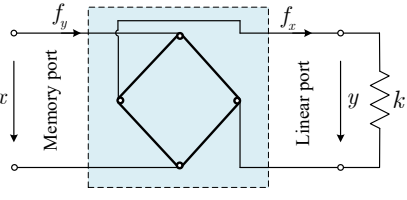
For elements located on a given Σ -diagonal, they share a constant sum of indices $\Sigma = \alpha + \beta$. Consequently, their potential functions exhibit identical physical dimensions. For example, the elements along the diagonals $\Sigma = -1$, -2 , and -3 possess potential functions corresponding to *energy*, *action*, and *evolution*, with physical dimensions of J , $J \cdot s$, and $J \cdot s^2$, respectively.

Correct location for DSMEs. Due to passivity of the mem-converter two-port, the DSMEs transformed from the linear passive elements via the converter two-port are inherently passive. Consequently, these memory elements should only be located on the three main diagonals in the table, i.e., $\Delta = 0, \pm 1$. This is because the elements on the other diagonals are all active, according to [45, 46].

Owing to the decisive role of CR, the across variables of both ports of the mem-converter two-port must be displacement; thus, the across variables of DSMEs are all displacement. That is, these elements are located along the axis $\alpha = -1$ in the table. Moreover, since these elements are transformed from the linear spring, damper, and inerter, they must retain the stiffness, damping, and inertance characteristics of the linear counterparts. Consequently, these

Table 3

Comparison of variable choices for DSMEs under the linear and mem-nonlinear modeling paradigms.

DSMEs	Variable choices for mem-nonlinear paradigm		Variable choices for linear paradigm	
	Memory ports	Linear ports	Memory ports	Linear ports
 <p>Mem-inerter</p>	Across variable $x = \int v_x dt$ Through variable $\delta_x = \int p_x dt$	Across variable $y = \int v_y dt$ Through variable $\delta_y = \int p_y dt$	Across variable $a_x = \dot{v}_x = \ddot{x}$ Through variable $f_x = \dot{p}_x = \ddot{\delta}_x$	Across variable $a_y = \dot{v}_y = \ddot{y}$ Through variable $f_y = \dot{p}_y = \ddot{\delta}_y$
 <p>Mem-damper</p>	Across variable $x = \int v_x dt$ Through variable $p_x = \dot{\delta}_x = \int f_x dt$	Across variable $y = \int v_y dt$ Through variable $p_y = \dot{\delta}_y = \int f_y dt$	Across variable $v_x = \dot{x} = \int a_x dt$ Through variable $f_x = \dot{p}_x = \ddot{\delta}_x$	Across variable $v_y = \dot{y} = \int a_y dt$ Through variable $f_y = \dot{p}_y = \ddot{\delta}_y$
 <p>Nonlinear spring</p>	Across variable $x = \int v_x dt$ Through variable $f_x = \dot{p}_x = \ddot{\delta}_x$	Across variable $y = \int v_y dt$ Through variable $f_y = \dot{p}_y = \ddot{\delta}_y$	Same as the mem-nonlinear paradigm	

memory elements are constrained to the diagonals $\Delta = 1, 0, -1$. The intersection points of the axis and the diagonals, namely, $(-1, 0)$, $(-1, -1)$, and $(-1, -2)$, designate the correct locations of the three elements, as indicated by the red ellipse in Fig. 6. The corresponding memory elements are the nonlinear spring, mem-damper, and mem-inerter. It should be noted that although the element located at $(-1, 0)$ is called the nonlinear spring, it is actually a real memory element characterized by the force-displacement CR and exhibiting the PHL in the yank-velocity plane.

Correct variable choices for DSMEs. In light of the locations of DSMEs in the periodic table, their CR variable choices are summarized in Tab. 3. For comparison, variable choices derived from the linear modeling paradigm are also included. As shown in Tab. 3, a clear discrepancy exists between these two modeling paradigms regarding the variable choices of the mem-inerter and mem-damper transformed from linear elements. The linear paradigm rigidly adheres to the idea that the across- and through-variable pairs at the nonlinear ports of the memory elements should directly inherit those of their linear counterparts (see the third column of Tab. 3). However, except for the nonlinear spring, whose variable choice merely coincided with the correct one, all other choices under the linear paradigm are incorrect, stemming from cognitive inertia. In contrast, the mem-nonlinear paradigm proposed in this work respects the decisive role of the two-port CR of the mem-converters and leverages the predictive capability of the periodic table of mechanical elements. Consequently, it yields the correct variable choices for DSMEs, as presented in the second column of Tab. 3.

3.2. CR Modeling for DSMEs

3.2.1. CRs of linear elements

The linear modeling paradigm dictates that, under all circumstances, the CRs of linear inerters, dampers, and springs should be expressed as

$$f_y = ba_y, f_y = cv_y, f_y = ky, \quad (17)$$

where f_y, y, v_y, a_y denote the forces applied to the ports of these elements and the corresponding displacement, velocity, acceleration resulting from these forces, respectively, while b, c, k represent inertance, damping, stiffness coefficients, respectively, as shown in the fifth column of Tab. 4.

In contrast, the mem-nonlinear modeling paradigm maintains that, owing to the port dimension consistency and the decisive role of the CR of the mem-converter two-port, the CRs of linear inerters, dampers, and springs connected to its ports should be expressed as

$$\delta_y = by, p_y = cy, f_y = ky, \quad (18)$$

where δ_y and p_y denote the momentement and momentum applied to the ports of these elements, as shown in the third column of Tab. 4.

3.2.2. Force analysis vs. through-variable analysis

In the field of electricity, with Chua's discovery of the periodic table of circuit elements, the concept of higher-order nonlinear elements has become deeply ingrained in the minds of scholars. Kirchhoff's Voltage^(a) Law and Kirchhoff's Current^(b) Law, along with the corresponding higher-order Euler-Lagrange equations are becoming increasingly accepted [47]. In the field of mechanics, however, the equation of force balance is well-known, while the equations of momentum and integrated momentum balance, corresponding to Kirchhoff's Charge Laws and their integrated versions, remain practically unknown, let alone the equation of higher-order integrated force balance. A similar situation exists for decomposition and composition of force. Such an approach of force analysis is common knowledge and deeply rooted in everyone's minds, to the extent that the decomposition and composition of momentum and higher-order integrated force cannot be tolerated. This is the typical linear modeling paradigm in which force is everything. In contrast, the mem-nonlinear modeling paradigm respects the decisive role of the CR and adopts the decomposition and composition of its variables. This approach will be used in the subsequent modeling of DSMEs.

3.2.3. CR models for DSMEs

Based on the through-variable triangle shown in the second column of Tab. 4, with the help of Eqs. (5), (6) and (18), the CRs of the diamond-shaped mem-inerter, mem-damper, nonlinear spring are derived as

$$\delta_x = \delta_y \tan \theta = by \tan \theta = b\hat{y}(x) \tan \hat{\theta}(x) = \hat{\delta}(x), \quad (19)$$

$$p_x = p_y \tan \theta = cy \tan \theta = c\hat{y}(x) \tan \hat{\theta}(x) = \hat{p}(x), \quad (20)$$

$$f_x = f_y \tan \theta = ky \tan \theta = k\hat{y}(x) \tan \hat{\theta}(x) = \hat{f}(x). \quad (21)$$

Dividing Eqs. (19), (21), and (20) by constant coefficients b, k , and c , respectively, results in

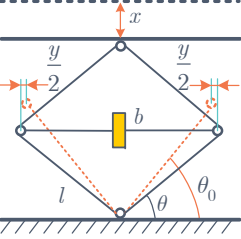
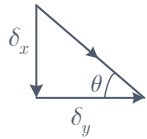
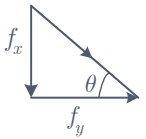
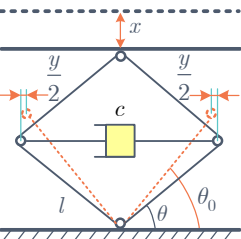
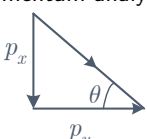
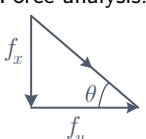
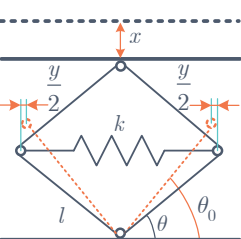
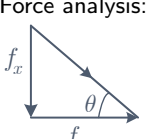
$$\frac{\hat{\delta}(x)}{b} = \frac{\hat{p}(x)}{c} = \frac{\hat{f}(x)}{k} = \hat{y}(x) \tan \hat{\theta}(x). \quad (22)$$

Defining $\eta(x)$ as a coefficient-free CR function for DSMEs and substituting Eqs. (5) and (6) into Eq. (22) yields

$$\eta = \hat{\eta}(x) = \hat{y}(x) \tan \hat{\theta}(x) = 2l \sin \theta_0 - x - \frac{4l^2 \sin \theta_0 \cos \theta_0 - 2xl \cos \theta_0}{\sqrt{4l^2 - (2l \sin \theta_0 - x)^2}}. \quad (23)$$

Table 4

Comparison of through-variable analysis and CRs for DSMEs under the linear and mem-nonlinear modeling paradigms.

DSMEs	Through-variable analysis and CR for mem-nonlinear paradigm		Force analysis and CR for linear paradigm	
	Through-variable analysis	CR	Force analysis	CR
 <p>Mem-inerter</p>	<p>Momentement analysis:</p>  <p>Triangle of Momentements</p>	<p>Linear port CR</p> $\delta_y = by$ <p>Memory port CR</p> $\begin{aligned}\delta_x &= \delta_y \tan \theta \\ &= by \tan \theta \\ &= b\hat{y}(x) \tan \hat{\theta}(x) \\ &= b\hat{\eta}(x) \\ &= \hat{\delta}(x)\end{aligned}$	<p>Force analysis:</p>  <p>Triangle of Forces</p>	<p>Linear port CR</p> $f_y = ba_y$ <p>Memory port CR</p> $\begin{aligned}f_x &= f_y \tan \theta \\ &= ba_y \tan \theta \\ &= b\hat{a}_y(x, v_x, a_x) \cdot \tan \hat{\theta}(x) \\ &= \hat{f}(x, v_x, a_x)\end{aligned}$
 <p>Mem-damper</p>	<p>Momentum analysis:</p>  <p>Triangle of Momentums</p>	<p>Linear port CR</p> $p_y = cy$ <p>Memory port CR</p> $\begin{aligned}p_x &= p_y \tan \theta \\ &= cy \tan \theta \\ &= c\hat{y}(x) \tan \hat{\theta}(x) \\ &= c\hat{\eta}(x) \\ &= \hat{p}(x)\end{aligned}$	<p>Force analysis:</p>  <p>Triangle of Forces</p>	<p>Linear port CR</p> $f_y = cv_y$ <p>Memory port CR</p> $\begin{aligned}f_x &= f_y \tan \theta \\ &= cv_y \tan \theta \\ &= c\hat{v}_y(x, v_x) \cdot \tan \hat{\theta}(x) \\ &= \hat{f}(x, v_x)\end{aligned}$
 <p>Nonlinear spring</p>	<p>Force analysis:</p>  <p>Triangle of Forces</p>	<p>Linear port CR</p> $f_y = ky$ <p>Memory port CR</p> $\begin{aligned}f_x &= f_y \tan \theta \\ &= ky \tan \theta \\ &= k\hat{y}(x) \tan \hat{\theta}(x) \\ &= k\hat{\eta}(x) \\ &= \hat{f}(x)\end{aligned}$	<p>Same as the mem-nonlinear paradigm</p>	

Differentiating Eq. (22) with respect to time leads to

$$\dot{\eta} = \frac{d\eta}{dt} = \frac{d\hat{\eta}(x)}{dx} \frac{dx}{dt} = R(x)\dot{x}, \quad (24)$$

where

$$R(x) = \frac{d\hat{\eta}(x)}{dx} = \frac{8l^3 \cos \theta_0}{[4l^2 - (2l \sin \theta_0 - x)^2]^{3/2}} - 1 \quad (25)$$

 is called the coefficient-free mem-impedance at x for the memory port of DSMEs. See Eq. A.1 in Appendix A for the detailed derivation of $R(x)$ and Tab. C.1 in Appendix C for the port governing and gain equations of DSMEs.

3.2.4. Memory element fingerprints

 Consider the same velocity input as Eq. (8) across the memory port of DSMEs, as illustrated by the magenta curve in Fig. 7d for $A = 0.5$ m/s and $\Omega = 2\pi$ rad/s. With an initial angle $\theta_0 = \pi/4$ and an initial displacement $x_0 = x(0) = 0$,

$x(t)$, $\eta(t)$, $\dot{\eta}(t)$ can be calculated using Eqs. (9), (23), (24), respectively. These results are plotted in green and red in Fig. 7c, in blue in Fig. 7d.

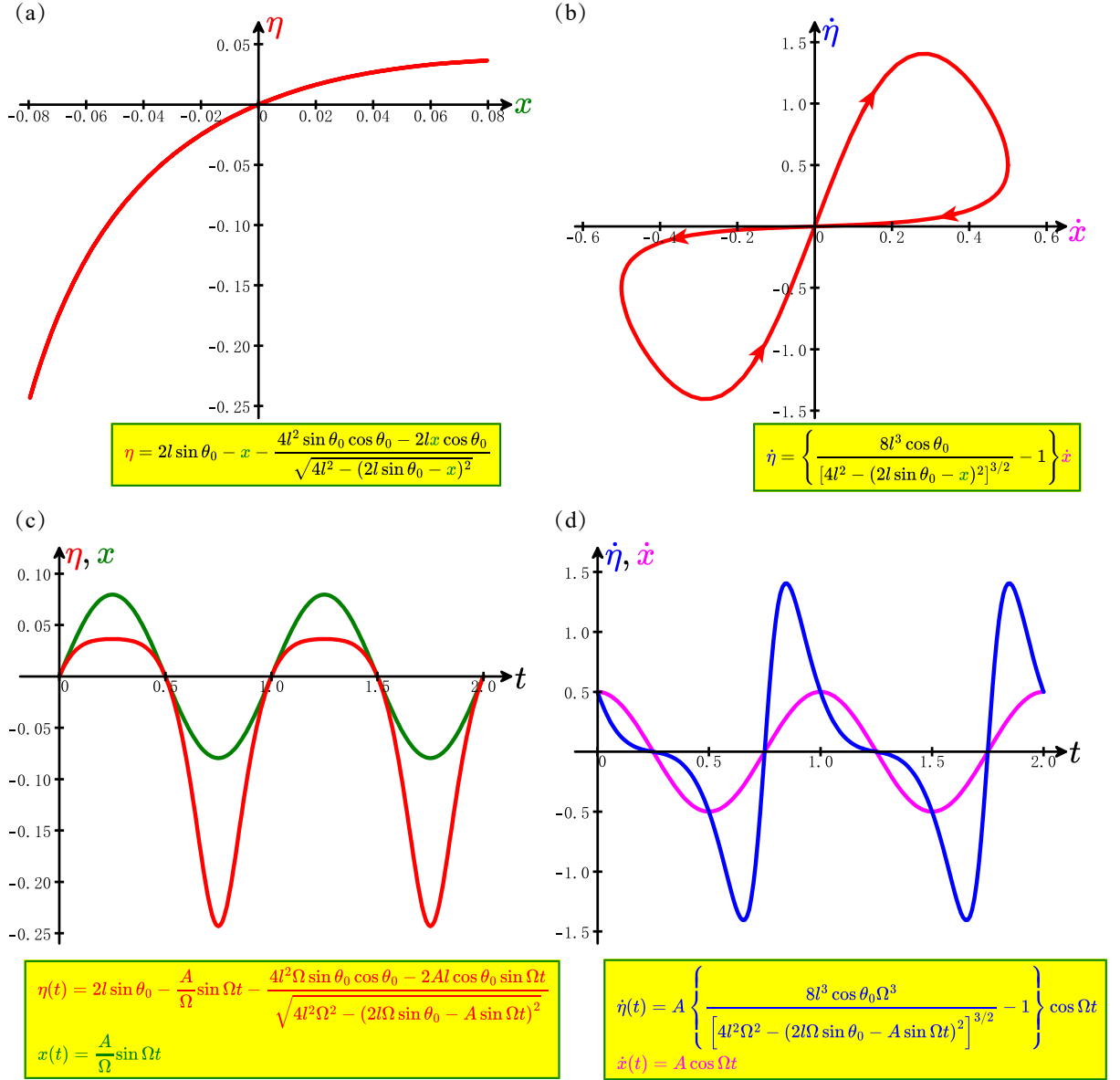


Figure 7: Coefficient-free CR and PHL of DSMEs, together with the associated waveforms, for $A = 0.5$ m/s and $\Omega = 2\pi$ rad/s: (a) CR, $\eta = \hat{\eta}(x)$. (b) PHL in the $\dot{x}(t) - \dot{\eta}(t)$ plane. (c) Periodic waveforms of $\eta(t) = \int_0^t \dot{\eta}(\tau) d\tau$ and $x(t) = \int_0^t \dot{x}(\tau) d\tau$. (d) Periodic waveforms of $\dot{\eta}(t)$ and $\dot{x}(t)$.

Plotting the loci of $(\dot{\eta}(t), \dot{x}(t))$ in the $\dot{\eta} - \dot{x}$ plane yields the PHL shown in Fig. 7b. Observing Fig. 7d, one can find the known Coincident Zero-Crossing signature that the waveform of $\dot{\eta}(t)$ must cross the time axis at the same instants of time whenever the waveform of $\dot{x}(t)$ crosses the time axis. As shown in Fig. 7, both the PHL and the Coincident Zero-Crossing signature confirm that DSMEs (comprising the diamond-shaped mem-inerter, mem-damper, and nonlinear spring) are memory elements. The coefficient-free CR in Eq. (22) further reveals the commonality of these elements.

3.3. Self-consistency analysis of the mem-nonlinear modeling paradigm

3.3.1. Energy, action, and evolution

The potential function and co-function are generally accepted in the electrical domain. By means of the force-current analogy, these functions can be easily translated from the electrical to mechanical domain: Cherry's energy and co-energy for inerters and springs [48]; the content and co-content introduced by Millar for dampers [49]; the action and co-action adopted by Chua for mem-dampers [7]; Biolek's evolution and co-evolution for mem-inerters and mem-springs [47].

The theoretical verification of the mem-nonlinear modeling paradigm via self-consistency analysis necessitates the introduction of the following concepts:

Definition 3.1 The coefficient-free potential function and co-function S_M and $\hat{S}_M$ associated with the memory ports of DSMEs are defined in the CR η - x plane by

$$S_M = \int \eta dx, \quad \hat{S}_M = \int x d\eta. \quad (26)$$

The integrals in Eq. 26 represent the relevant areas enclosed between the CR curve and the coordinate axes, as illustrated in Fig. 8a. They behave as potential functions in the sense that their gradients correspond to quantities η and x :

$$\eta = \frac{dS_M}{dx}, \quad x = \frac{d\hat{S}_M}{d\eta}. \quad (27)$$

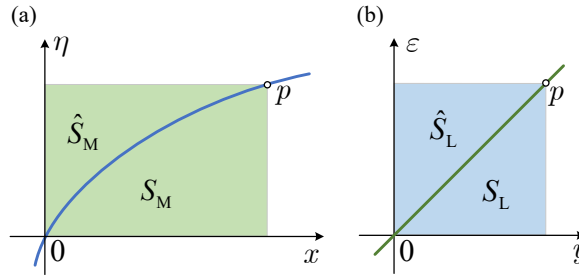


Figure 8: Relationship between the coefficient-free CR and potential function for DSMEs. (a) Memory port. (b) Linear port.

For the linear ports of DSMEs, $\varepsilon(y)$ is defined as a coefficient-free CR function following the form of Eq. (23) and utilizing Eq. (18), which results in

$$\varepsilon = \hat{\varepsilon}(y) = \frac{\delta_y}{b} = \frac{f_y}{k} = \frac{p_y}{c} = y. \quad (28)$$

Definition 3.2 The coefficient-free potential function and co-function S_L , $\hat{S}_L$ associated with the linear ports of DSMEs are defined in the CR ε - y plane by

$$S_L = \int \varepsilon dy = \frac{1}{2}y^2, \quad \hat{S}_L = \int y d\varepsilon = \frac{1}{2}y^2. \quad (29)$$

As depicted in Fig. 8b and Eq. (29), the coefficient-free potential function and co-function for the linear ports are identical, i.e., $S_L = \hat{S}_L$.

From Eq. (26) and the arrangement of potential functions and co-functions in the electrical domain [22], the corresponding functions of DSMEs can be derived, as summarized in Tab. 5.

Definition 3.3 The coefficient-free power functions P_M and P_L associated with the memory and linear ports of DSMEs are defined in the η - v_x and y - v_y planes, respectively, by

$$P_M = \eta v_x, \quad (30)$$

and

$$P_L = y v_y. \quad (31)$$

Table 5

Potential functions and co-functions for DSMEs.

DEMEs	Potential functions	Potential co-functions
Mem-inerter	Evolution: $Y_M = b\hat{S}_M$	Co-evolution: $Y_M^* = bS_M$
Mem-damper	Action: $A_M = c\hat{S}_M$	Co-action: $A_M^* = cS_M$
Nonlinear spring	Energy: $T_M = k\hat{S}_M$	Co-energy: $T_M^* = k\hat{S}_M$

3.3.2. Conservation and balance

Lemma 3.1 *The coefficient-free potential and power functions associated with the memory and linear ports are identical for DSMEs.*

Proof Identical coefficient-free potential function. With the help of Eqs. (22) and Eq. (26), the coefficient-free potential function for the memory ports of DSMEs is derived as

$$S_M = \int_0^x \eta dx = \int_0^x \left(2l \sin \theta_0 - x - \frac{4l^2 \sin \theta_0 \cos \theta_0 - 2l \cos \theta_0 x}{\sqrt{4l^2 - (2l \sin \theta_0 - x)^2}} \right) dx. \quad (32)$$

Let $u = l \sin \theta + \frac{x}{2}$. Applying integration by substitution yields

$$\begin{aligned} S_M &= \int_{2l \sin \theta_0}^{2l \sin \theta_0 - x} \left(u - \frac{2l \cos \theta_0 u}{\sqrt{4l^2 - u^2}} \right) du \\ &= \left(-\frac{1}{2}u^2 - 2l \cos \theta_0 \sqrt{4l^2 - u^2} \right) \Big|_{2l \sin \theta_0}^{2l \sin \theta_0 - x} \\ &= 2l^2 \sin^2 \theta_0 + 4l^2 \cos^2 \theta_0 - \frac{1}{2}(2l \sin \theta_0 - x)^2 - 2l \cos \theta_0 \sqrt{4l^2 - (2l \sin \theta_0 - x)^2}. \end{aligned} \quad (33)$$

By substituting Eq. (5) into Eq. (29), the coefficient-free potential function for the linear ports of DSMEs is expressed as

$$\begin{aligned} S_L &= \int_0^y \varepsilon dy \\ &= \frac{1}{2}y^2 \\ &= \frac{1}{2} \left(\sqrt{4l^2 - (2l \sin \theta_0 - x)^2} - 2l \cos \theta_0 \right)^2 \\ &= 2l^2 \sin^2 \theta_0 + 4l^2 \cos^2 \theta_0 - \frac{1}{2}(2l \sin \theta_0 - x)^2 - 2l \cos \theta_0 \sqrt{4l^2 - (2l \sin \theta_0 - x)^2}. \end{aligned} \quad (34)$$

By comparing Eq. (33) and Eq. (34), it follows that the coefficient-free potential functions for the memory and linear ports are identical. $\square$

Identical coefficient-free power function. Multiplying the opposite sides of Eqs. (23) and (7), and eliminating $\tan \theta$ leads to

$$P_M = \eta v_x = y v_y = P_L. \quad (35)$$

Equation (35) demonstrates that the coefficient-free power function for the memory and linear ports is identical for DSMEs. $\square$

Table 6

Consistency of conclusions on DSMEs: content and form.

DEMEs	Conservation	Balance
Coefficient-free DSMEs	Potential function: $S_M = S_L$	Power function: $\eta_x v_x = \eta_y v_y$
Mem-inerter	Co-evolution: $Y_M^* = Y_L^*$	instantaneous action: $\delta_x v_x = \delta_y v_y$
Mem-damper	Co-action: $A_M = A_L$	instantaneous energy: $p_x v_x = p_y v_y$
Nonlinear spring	Energy: $T_M = T_L$	instantaneous power: $f_x v_x = f_y v_y$

Theorem 3.1 *Co-evolution conservation and instantaneous action balance for the two ports of the diamond-shaped mem-inerter.*

Proof Co-evolution conservation. From Lemma 3.1, it follows that $S_M = S_L$. According to Tab. 5 and Eq. (28), the co-evolution for the memory port of the diamond-shaped mem-inerter is derived as

$$Y_M^* = bS_M = bS_L = b \int_0^y \epsilon dy = \int_0^y \delta_y dy = Y_L^*. \quad (36)$$

where Y_L^* denotes the co-evolution for the linear port.

It follows from Eq. (36) that the co-evolution conservation for the two-ports of the mem-inerter. $\square$

Instantaneous action balance. By Lemma 3.1, $P_M = P_L$. Multiplying both sides of Eq. (35) by the inertance coefficient b gives

$$\delta_x v_x = b\eta v_x = bP_M = bP_L = b\eta v_y = \delta_y v_y, \quad (37)$$

where $\delta_x v_x$ and $\delta_y v_y$ represent the instantaneous actions for the memory and linear ports of the mem-inerter, respectively.

This formula indicates that the action input from one port is equal to the action output from the other. Therefore, the instantaneous actions for the two ports of the mem-inerter are balanced. $\square$

Similarly, it can be proven that the following theorems holds.

Theorem 3.2 *Co-action conservation and instantaneous energy balance for the two ports of the diamond-shaped mem-damper.*

Theorem 3.3 *Energy conservation and instantaneous power balance for the two ports of the diamond-shaped nonlinear spring.*

The proofs of Theorems 3.2 and 3.3 follow a procedure analogous to that of Theorem 3.1 and are omitted here for brevity. For the proofs of Theorems 3.2 and 3.3, refer to Theorem 3.1. The conclusions of Lemma 3.1 and the Theorems 3.1–3.3 are summarized in Tab. 6.

As shown in Tab. 6, the conservation equations and the balance equations are fully consistent in content and identical in form. The same applies to the port and gain equations of DSMEs (see Tab. C.1 in Appendix C). Furthermore, the mem-nonlinear modeling paradigm is grounded in the framework of the periodic table of mechanical elements, and the locations of DSMEs in the table and the choice of their variables are therefore well-founded. The lemmas and theorems are mutually compatible and logically consistent. Consequently, the entire modeling and analysis process forms a coherent and logically consistent framework. These observations demonstrate that the mem-nonlinear modeling paradigm proposed in this work is self-consistent.

3.4. Comparison of the linear and mem-nonlinear modeling paradigms

For the nonlinear spring among the DSMEs, there is no difference in modeling between the linear and mem-nonlinear paradigms, since both rely on force analysis. In contrast, the mem-inerter and mem-damper are modeled differently under the mem-nonlinear paradigm, which relies on momentement and momentum analysis, respectively. Therefore, a comparison between the two paradigms is only necessary for the mem-inerter and mem-damper.

Models based on the mem-nonlinear paradigm are expressed by the CRs $\delta_x = \hat{\delta}(x)$ for the mem-inerter and $p_x = \hat{p}(x)$ for the mem-damper (Eqs. (19) and (20)). In contrast, models constructed from the linear paradigm are force-based, represented by $f_x = \hat{f}(x, v_x, a_x)$ for the mem-inerter and $f_x = \hat{f}(x, v_x)$ for the mem-damper (See Tab. 4). To ensure a fair comparison, the force models need to be derived from the CR formulations. The process is as follows.

With the help of Eqs. (A.1) and (A.2) in Appendix A, the second derivative of the coefficient-free CR for DSMEs is derived as

$$\begin{aligned} \ddot{\eta} &= \frac{d\dot{\eta}}{dt} \\ &= \frac{d(R(x)\dot{x})}{dt} \\ &= R(x)\ddot{x} + \dot{R}(x)\dot{x} \\ &= R(x)a_x + \dot{R}(x)v_x \\ &= \left\{ \frac{8l^3 \cos \theta_0}{[4l^2 - (2l \sin \theta_0 - x)^2]^{3/2}} - 1 \right\} a_x - \frac{24l^3 \cos \theta_0 (2l \sin \theta_0 - x)}{[4l^2 - (2l \sin \theta_0 - x)^2]^{5/2}} v_x^2. \end{aligned} \quad (38)$$

where $v_x = \dot{x}$ and $a_x = \ddot{x}$ denote the velocity and acceleration of the memory ports for DSMEs, respectively.

Force model for the mem-inerter. By substituting Eq. (23) into Eq.(19), the CR model of the mem-inerter can be expressed in terms of the coefficient-free CR function $\eta(x)$ as $\delta_x = b\hat{\eta}(x)$. Differentiating this equation twice with respect to time yields the corresponding force model as

$$f_x = \ddot{\delta}_x = b\ddot{\eta} = b\dot{R}(x)v_x + bR(x)a_x. \quad (39)$$

Force model for the mem-damper. Similarly, by substituting Eq. (23) into Eq.(20), the CR model of the mem-damper is also expressed in terms of the coefficient-free CR function as $p_x = c\hat{\eta}(x)$. Differentiating this equation twice with respect to time yields the corresponding force model as

$$f_x = \dot{p}_x = c\dot{\eta} = R(x)\dot{x} = R(x)v_x. \quad (40)$$

Table 7 summarizes the models for the mem-inerter and mem-damper under both modeling paradigms to facilitate comparison. It is evident from this table that the force models under the two paradigms differ significantly in terms of the coefficients of a_x and v_x in their mathematical expressions. This distinction is intuitive because the force models for the mem-nonlinear paradigm are derived from the CR models, which result from the triangular decomposition of momentement and momentum, whereas the force models for the linear paradigm are the direct results of the triangular decomposition of force.

In order to clearly demonstrate the qualitative and quantitative differences between the two types of force models, a sinusoidal displacement excitation $x(t) = A_m \sin \Omega t$ is applied as the input, with an initial displacement of $x(0) = 0$ and an angular frequency of $\Omega = 2\pi$ rad/s. The inertial output forces F_b for the diamond-shaped mem-inerter, with an inertance of $b = 30$ kg in the linear port, are plotted in Fig. 9, and the corresponding quantitative comparisons are presented in Tab. 8 under the sinusoidal displacement excitations with amplitudes of $A_m = 16$ mm, 80 mm, 111 mm. Similarly, the damping output forces F_c for the diamond-shaped mem-damper, with a damping coefficient of $c = 500$ N·m/s in the linear port, are plotted in Fig. (10), and the corresponding quantitative comparisons are presented in Tab. (9).

For the mem-inerter, It is observed from Fig. 9 and Tab. 8 that the peak inertial forces predicted by the two modeling paradigm exhibit only minor differences under the small displacement excitation of 16 mm, with discrepancies of 10.2% and 3.8% in the positive and negative peak values, respectively. These discrepancies increase significantly with increasing excitation amplitude. Under the medium and large displacement excitations of 80 mm and 111 mm, the positive peak inertial forces predicted by the linear modeling paradigm are 45% and 69.5% lower, respectively, than those predicted by the mem-nonlinear modeling paradigm. Similarly, the negative peak inertial forces are 35.5% and 70.2% lower, respectively.

For the mem-damper, it can be seen from Fig. 10 and Tab. 9 that, apart from a slight difference in phase, the damping force curves predicted by the two paradigm are almost coincident under the small displacement excitation of

Table 7

Model comparison under the linear and mem-nonlinear modeling paradigms.

	Mem-nonlinear paradigm		Linear paradigm
	CR models	Force models	Force models
Mem-inerter	$\begin{aligned} \delta_x &= by \tan \theta \\ &= b\dot{\eta}(x) \\ &= b(2l \sin \theta_0 - x) \\ &\quad - \frac{b(4l^2 \sin \theta_0 \cos \theta_0 - 2xl \cos \theta_0)}{\sqrt{4l^2 - (2l \sin \theta_0 - x)^2}} \end{aligned}$ <p>(See Eqs. (19) and (23))</p>	$\begin{aligned} f_x &= \ddot{\delta}_x \\ &= b\ddot{\eta} \\ &= \left\{ \frac{8bl^3 \cos \theta_0}{[4l^2 - (2l \sin \theta_0 - x)^2]^{3/2}} - 1 \right\} a_x \\ &\quad - \frac{24bl^3 \cos \theta_0 (2l \sin \theta_0 - x)}{[4l^2 - (2l \sin \theta_0 - x)^2]^{5/2}} v_x^2 \end{aligned}$ <p>(See Eqs. (38) and (39))</p>	$\begin{aligned} f_x &= ba_y \tan \theta \\ &= \frac{b(2l \sin \theta_0 - x)^2}{4l^2 - (2l \sin \theta_0 - x)^2} a_x \\ &\quad - \frac{4bl^2 (2l \sin \theta_0 - x)}{[4l^2 - (2l \sin \theta_0 - x)^2]^2} v_x^2 \end{aligned}$ <p>(See Eq. (B.2) in Appendix B)</p>
Mem-damper	$\begin{aligned} p_x &= cy \tan \theta \\ &= c\dot{\eta}(x) \\ &= c(2l \sin \theta_0 - x) \\ &\quad - \frac{c(4l^2 \sin \theta_0 \cos \theta_0 - 2xl \cos \theta_0)}{\sqrt{4l^2 - (2l \sin \theta_0 - x)^2}} \end{aligned}$ <p>(See Eqs. (20) and (23))</p>	$\begin{aligned} f_x &= \dot{p}_x \\ &= c\dot{\eta} \\ &= \left\{ \frac{8l^3 \cos \theta_0}{[4l^2 - (2l \sin \theta_0 - x)^2]^{3/2}} - 1 \right\} v_x \end{aligned}$ <p>(See Eqs. (20) and (24))</p>	$\begin{aligned} f_x &= cv_y \tan \theta \\ &= \frac{c(2l \sin \theta_0 - x)^2}{4l^2 - (2l \sin \theta_0 - x)^2} v_x \end{aligned}$ <p>(See Eq. (B.3) in Appendix B)</p>

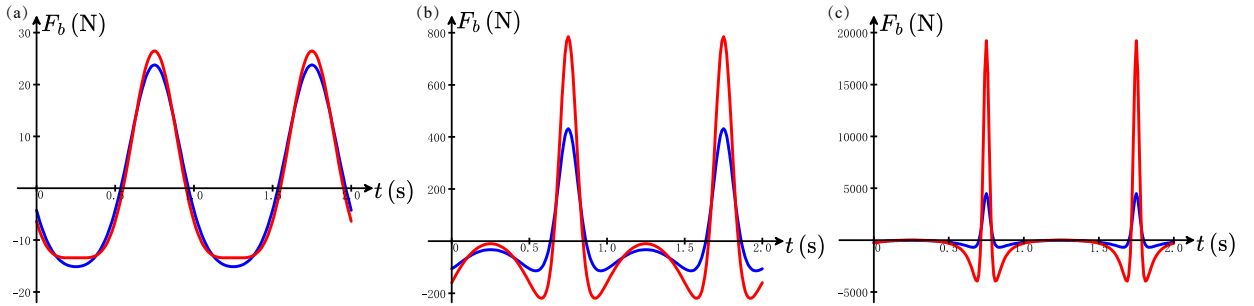


Figure 9: Inertial forces F_b for the diamond-shaped mem-inerter with an inertance of $b = 30$ kg in the linear port under the sinusoidal displacement excitations of with amplitudes of $A_m = 16$ mm, 80 mm, 111 mm, with an initial displacement $x(0) = 0$, angular frequency $\Omega = 2\pi$ rad/s: (a) $A_m = 16$ mm; (b) $A_m = 80$ mm; (c) $A_m = 111$ mm; — Force model for the mem-nonlinear paradigm; — Force model for the linear paradigm.

16 mm. Consequently, the peak-to-peak (*PTP*) values of the damping forces predicted by the two paradigms differ by only 3.8%. Under the medium and large displacement excitations of 80 mm and 111 mm, the *PTP* values of the damping forces predicted by the linear paradigm are 35.3% and 70.2% lower, respectively, than those predicted by the mem-nonlinear paradigm.

From the above comparison, it follows that, under medium and large displacement excitations, the discrepancies between the models based on the two paradigms become significant, with the forces predicted by the linear paradigm being significantly lower than those predicted by the mem-nonlinear paradigm. These results highlight the need for the mem-nonlinear modeling paradigm and its potential advantages over the linear modeling paradigm. The models based on the mem-nonlinear modeling paradigm will be validated experimentally in the next section.

Table 8

Comparison of the inertial force peak values of the models between the linear and mem-nonlinear paradigm.

Amplitudes	Positive peak value (kN)			Negative peak value (kN)		
	Linear paradigm	Mem-nonlinear paradigm	Relative discrepancy	Linear paradigm	Mem-nonlinear paradigm	Relative discrepancy
$A_m = 16\text{mm}$	0.024	0.026	10.2%	0.039	0.040	3.8%
$A_m = 80\text{mm}$	0.432	0.0785	45.0%	0.546	1.005	35.3%
$A_m = 111\text{mm}$	4.492	19.221	69.5%	5.191	23.189	70.2%

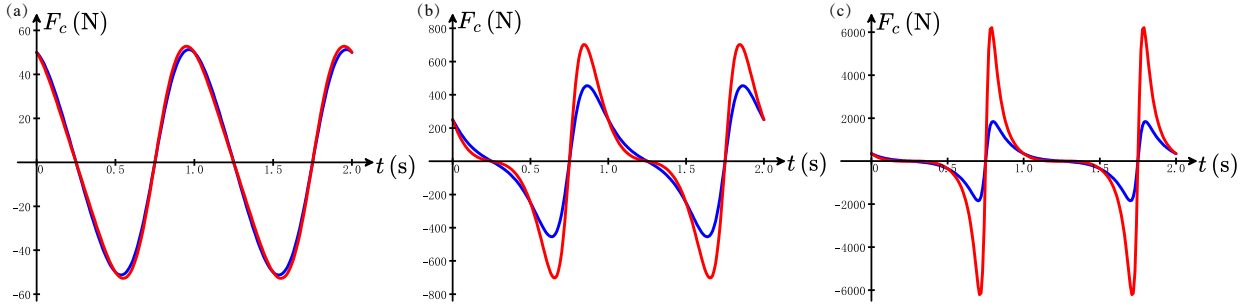


Figure 10: Damping forces F_c for the diamond-shaped mem-damper with a damping coefficient $c = 500\text{N}\cdot\text{m/s}$ in the linear port under the sinusoidal displacement excitations with amplitudes of $A_m = 16\text{ mm}$, 80 mm , 111 mm , with an initial displacement $x(0) = 0$, angular frequency $\Omega = 2\pi\text{ rad/s}$: (a) $A_m = 16\text{ mm}$; (b) $A_m = 80\text{ mm}$; (c) $A_m = 111\text{ mm}$; — Force model for the mem-nonlinear paradigm; — Force model for the linear paradigm.

Table 9Comparison of the damping force PTP values of the models between the linear and memory-nonlinearity paradigm.

Amplitudes	PTP value (kN)		
	Linear paradigm	Mem-nonlinear paradigm	Relative discrepancy
$A_m = 16\text{mm}$	0.102	0.106	3.8%
$A_m = 80\text{mm}$	0.908	1.404	35.3%
$A_m = 111\text{mm}$	3.701	12.422	70.2%

3.5. Experimental validation and evaluation of the mem-nonlinear modeling paradigm

To validate the proposed modeling paradigm, experiments were performed on the diamond-shaped mem-inerter with the rod length $l = 0.2\text{ m}$ illustrated in Fig. 11, and the corresponding inertial force results are given in Fig. 12.

The force model for the mem-nonlinear paradigm exhibits substantially better agreement with the experimental results than the force models for the linear paradigm. As shown in Figs. 12(a)–(c), the model for the mem-nonlinear paradigm accurately captures the main characteristics of the experimental responses, including the force peaks, the fluctuations and local extrema of the force in the valley regions, whereas the model for the linear paradigm tends to underestimate the force peaks. Moreover, it deviates from the experimental local extrema in the valley regions, since it exhibits little fluctuation there, especially under small displacement excitations. At $A=40\text{ mm}$, for instance, no fluctuation is observed in the valleys at all. In addition, it can be observed from Fig. 12(a) to (c) that as the displacement excitation increases, the discrepancy between the model for the linear paradigm and the experimental results becomes more pronounced, whereas the model for the mem-nonlinear paradigm remains closely correlated with the experimental measurements. Therefore, the experimental results demonstrate that the model developed based on the mem-nonlinear paradigm can achieve a more accurate description of the observed force response than the model for the linear paradigm.

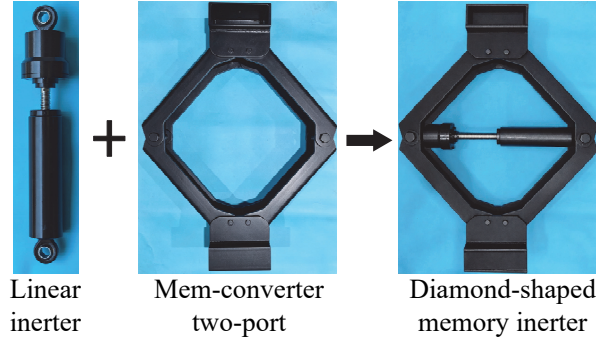


Figure 11: Diamond-shaped mem-inerter device

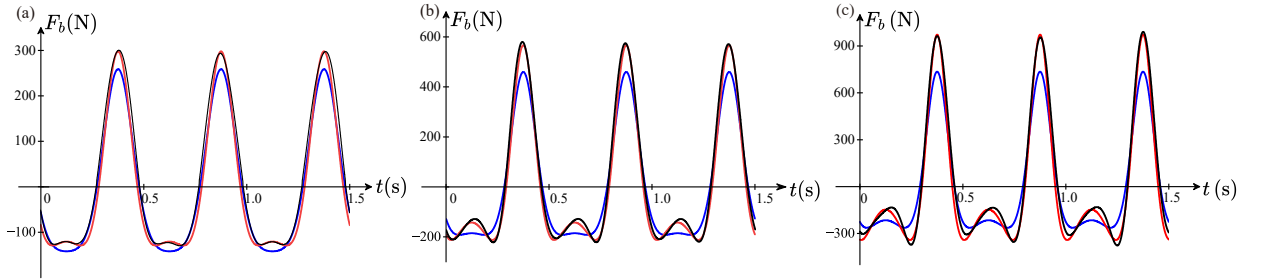


Figure 12: Inertial forces F_b for the diamond-shaped mem-inerter with an inertance $b = 30$ kg in the linear port under the sinusoidal displacement excitations with amplitudes of $A_m = 40$ mm, 60 mm, 80 mm, with an initial displacement $x(0) = 0$, angular frequency $\Omega = 4\pi$ rad/s: (a) $A_m = 40$ mm; (b) $A_m = 60$ mm; (c) $A_m = 80$ mm; — Force model for the mem-nonlinear paradigm; — Force model for the linear paradigm; — Force output for the experiment.

4. Relativistic converter

4.1. Relativistic mass model

4.1.1. CR and PHL for the relativistic mass model

According to [10] and the relativistic mechanics, the relativistic mass of a body can be modeled as an ideal mechanical element described by a constitutive relation between momentum p_m and velocity v , as follows:

$$p_m = \hat{p}_m(v) = m(v)v = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} v, \quad (41)$$

where $m(v) = (m_0 / \sqrt{1 - v^2/c^2})$ is the relativistic mass and m_0 denotes the rest mass.

Differentiating Eq. (41) with respect to time yields

$$F_m = \frac{dp_m}{dt} = \frac{d\hat{p}_m(v)}{dv} \frac{dv}{dt} = m_R(v)\dot{v} = \left(\frac{m_0}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)^3}} \right) a, \quad (42)$$

where

$$m_R(v) = \frac{d\hat{p}_m(v)}{dv} = \frac{m_0}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)^3}}, \quad (43)$$

is called the relativistic mass with memory, from the perspective of memory elements, to distinguish it from $m(v)$. Here, a denotes the acceleration of the body, F_m denotes the applied force, c is the speed of light in vacuum.

Assuming a force F_m acts on the body, its acceleration can be written as

$$\begin{cases} a(t) = A_l \cos \Omega t, & t \geq 0 \\ = 0, & t < 0 \end{cases} \quad (44)$$

where $A_l = 0.1\pi c \text{ m/s}^2$ and $\Omega = 2\pi \text{ rad/s}$.

By plotting Eq. (41) in the $p_m - v$ plane and Eq. (42) in the $F_m - a$ plane with the initial velocity $v = v(0) = 0.9c$, the loci of the two equations are obtained as shown in Fig. 13. Observe that the $p_m - v$ plot exhibits a single-valued curve (Fig. 13a), resulting from the one-to-one correspondence inherent in the relativistic mass CR, while $F_m - a$ Lissajous figure is a PHL (Fig. 13b), which is the fingerprint of memory elements. Consequently, the relativistic mass is identified as a nonlinear element located at $(0, -1)$ in the periodic table of mechanical elements shown in Fig. 6.

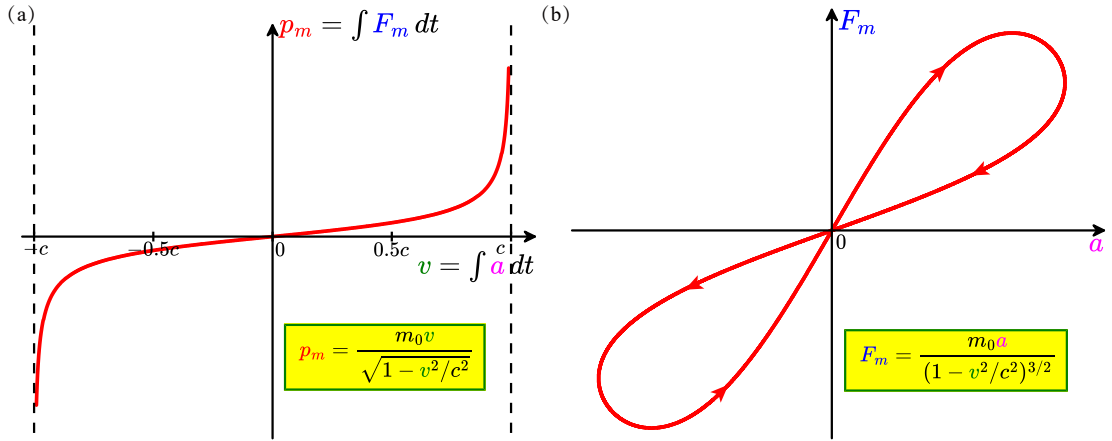


Figure 13: CR and PHL of the relativistic mass for $A_l = 0.1\pi c \text{ m/s}^2$ and $\Omega = 2\pi \text{ rad/s}$, with an initial velocity $0.9c$: (a) CR, $p_m = \hat{p}_m(v)$; (b) PHL in the $F_m - a$ plane.

4.1.2. Two-port model of the relativistic mass

By substituting the Lorentz factor $\gamma = \gamma(v) = (1/\sqrt{1 - v^2/c^2})$ into Eq. (41), the equation can be recast as

$$p_m = \gamma m_0 v = \gamma p = \gamma(v) \hat{p}(v) = \hat{p}_m(v), \quad (45)$$

where $p = \hat{p}(v) = m_0 v$ is a linear function of v .

Following the two-port modeling approach for DSMEs introduced in the previous section, the relativistic mass can be constructed as a two-port model. As shown in Tab. 10, the momentum p and the velocity v constitute the variable pair of the linear port, while the momentum p_m and the velocity v_m constitute the variable pair of the memory port. From comparing this table and Tab. 4, it follows that the momentums p_m and p of the two ports of the element form a through-variable triangle similar to that of DSMEs. In this model, Lorentz factor γ represents the momentum gain from the linear to memory port, while the memory port velocity v_m and the acceleration gain from the memory to linear port must be derived within the mem-nonlinear modeling paradigm.

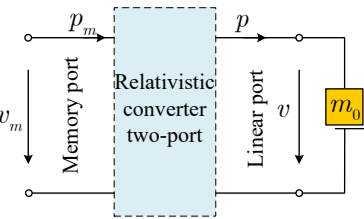
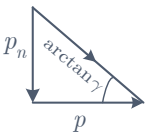
4.1.3. Relativistic port velocity

As established by Theorems 3.1–3.3, the two-port model of DSMEs satisfies conservation laws, including co-evolution conservation for the mem-inerter. Specifically, the co-evolution exiting one port is equal to that entering the other. Similarly, under the mem-nonlinear modeling paradigm, the relativistic mass two-port model satisfies a corresponding conservation law, namely co-energy conservation. Hence, calculating the co-energies of both ports yields

$$\int p_m dv_m = \int p_m \frac{dv_m}{dv} dv = \int p dv, \quad (46)$$

Table 10

Two-port model of the relativistic mass.

Relativistic mass	Momentum analysis	CR	Gain
	 <p>Triangle of Momentums</p>	<p>Linear port CR $p = m_0 v$</p> <p>Memory port CR $p_m = \gamma m_0 v$ $= \gamma p$ $= \gamma(v) \hat{p}(v)$ $= \hat{p}_m(v)$</p>	<p>Momentum gain: γ (from linear to memory port) $p_m = \gamma p$ (See Eq. (45))</p> <p>Acceleration gain: γ (from memory to linear port) $a = \gamma a_m$ (See Eq. (49))</p>

where v_m is a function of v , i.e., $v_m = \hat{v}_m(v)$.

For the above equation to hold, the integrands on both sides must be equal, i.e.,

$$p_m \frac{dv_m}{dv} = p. \quad (47)$$

Substituting Eq. (41) and $p = m_0 v$ into Eq. (47) leads to

$$dv_m = \sqrt{1 - \frac{v^2}{c^2}} dv. \quad (48)$$

Dividing both sides of Eq. (48) by dt gives

$$a_m = a \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{\gamma} a, \text{ or } a = \gamma a_m, \quad (49)$$

where $a_m = dv_m/dt$ and $a = dv/dt$ denote the memory and linear port accelerations, respectively. Here γ represents the acceleration gain from the memory to linear port, as shown in Tab. 10 and Tab. C.1 in Appendix C.

Integrating both sides of Eq. (49) with respect to t yields

$$v_m = \frac{v}{2} \sqrt{1 - \frac{v^2}{c^2}} + \frac{c}{2} \arcsin \frac{v}{c}, \quad (50)$$

where v_m represents the memory port velocity, which is derived within the mem-nonlinear modeling paradigm in this work. As no such derivation has been established previously, the authors formally defined v_m as the relativistic port velocity.

Theorem 4.1 *The relativistic port velocity does not exceed $\pi c/4$.*

Proof Calculating the limit of Eq. (50) as v approaches c gives

$$\lim_{v \rightarrow c} v_m = \lim_{v \rightarrow c} \left(\frac{v}{2} \sqrt{1 - \frac{v^2}{c^2}} \right) + \lim_{v \rightarrow c} \left(\frac{c}{2} \arcsin \frac{v}{c} \right) = \frac{\pi}{4} c. \quad (51)$$

Equation (51) indicates that the upper limit of the relativistic port velocity is $\pi c/4$ when the line port velocity v reaches the maximum, i.e., the speed of light. $\square$

Since no such constant has been previously reported in the literature, $\pi c/4$ is called the relativistic port velocity limit.

Theorem 4.2 *Instantaneous content balance for the two-ports of the relativistic mass.*

Proof Multiplying the opposite sides of the two equations, Eq. (45) and $a = \gamma a_m$ from Eq. (49), and eliminating γ leads to

$$p_m a_m = p a, \quad (52)$$

where $p_m a_m$ and $p a$ represent the instantaneous contents for the memory and linear ports of the relativistic mass, respectively.

This relation implies that the content input from one port is equal to the content output from the other. Therefore, the instantaneous contents for the two ports of the relativistic mass are balanced. $\square$

4.2. Relativistic converter two-port

Based on the two-port model of the relativistic mass illustrated in the first column in Tab. 10, removing the rest mass m_0 and the through variables p_m and p , while focusing on the across variables v_0 and v , yields an ideal two-port model of relativistic converter, as shown in Fig. 14. Such a model is precisely expressed by Eq.(50), which serves as the CR of the relativistic converter.

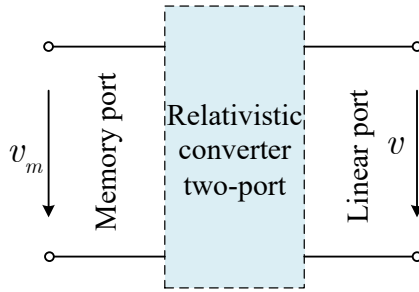


Figure 14: Relativistic converter: a nonlinear motion two-port with memory.

Consider an acceleration input defined by Eq. (44) across the memory port of the converter. By plotting Eq.(50) in the $v_m - v$ plane and Eq. (49) in the $a_m - a$ plane with an initial velocity $v = v(0) = 0.9c$, the locus of the two equations is obtained as shown in Fig. 15. Observe that $v_m - v$ plot exhibits a single-valued curve (Fig. 15a), resulting from the one-to-one correspondence inherent in the relativistic converter CR, while $a_m - a$ Lissajous figure is a PHL (Fig. 15b), which is the fingerprint of memory elements. Consequently, the relativistic converter is identified as a nonlinear motion two-port with memory.

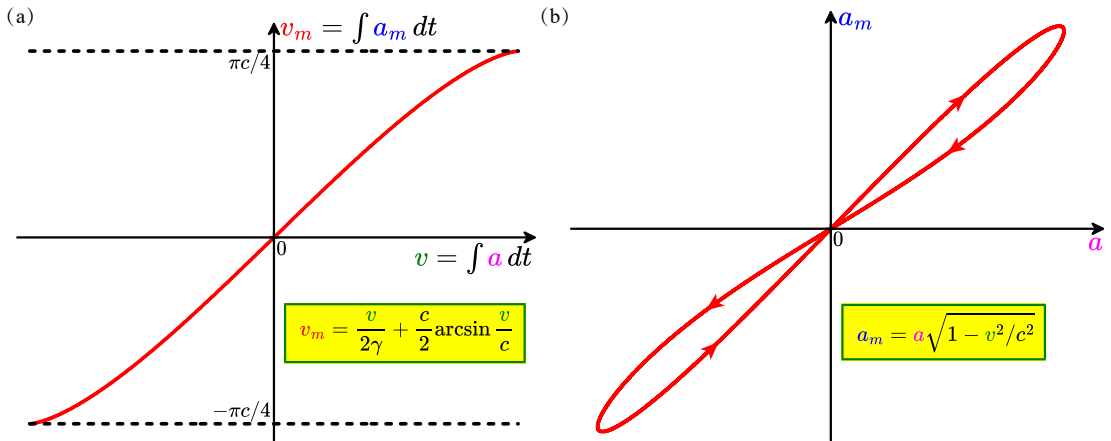


Figure 15: CR and PHL of the relativistic mass converter for $A_l = 0.1\pi c \text{ m/s}^2$ and $\Omega = 2\pi \text{ rad/s}$, with an initial velocity $0.9c$: (a) CR, $v_m = \hat{v}_m(v)$; (b) PHL in the $a_m - a$ plane.

4.3. Defining an ideal motion converter

Extending the relativistic converter to general case yields an ideal motion converter. This converter is a two-port characterized by a CR between two variables v_m and v representing the time integral of the two port accelerations a_m and a , respectively; namely,

$$v_m(t) \triangleq \int_{-\infty}^t a_m(\tau) d\tau, \quad (53)$$

$$v(t) \triangleq \int_{-\infty}^t a(\tau) d\tau. \quad (54)$$

The CR of such a converter can be expressed by

$$v_m = \hat{v}_m(v). \quad (55)$$

Differentiating Eq. (55) with respect to time yields

$$a_m = \frac{dv_m}{dt} = \frac{d\hat{v}_m(v)}{dv} \frac{dv}{dt} = \gamma_r(v)a, \quad (56)$$

where

$$\gamma_r(v) = \frac{d\hat{v}_m(v)}{dv} \quad (57)$$

is called the acceleration gain with memory from the linear to the memory port, except that the gain $\gamma_r(v)$ at any time $t = t_0$ depends on the entire past history of $a(t)$ from $t = -\infty$ to $t = t_0$. If the gain $\gamma_r(v)$ is a constant, the converter degenerates into a linear converter, just as the mem-converter does.

5. From converters and mem-converters to a two-port mechanical-element array

5.1. Two-port mechanical-element array

As demonstrated in Sect. 3, the mem-converter two-port transforms three linear elements into three displacement-dependent memory elements, which occupy adjacent positions along the axis $\alpha = -1$ in the periodic table of mechanical elements (Fig. 6). Conversely, the three adjacent memory elements uniquely map to a mem-converter two-port. This implies the existence of a class of two-port elements where each element corresponds to three adjacent elements along the axis $\alpha = n$ ($n = 0, \pm 1, \pm 2, \dots, \pm N$). Such a family is illustrated as an array in Fig. 16.

In this two-port mechanical-element array, the axes x and y are associated with the variables $v^{(x)}$ and $v^{(y)}$ of the primary and secondary ports, respectively, as shown in Fig. 17. Each coordinate (x, y) represents a $v^{(x)} - v^{(y)}$ two-port element. Hence, the coordinates $(0, 0)$ and $(-1, -1)$ in this table correspond to the converter and mem-converter, respectively.

Since $\Delta_{xy} = y - x$ can be any positive, zero or negative integer, a $v^{(x)} - v^{(y)}$ element will be called a k th-order two-port element if $\Delta_{xy} = \pm k$. Thus, the converter and mem-converter are classified as zeroth-order elements, whereas all other elements in the array are referred to as first-order and higher-order elements.

As a two-port, the $v^{(x)} - v^{(y)}$ element is characterized by a CR between two port variables $v^{(x)}$ and $v^{(y)}$ representing the time integral of the two port variables $v^{(x-1)}$ and $v^{(y-1)}$, respectively; namely,

$$v^{(x)}(t) \triangleq \int_{-\infty}^t v^{(x-1)}(\tau) d\tau, \quad (58)$$

$$v^{(y)}(t) \triangleq \int_{-\infty}^t v^{(y-1)}(\tau) d\tau. \quad (59)$$

The CR of the $v^{(x)} - v^{(y)}$ element is expressed by

$$v^{(x)} = f(v^{(y)}). \quad (60)$$

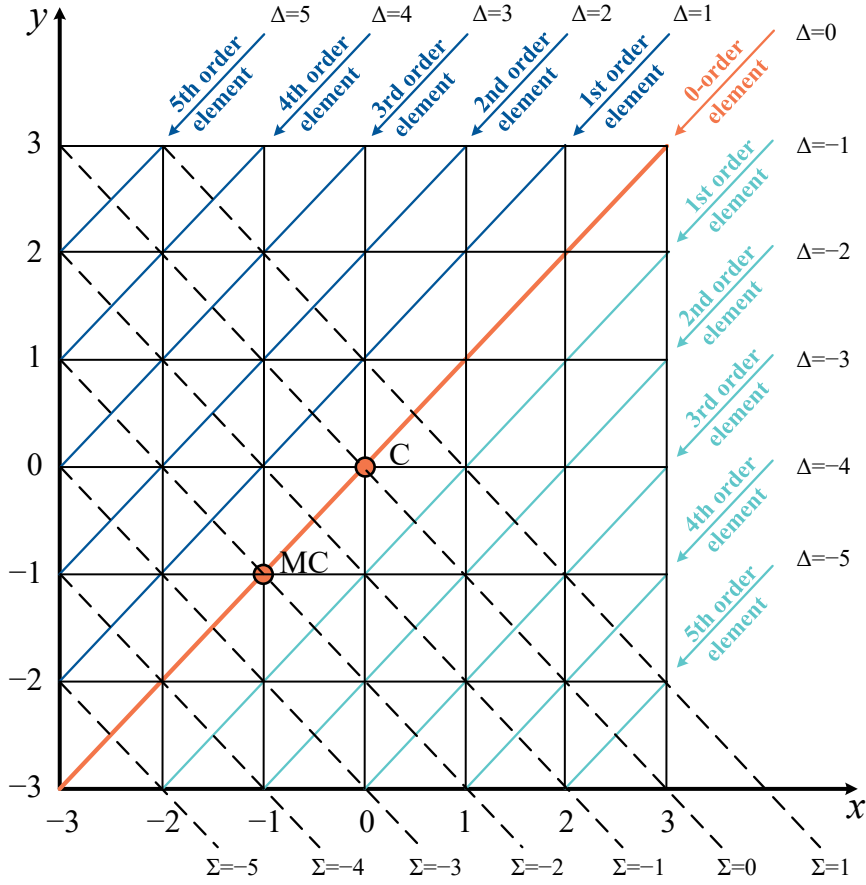


Figure 16: Two-port mechanical-element array: Each of coordinates (x, y) denotes a $v^{(x)} - v^{(y)}$ two-port element. ($\Delta_{xy} = y - x$, $\Sigma_{xy} = y + x$. Axes x and y are associated with port variables $v^{(x)}$ and $v^{(y)}$ of the primary and secondary ports of the two-port element, respectively. The tick mark values $-2, -1, 0, 1, 2, 3$ denote absement, displacement, velocity, acceleration, jerk, jounce, respectively. C and MC denote the converter and the mem-converter, respectively.)

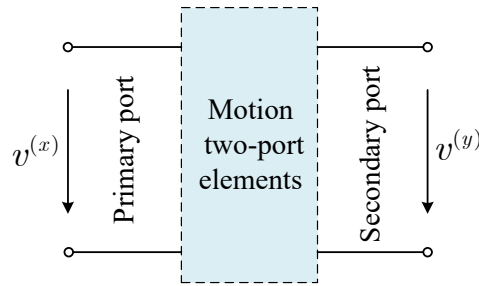


Figure 17: A two-port element $v^{(x)} - v^{(y)}$ defined by the CR between port variables $v^{(x)}$ and $v^{(y)}$.

Differentiating Eq. (60) with respect to time yields

$$v^{(x-1)} = \frac{dv^{(x)}}{dt} = \frac{df(v^{(y)})}{dv^{(y)}} \frac{dv^{(y)}}{dt} = G(v^{(y)})v^{(y-1)}, \quad (61)$$

where

$$G(v^{(y)}) = \frac{df(v^{(y)})}{dv^{(y)}} \quad (62)$$

is called the gain with memory from the secondary to primary port, except that the gain $G(v^{(y)})$ at any time $t = t_0$ depends on the entire past history of $v^{(y-1)}$ from $t = -\infty$ to $t = t_0$. If the gain $G(v^{(y)})$ is a constant, the $v^{(x)} - v^{(y)}$ element degenerates into a linear two-port element. In addition, all linear elements of the same order $k = |\Delta_{xy}|$ corresponding to all coordinates falling on Δ_{xy} diagonal in Fig. 16 are identical to each other.

5.2. From $v^{(x)} - v^{(y)}$ two-ports to $v^{(\alpha)} - v^{(\beta)}$ elements

The case study of the diamond-shaped mem-converter and the relativistic converter demonstrates that the $v^{(x)} - v^{(y)}$ two-port is capable of transforming a linear element into a $v^{(\alpha)} - v^{(\beta)}$ element. Moreover, the CR of the $v^{(x)} - v^{(y)}$ two-port plays a decisive role in determining the across variables of the resulting $v^{(\alpha)} - v^{(\beta)}$ element. Thus, the across variables of the primary and secondary ports of this $v^{(\alpha)} - v^{(\beta)}$ element are chosen as x and y , respectively. Specifically, α -coordinate of the linear element connected to the secondary port corresponds to y , and the primary port α -coordinate corresponds to x .

Then, determining the location of the $v^{(\alpha)} - v^{(\beta)}$ elements in the periodic table of mechanical elements, which are transformed from the linear elements via a $v^{(x)} - v^{(y)}$ two-port in the array, requires a two-step procedure, as shown in Fig. 18.

Step 1: Determine the coordinates of the linear elements connected to the secondary port based on the property that all elements located on the $\Delta_{\alpha\beta}$ diagonal have the same order.

Step 2: Determine the coordinates of the $v^{(\alpha)} - v^{(\beta)}$ elements from the primary port based on the property that the two ports of these elements share identical physical dimension. In the periodic table, this property implies that the linear element and the resulting $v^{(\alpha)} - v^{(\beta)}$ element are located on the same $\Sigma_{\alpha\beta}$ diagonal. Therefore, they share same $\Sigma_{\alpha\beta}$ value, i.e., $\Sigma_{\alpha\beta} = y + x$.

Following step 1, the coordinates of the linear elements connected to the secondary port can be calculated as $(\alpha, \beta) = (\alpha, \Delta_{\alpha\beta} + \alpha)$.

For the linear spring, $\alpha = y$ and $\Delta_{\alpha\beta} = 1$, then

$$(\alpha, \beta) = (y, y + 1).$$

For the linear damper: $\alpha = y$ and $\Delta_{\alpha\beta} = 0$, then

$$(\alpha, \beta) = (y, y).$$

For the linear inerter: $\alpha = y$ and $\Delta_{\alpha\beta} = -1$, then

$$(\alpha, \beta) = (y, y - 1).$$

Following step 2, the coordinates and orders of the $v^{(\alpha)} - v^{(\beta)}$ elements can be calculated as $(\alpha, \beta) = (\alpha, \Sigma_{\alpha\beta} - \alpha)$ and $\Delta_{\alpha\beta} = \beta - \alpha$.

For the $v^{(\alpha)} - v^{(\beta)}$ elements corresponding to the linear spring, $\alpha = x$ and $\Sigma_{\alpha\beta} = 2y + 1$, then

$$(\alpha, \beta) = (x, 2y - x + 1) \text{ and } \Delta_{\alpha\beta} = 2y - 2x + 1.$$

For the elements corresponding to the linear damper, $\alpha = x$ and $\Sigma_{\alpha\beta} = 2y$, then

$$(\alpha, \beta) = (x, 2y - x) \text{ and } \Delta_{\alpha\beta} = 2y - 2x.$$

For the elements corresponding to the linear inerter, $\alpha = x$ and $\Sigma_{\alpha\beta} = 2y - 1$, then

$$(\alpha, \beta) = (x, 2y - x - 1) \text{ and } \Delta_{\alpha\beta} = 2y - 2x - 1.$$

Furthermore, when $\Delta_{xy} = y - x = \pm n$, where $n = \text{even integer}$, the $v^{(\alpha)} - v^{(\beta)}$ elements corresponding to the linear spring, damper, and inerter are classified as stiffness, damping, and inertance elements, respectively. Conversely, when $n = \text{odd integer}$, these elements transform into inertance, negative damping, and stiffness elements, which are fundamentally distinct from the linear counterparts in terms of physical category. Consequently, as the order of the $v^{(x)} - v^{(y)}$ two-port element alternates between odd and even integers, the categories of the resulting $v^{(\alpha)} - v^{(\beta)}$ elements exhibit a period-2 alternation, cycling between the two sets, as illustrated in Fig. 18.

Observe that the $v^{(x)} - v^{(y)}$ two-port elements of order $(y - x)$ correspond to the $v^{(\alpha)} - v^{(\beta)}$ elements of orders $(2y - 2x + 1)$, $(2y - 2x)$, $(2y - 2x - 1)$. Specifically, zeroth-order two-port elements correspond to zeroth and first-order $v^{(\alpha)} - v^{(\beta)}$ elements. Since second-order and higher-order $v^{(\alpha)} - v^{(\beta)}$ elements are active, the $v^{(x)} - v^{(y)}$ elements of first order and higher are active two-ports.

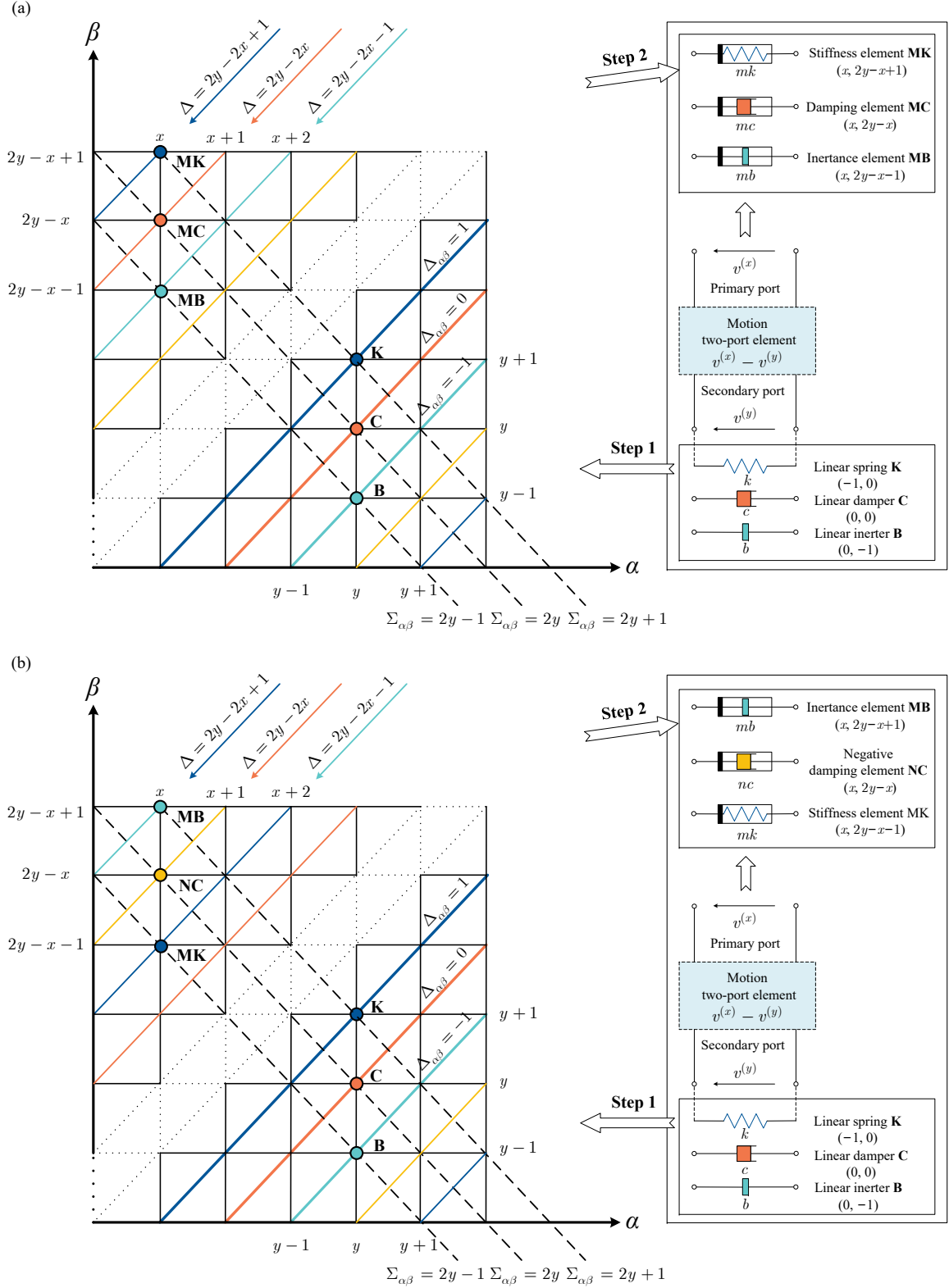


Figure 18: An illustration of transforming linear elements into $v^{(\alpha)} - v^{(\beta)}$ elements via a $v^{(x)} - v^{(y)}$ two-port: (a) For $\Delta_{xy} = y - x = \pm n$, where $n = \text{even integer}$; (b) For $\Delta_{xy} = y - x = \pm n$, where $n = \text{odd integer}$.

6. Conclusions

This work clarifies the root cause of the limitations of linear modeling paradigm through the concept of the mem-converter, using a case study of DSMEs. It finds that the diamond-shaped mechanism is a purely mechanical two-port memory element, termed the motion mem-converter, which is characterized by the CR between its two port displacements and exhibits the PHL in the plane of its port velocities. The mem-converter reveals that displacement (the across variable), rather than force (the through variable), should be the dominant quantity in modeling geometrically nonlinear elements such as DSMEs.

In this paper, a mem-nonlinear modeling paradigm is established within the framework of the periodic table of mechanical elements. A systematic approach for port-variable selection and through-variable analysis of DSMEs is proposed, which breaks free from the force-dominated tradition of variable selection and the long-entrenched practice of force analysis. Under its guidance, the correct CRs and port equations of DSMEs are derived. It has also been proved that, for the two ports of the diamond-shaped mem-inerter, mem-damper, and nonlinear spring, respectively: (i) the conservation of co-evolution, co-action, and energy holds; and (ii) the instantaneous balance of action, energy, and content is satisfied. These equations and conclusions are consistent in content and identical in form. Moreover, the entire modeling and analysis process is logically consistent. This implies that the mem-nonlinear modeling paradigm proposed in this work is self-consistent. Experimental results for the diamond-shaped mem-inerter demonstrate good agreement between the theoretical model and the experimental data, thereby validating the correctness and effectiveness of the proposed modeling paradigm.

Beyond defining the mem-converter, this work also reveals that the relativistic converter is a velocity-dependent nonlinear converter via decomposition of the relativistic mass into the relativistic converter and the rest mass. The relativistic converter has memory property in the sense that it is characterized by the CR between its two port velocities and exhibits the PHL in the plane of its port accelerations.

Finally, based on the variable selection approach developed within the paradigm, the converters and mem-converters are generalized to a motion two-port mechanical-element array, thereby enabling prediction of higher-order two-port elements.

A. Detailed derivation of the coefficient-free mem-impedance $R(x)$

Using eq. (23), $R(x)$ and $\dot{R}(x)$ are given by

$$\begin{aligned}
 R(x) &= \frac{d\hat{\eta}(x)}{dx} \\
 &= \frac{d}{dx} \left\{ 2l \sin \theta_0 - x - \frac{4l^2 \sin \theta_0 \cos \theta_0 - 2xl \cos \theta_0}{\sqrt{4l^2 - (2l \sin \theta_0 - x)^2}} \right\} \\
 &= \frac{2l \cos \theta_0 \sqrt{4l^2 - (2l \sin \theta_0 - x)^2} + (4l^2 \sin \theta_0 \cos \theta_0 - 2xl \cos \theta_0) \frac{2l \sin \theta_0 - x}{\sqrt{4l^2 - (2l \sin \theta_0 - x)^2}}}{4l^2 - (2l \sin \theta_0 - x)^2} - 1 \\
 &= \frac{2l \cos \theta_0 [4l^2 - (2l \sin \theta_0 - x)^2] + (4l^2 \sin \theta_0 \cos \theta_0 - 2xl \cos \theta_0)(2l \sin \theta_0 - x)}{[4l^2 - (2l \sin \theta_0 - x)^2]^{3/2}} - 1 \\
 &= \frac{8l^3 \cos \theta_0}{[4l^2 - (2l \sin \theta_0 - x)^2]^{3/2}} - 1
 \end{aligned} \tag{A.1}$$

and

$$\dot{R}(x) = \frac{dR(x)}{dt} = \frac{dR(x)}{dx} \frac{dx}{dt} = \frac{-24l^3 \cos \theta_0 (2l \sin \theta_0 - x)}{[4l^2 - (2l \sin \theta_0 - x)^2]^{5/2}} \dot{x} \tag{A.2}$$

Table C.1

Port governing equations for DSMEs and relativistic mass.

Elements	Port governing equations	Port gain equations
Coefficient-free DSMEs	$\begin{cases} \eta = y \tan \theta \\ y = \hat{y}(x) \end{cases}$	$\begin{cases} \eta = y \tan \theta \\ v_y = v_x \tan \theta \end{cases}$
Mem-inerter	$\begin{cases} \delta_x = \delta_y \tan \theta \\ y = \hat{y}(x) \end{cases}$	$\begin{cases} \delta_x = \delta_y \tan \theta \\ v_y = v_x \tan \theta \end{cases}$
Mem-damper	$\begin{cases} p_x = p_y \tan \theta \\ y = \hat{y}(x) \end{cases}$	$\begin{cases} p_x = p_y \tan \theta \\ v_y = v_x \tan \theta \end{cases}$
Nonlinear spring	$\begin{cases} f_x = f_y \tan \theta \\ y = \hat{y}(x) \end{cases}$	$\begin{cases} f_x = f_y \tan \theta \\ v_y = v_x \tan \theta \end{cases}$
Relativistic mass	$\begin{cases} p_m = \gamma p \\ v_m = \hat{v}_m(v) \end{cases}$	$\begin{cases} p_m = \gamma p \\ a = \gamma a_m \end{cases}$

B. DSME Modeling under the linear paradigm

Differentiating both sides of Eq. (7) with respect to time yields

$$a_y = \frac{2l \sin \theta_0 - x}{\sqrt{4l^2 - (2l \sin \theta_0 - x)^2}} a_x - \frac{4l^2}{[4l^2 - (2l \sin \theta_0 - x)^2]^{3/2}} v_x^2, \quad (\text{B.1})$$

where $a_x = dv_x/dt$ and $a_y = dv_y/dt$ denote the accelerations of the memory and linear ports, respectively.

Mem-inerter modeling. By following the linear paradigm in Tab. 4 and using Eq. B.1, the inertial force generated by the mem-inerter can be expressed as

$$f_x = ba_y \tan \theta = \frac{b(2l \sin \theta_0 - x)^2}{4l^2 - (2l \sin \theta_0 - x)^2} a_x - \frac{4bl^2(2l \sin \theta_0 - x)}{[4l^2 - (2l \sin \theta_0 - x)^2]^2} v_x^2. \quad (\text{B.2})$$

Mem-damper modeling. Similarly, with the help of Eq. 7, the damping force generated by the mem-damper can be expressed as

$$f_x = cv_y \tan \theta = \frac{c(2l \sin \theta_0 - x)^2}{4l^2 - (2l \sin \theta_0 - x)^2} v_x. \quad (\text{B.3})$$

C. Port governing equations for DSMEs and relativistic mass

Table C.1 presents the governing equations for the DSMEs and the relativistic mass. Specifically, the first equation represents the through-variable CR of the element, while the second equation represents the across-variable CR associated with the corresponding two-port. The port gain equations are derived from these governing equations and are concise, exhibiting clear physical significance. Overall, the equations in this table are fully consistent in content and identical in form, thereby demonstrating the self-consistency of the proposed mem-nonlinear modeling paradigm.

References

- [1] L. O. Chua, Memristors on ‘edge of chaos’, *Nature Reviews Electrical Engineering* 1 (9) (2024) 614–627.

- [2] The Editors of Nature Reviews Electrical Engineering, From memristor to the edge of chaos, *Nature Reviews Electrical Engineering* 1 (9) (2024) 559–560, editorial.
- [3] L. O. Chua, Homemade us\$10 chua corsage memristor: Use it to make the poor man's biomimetic neurons, *IEEE Electron Devices Magazine* 1 (2) (2023) 10–22.
- [4] J.-M. Ginoux, R. Concas, E. Pugliese, R. Meucci, L. O. Chua, From relaxation to chaotic oscillations: A new paradigm for memristor circuits, *IEEE Transactions on Circuits and Systems I: Regular Papers* 72 (10) (2025) 4123–4134.
- [5] D. Biolek, Z. Biolek, About fingerprints of chua's memristors, *IEEE Circuits and Systems Magazine* 18 (2019) 35–47.
- [6] P. Jin, G. Wang, Y. Liang, H. H. C. Iu, L. O. Chua, Neuromorphic dynamics of chua corsage memristor, *IEEE Transactions on Circuits and Systems I: Regular Papers* 68 (11) (2021) 4419–4432.
- [7] L. Chua, Memristor — the missing circuit element, *IEEE Transactions on circuit theory* 18 (5) (1971) 507–519.
- [8] L. O. Chua, S. M. Kang, Memristive devices and systems, *Proceedings of the IEEE* 64 (2) (1976) 209–223.
- [9] M. Di Ventra, Y. V. Pershin, L. O. Chua, Circuit elements with memory: memristors, memcapacitors, and meminductors, *Proceedings of the IEEE* 97 (10) (2009) 1717–1724.
- [10] L. O. Chua, Nonlinear circuit foundations for nanodevices, i: The four-element torus, *Proceedings of the IEEE* 91 (11) (2003) 1830–1859.
- [11] D. Biolek, Leon chua and his double-eight fingerprint [cas education], *IEEE Circuits and Systems Magazine* 24 (4) (2024) 70–78.
- [12] L. Chua, If it's pinched it's a memristor, *Semiconductor Science and Technology* 29 (10) (2014) 104001.
- [13] D. B. Strukov, G. S. Snider, D. R. Stewart, R. S. Williams, The missing memristor found, *nature* 453 (7191) (2008) 80.
- [14] Z. Jiang, H. Zhao, J. Tang, Y. Lu, Q. Qin, Z. Wang, R. Hu, R. Yu, Y. He, J. Zhang, M. Shi, N. Deng, B. Gao, H. Qian, H. Wu, Strategies of high-accuracy memristor-based analogue computing in memory for artificial intelligence, *Nature Materials* 25 (7) (2026) 1110–1124.
- [15] T.-Y. Wang, J.-L. Meng, Q.-X. Li, Z.-Y. He, H. Zhu, L. Ji, Q.-Q. Sun, L. Chen, D. W. Zhang, Reconfigurable optoelectronic memristor for in-sensor computing applications, *Nano Energy* 89 (2021) 106291.
- [16] Z. Xia, X. Sun, Z. Wang, J. Meng, B. Jin, T. Wang, Low-power memristor for neuromorphic computing: From materials to applications, *Nano-Micro Letters* 17 (2025) 217.
- [17] The Editors of Nature Material, High-precision memristor-based computing, *Nature Materials* 25 (7) (2026) 1069, editorial.
- [18] Z. Liu, C. Gao, J. Yang, Z. Chen, E. Li, J. Li, M. Li, J. Zhang, Memristor devices for next-generation computing: from performance optimization to application-specific co-design, *International Journal of Extreme Manufacturing* 8 (1) (2025) 012004.
- [19] S. Mann, R. Janzen, M. Post, Hydraulicphone design considerations: absement, displacement, and velocity-sensitive music keyboard in which each key is a water jet, in: *Proceedings of the 14th ACM International Conference on Multimedia*, Association for Computing Machinery, New York, NY, USA, 2006, p. 519–528.
- [20] S. Mann, C. Pierce, B. C. Zheng, J. Hernandez, C. Scavuzzo, C. Mann, Integral kinesiology feedback for weight and resistance training, in: *2019 15th International Conference on Signal-Image Technology & Internet-Based Systems SITIS*, 2019, pp. 319–326.
- [21] Z. Biolek, D. Biolek, V. Biolková, Z. Kolka, Lagrangian and hamiltonian formalisms for coupled higher-order elements: theory, modeling, simulation, *Nonlinear Dynamics* 104 (4) (2021) 3547–3560.
- [22] Z. Biolek, D. Biolek, V. Biolková, Euler-Lagrange equations of networks with higher-order elements, *Radioengineering* 26 (2) (2017) 397–405.
- [23] Z. Biolek, V. Biolková, D. Biolek, Z. Kolka, Modeling of the generic memcapacitors using higher-order multi-ports, *Communications in Nonlinear Science and Numerical Simulation* 113 (2022) 106497.
- [24] X.-L. Zhang, C. Geng, J.-M. Nie, Q. Gao, The missing mem-inerter and extended mem-dashpot found, *Nonlinear Dynamics* 101 (2) (2020) 835–856.
- [25] X.-L. Zhang, Y.-L. Chen, J.-M. Nie, W.-A. Zhu, Modeling and characteristic test for a crank-connecting rod mem-inerter device, *Machines* 12 (12) (2024) 938.
- [26] L.-L. Zhang, Z.-X. Huang, X.-L. Zhang, F.-L. Zeng, A generic design method of transforming linear elements into memory elements with the diamond-shaped structures, *Proceedings of the Institution of Mechanical Engineers, Part D: Journal of Automobile Engineering* 240 (2-3) (2025) –.
- [27] D. Jeltsema, A. Dòria-Cerezo, Mechanical memory elements: modeling of systems with position-dependent mass revisited, in: *2010 49th IEEE Conference on Decision and Control (CDC)*, 2010, pp. 3511–3516.
- [28] D. Jeltsema, A. Doria-Cerezo, Port-hamiltonian formulation of systems with memory, *Proceedings of the IEEE* 100 (6) (2011) 1928–1937.
- [29] D. Biolek, Z. Biolek, V. Biolkova, Memristors and other higher-order elements in generalized through-across domain, in: *2016 IEEE International Conference on Electronics, Circuits and Systems (ICECS)*, IEEE, 2016, pp. 604–607.
- [30] J.-S. Pei, J. P. Wright, M. D. Todd, S. F. Masri, F. Gay-Balmaz, Understanding memristors and memcapacitors in engineering mechanics applications, *Nonlinear Dynamics* 80 (1-2) (2015) 457–489.
- [31] J.-S. Pei, Mem-spring models combined with hybrid dynamical system approach to represent material behavior, *Journal of Engineering Mechanics* 144 (12) (2018) 04018115.
- [32] J.-S. Pei, B. Carboni, W. Lacarbonara, Mem-models as building blocks for simulation and identification of hysteretic systems, *Nonlinear Dynamics* 100 (2) (2020) 973–998.
- [33] G. F. Oster, D. M. Auslander, The memristor: A new bond graph element, *Journal of Dynamic Systems, Measurement, and Control* 94 (3) (1972) 249–252.
- [34] X.-L. Zhang, S.-J. Zhang, F.-L. Wang, J.-M. Nie, Understanding the inherently self-adjusting mechanism and load adaptability of the mem-damper, *International Journal of Structural Stability and Dynamics* 26 (05) (2026) 2650024.
- [35] D. Jeltsema, J. M. Scherpen, Multidomain modeling of nonlinear networks and systems, *IEEE Control Systems Magazine* 29 (4) (2009) 28–59.
- [36] J.-M. Nie, X.-B. Liu, X.-R. Huang, X.-L. Zhang, Load adaptivity of the mem-spring and its performance benefits in vehicle suspensions, *Nonlinear Dynamics* 112 (24) (2024) 21645–21666.
- [37] L. Chua, Device modeling via nonlinear circuit elements, *IEEE Transactions on Circuits and Systems* 27 (11) (1980) 1014–1044.

- [38] B. Shi, W. Dai, J. Yang, Performance enhancement of vehicle suspension system with geometrically nonlinear inerter, *Archive of Applied Mechanics* 94 (1) (2024) 39–55.
- [39] B. Shi, W. Dai, J. Yang, Performance analysis of a nonlinear inerter-based vibration isolator with inerter embedded in a linkage mechanism, *Nonlinear Dynamics* 109 (2) (2022) 419–442.
- [40] Y. Wang, H.-X. Li, C. Cheng, H. Ding, L.-Q. Chen, Dynamic performance analysis of a mixed-connected inerter-based quasi-zero stiffness vibration isolator, *Structural Control and Health Monitoring* 27 (10) (2020) e2604.
- [41] J. Bian, X.-h. Zhou, K. Ke, M. C. Yam, Y.-h. Wang, Y. Qiu, A passive vibration isolator with bio-inspired structure and inerter nonlinear effects, *Structural engineering and mechanics: An international journal* 88 (3) (2023) 221–238.
- [42] X. Feng, X. Jing, Human body inspired vibration isolation: beneficial nonlinear stiffness, nonlinear damping & nonlinear inertia, *Mechanical Systems and Signal Processing* 117 (2019) 786–812.
- [43] L. Zhang, S. Xue, R. Zhang, L. Hao, C. Pan, L. Xie, A novel crank inerter with simple realization: Constitutive model, experimental investigation and effectiveness assessment, *Engineering Structures* 262 (2022) 114308.
- [44] C. Leon, Everything you wish to know about memristors but are afraid to ask, *Radioengineering* 24 (2) (2015) 319–368.
- [45] F. Z. Wang, A triangular periodic table of elementary circuit elements, *IEEE Transactions on Circuits and Systems I: Regular Papers* 60 (3) (2013) 616–623.
- [46] F. Z. Wang, Topological electronics: from infinity to six, *Journal of Computational Electronics* 22 (4) (2023) 913–920.
- [47] Z. Bielek, D. Bielek, V. Biolková, Utilization of euler-lagrange equations in circuits with memory elements, *Radioengineering* 25 (4) (2016) 783–789.
- [48] C. Cherry, Cxvii. some general theorems for non-linear systems possessing reactance, *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 42 (333) (1951) 1161–1177.
- [49] W. Millar, Cxvi. some general theorems for non-linear systems possessing resistance, *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 42 (333) (1951) 1150–1160.