Resolving Ambiguity in Multi-objective Linear Programming

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Abstract

We show that for multi-objective linear programming problems, maximizing the Hurwicz criterion when the pessimism index is greater than half, is equivalent to solving a related linear programming problem.

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1. **Introduction:** Multi-objective linear programming is discussed in Luc (1989, 2016) , Jahn (2004) and Luan (2019), the last one being extensions of results on vector optimization to infinite dimensional spaces. Much of the theory of multi-objective linear programming is concerned with general properties of set of efficient points of such problems. An exception in this respect, is the work reported in Lahiri (2026), where in the context of a bi-objective linear programming (which is a special case of multi-objective linear programming) two models are introduced, namely lexicographic and lexicographic egalitarian. For lexicographic bi-objective linear programming, there is an endogenously determined pair of weights, such that an alternative solves the bi-objective linear programming problem if and only if it solves the linear programming problem whose objective function is the weighted sum of the two linear objective functions. For lexicographic egalitarian bi-objective linear programming, there is an endogenously determined pair of weights, that evaluates each alternative by considering the weighted sum of the minimum and the maximum of the values attained by the two objective functions and chooses the alternative that maximizes such an evaluation subject to the linear constraints. As such, the second problem has a non-linear objective function, since given two different alternatives it is quite possible that the minimum value for the first alternative is attained by one objective function, whereas the minimum value for the second alternative is attained by the other objective function. It turns out, that under some mild assumptions that guarantees the requirement that the weight assigned to the minimum value is greater than the weight assigned to the maximum value, the second bi-objective linear programming problem can be reformulated as a linear programming problem.

The purpose of the work reported here is to generalize the result concerning the possibility of formulating a linear programming problem for the second category of bi-objective linear programming problems to the situation where there are possibly more than two linear objective functions. We assume that there are exogenous non-negative weights with the weight assigned to the worst value of the objective functions being strictly greater than the weight assigned to the best value of the objective functions. We show that with linear inequality constraints, there is a one-to-one (non-empty set-valued) correspondence between the set of solutions for the problem related to maximizing such a non-linear objective function and the set of solutions of a related linear programming problem.

Essentially, the kind of problem we are concerned with, arises due to "the informational ambiguity" (as in Ellsberg(1961) or K*ö*nig-Kersting and Trautmann (2016)) that a decision maker faces, when the exact objective function, which may depend on the realization of a future state of nature, is unknown to it, but it knows that the state-dependent objective function belongs to a non-empty finite set of objective functions. Similar informational ambiguity in the context of available alternatives being defined by a budget constraint is discussed in Lahiri (2024).

The criterion that evaluates the array of state-dependent pay-offs as the weighted sum of the minimum pay-off and the maximum pay-off, is due to Hurwicz (1951). In chapter 2 of Resnik (1987), the Hurwicz criterion is referred to as "optimism-pessimism rule" and the rule is discussed there along with some illustrative examples and related possibilities.

The natural question that arises at this point is: Does linearly constrained maximization with respect to lexicographic ordering (see Varian (1978)) over pairs of (minimum, maximum) with first priority being given to the first coordinate lead to a set of solutions that is equal to the set of solutions obtained by constrained maximization using the Hurwicz criterion for some pessimism index, as in the case of bi-objective linear programming? While we do not address this question here, it may be a query worth pursuing in future research.

2. Some General Notations: For natural numbers m, n, let $\mathbb{R}^{m\times n}$ be the set of real-valued m×n matrices. $\mathbb{R}^{m\times 1}$ denotes the set of all real-valued m-dimensional column vectors and $\mathbb{R}^{1\times n}$ denotes the set of all real-valued n-dimensional row vectors.

The transpose of any finite dimensional real-valued vector x is denoted by x^T.

Given $B\epsilon\mathbb{R}^{m\times n}$, $i\epsilon\{1, \ldots, m\}$ and $j\epsilon\{1, \ldots, n\}$, let $B_i \epsilon \mathbb{R}^{1\times n}$ denote the i^{th} row of B and let $B^j \epsilon\mathbb{R}^{m\times 1}$ denote the j^{th} column of B. Thus, for $x\epsilon\mathbb{R}^n$, Bx is the m-dimensional column vector whose i^{th} coordinate is B_ix.

When there is no scope for confusion, we will denote $\mathbb{R}^{m\times 1}$ by $\mathbb{R}^m$.

3. The Context: Given natural numbers m, n, let $(A, b)\epsilon\mathbb{R}^{m\times n} \times \mathbb{R}^m$ and let $X(A, b) = \{x\epsilon\mathbb{R}^n | Ax \leq b\} = \{x\epsilon\mathbb{R}^n | A_ix \leq b_i \text{ for all } i\epsilon\{1, \ldots, m\}\}$.

For a natural number K, let $<p^{(k)}|\ k = 1, \ldots, K>$ be an array in $\mathbb{R}^n$.

The triplet $((A, b), <p^{(k)}|\ k = 1, \ldots, K>)$ is said to be a **multi-objective linear programming problem (MOLP)**.

$x \epsilon X(A, b)$ is said to be **efficient** if *there does not exist* any $y \epsilon X(A, b)$ satisfying $p^{(k)T}y \geq p^{(k)T}x$ for all $k = 1, \ldots, K$, with strict inequality for at least one $k \epsilon \{1, \ldots, K\}$.

In such an MOLP, the decision maker is not sure about the objective function that will be realized although, regardless of which objective function is realized, it prefers a higher value of the objective function. This is clearly a decision-making problem under ambiguity.

4. The Problem: Let $\alpha \epsilon [0, 1]$.

Our main problem is the following optimization problem denoted **OPT-α**: Maximize $\{\alpha \min_{k \epsilon \{1, \ldots, K\}} p^{(k)T}x + (1- \alpha) \max_{k \epsilon \{1, \ldots, K\}} p^{(k)T}x\}$ subject to $x \epsilon X(A, b)$.

The kind of objective function defined in OPT- α is known as the **Hurwicz (1951) criterion**, with α being the **pessimism index**.

5. A First Lemma Towards Formulating OPT-α as a Linear Programming Problem: While the OPT- α is not a linear programming problem, it is possible to solve OPT- α for $\alpha \epsilon (\frac{1}{2}, 1]$ by solving a related linear programming problem.

Let $[K] = \{1, \ldots, K\}^2$. This implies $(j, k) \epsilon [K]$ for all $j, k \epsilon \{1, \ldots, K\}$ and this allows for the possibility of $j = k$.

For $(j, k) \epsilon [K]$, let $X(j, k) = \{x \epsilon X(A, b) | p^{(j)T}x \leq p^{(h)T}x \leq p^{(k)T}x \text{ for all } h \epsilon \{1, \ldots, K\}\}$.

For $(j, k) \epsilon [K]$, $X(j, k) = \{x | Ax \leq b, p^{(j)T}x \leq p^{(h)T}x \leq p^{(k)T}x \text{ for all } h \epsilon \{1, \ldots, K\}\}$.

Thus, if $x \epsilon X(j, k)$, then $\min\{p^{(j)T}x, p^{(k)T}x\} = p^{(j)T}x$ and $\max\{p^{(j)T}x, p^{(k)T}x\} = p^{(k)T}x$.

Let $[K]^+ = \{(j, k) \epsilon [K] | X(j, k) \neq \phi\}$. Clearly $X(A, b) = \bigcup_{(j,k) \epsilon [K]^+} X(j, k)$.

In this section we provide a lemma that is a stepping stone towards formulating OPT-α as a linear programming problem.

Lemma 1: Suppose $\alpha \epsilon (\frac{1}{2}, 1]$ and $(j, k) \epsilon [K]^+$.

$x^* \epsilon \underset{x \epsilon X(j,k)}{\operatorname{argmax}} \{\alpha p^{(j)T}x + (1 - \alpha)p^{(k)T}x\}$ <u>if and only if</u> there exists a real number u^*, such that x^*, u^* solve the following linear programming problem: Maximize $[(2\alpha-1)u + (1-\alpha)(p^{(k)T} + p^{(j)T})x]$ subject to $u \leq p^{(j)T}x$, $x \epsilon X(j, k)$.

Further $u^* = p^{(j)T}x^*$.

Proof: Suppose $x^* \epsilon \underset{x \epsilon X(j,k)}{\operatorname{argmax}} [\alpha p^{(j)T}x + (1 - \alpha)p^{(k)T}x\}]$ and let $u^* = p^{(j)T}x^*$. Thus, $p^{(k)T}x^* = \max\{p^{(j)T}x^*, p^{(k)T}x^*\} = p^{(j)T}x^* + p^{(k)T}x^* - u^*$.

Towards a contradiction suppose there exists $x\epsilon X(j, k)$, $u\epsilon\mathbb{R}$ satisfying $u < p^{(j)T}x$ and $(2\alpha-1)u + (1-\alpha)(p^{(k)T} + p^{(j)T})x > (2\alpha-1)u^* + (1-\alpha)(p^{(k)T} + p^{(j)T})x^*$.

Since, $2\alpha - 1 > 0$, $(2\alpha-1)p^{(j)T}x$,+ $(1-\alpha)(p^{(k)T} + p^{(j)T})x > (2\alpha-1)u + (1-\alpha)(p^{(k)T} + p^{(j)T})x$, so that without loss of generality we may assume $u = p^{(j)T}x$.

Thus, $(2\alpha-1)u + (1-\alpha)(p^{(k)T} + p^{(j)T})x = \alpha u + (1-\alpha)(p^{(k)T}x + p^{(j)T}x- u) = \alpha p^{(j)T}x + (1-\alpha)p^{(k)T}x$.

Since $(2\alpha-1)u^* + (1-\alpha)(p^{(k)T} + p^{(j)T})x^* = \alpha p^{(j)T}x^*+ (1-\alpha)p^{(k)T}x^*$, we get $\alpha p^{(j)T}x+ (1-\alpha)p^{(k)T}x > \alpha p^{(j)T}x^* + (1-\alpha)p^{(k)T}x^*$, contrary to our assumption $x^*\epsilon\underset{x\epsilon X(j,k)}{\operatorname{argmax}}[\alpha p^{(j)T}x + (1-\alpha)p^{(k)T}x\}]$.

Thus, x^*, u^* solve the linear programming problem in the statement of this lemma.

Conversely, suppose x^*, u^* solve: Maximize $[(2\alpha-1)u + (1-\alpha)(p^{(k)T} + p^{(j)T})x]$ subject to $u \leq p^{(j)T}x$, $x\epsilon X(j, k)$, $u\epsilon\mathbb{R}$.

If $u^* < p^{(j)T}x^*$, then since x^*, $p^{(j)T}x^*$ satisfy all the constraints of the linear programming problem and $2\alpha - 1 > 0$, we get $(2\alpha-1)p^{(j)T}x^* + (1-\alpha)(p^{(k)T} + p^{(j)T})x^* > (2\alpha-1)u^* + (1-\alpha)(p^{(k)T} + p^{(j)T})x^*$, contrary to our assumption that x^*, u^* solve the linear programming problem.

Thus, it must be the case that $u^* = p^{(j)T}x^*$.

Thus, $(2\alpha-1)u^* + (1-\alpha)(p^{(k)T} + p^{(j)T})x^* = \alpha p^{(j)T}x^* + (1-\alpha)p^{(k)T}x^*$.

Now suppose there exists x, u satisfying the constraints of the linear programming problem and $(2\alpha-1)u + (1-\alpha)(p^{(k)T} + p^{(j)T})x > (2\alpha-1)u^* + (1-\alpha)(p^{(k)T} + p^{(j)T})x^*$.

Since $u < p^{(j)T}x$ would imply $(2\alpha-1)p^{(j)T}x + (1-\alpha)(p^{(k)T} + p^{(j)T})x > (2\alpha-1)u + (1-\alpha)(p^{(k)T} + p^{(j)T})x$, we may without loss of generality assume $u = p^{(j)T}x$.

Thus, $(2\alpha-1)u + (1-\alpha)(p^{(k)T} + p^{(j)T})x = \alpha p^{(j)T}x + (1-\alpha)p^{(k)T}x > \alpha p^{(j)T}x^* + (1-\alpha)p^{(k)T}x^* = (2\alpha-1)u^* + (1-\alpha)(p^{(k)T} + p^{(j)T})x^*$, leading to $(2\alpha-1)u + (1-\alpha)(p^{(k)T} + p^{(j)T})x > (2\alpha-1)u^* + (1-\alpha)(p^{(k)T} + p^{(j)T})x^*$, contrary to our assumption that x^*, u^* solve the linear programming problem.

Thus, it must be the case that $x^*\epsilon\underset{x\epsilon X(j,k)}{\operatorname{argmax}}\{\alpha p^{(j)T}x + (1-\alpha)p^{(k)T}x\}$.

This proves the lemma. Q.E.D.

Note 1: A special case of lemma 1(i.e., K = 2) is "Note 8" in Lahiri (2026). ▌

Notation 1: For all $\alpha\epsilon(\frac{1}{2}, 1]$ and $(j, k)\epsilon[K]^+$, let $X^*(j, k) = \{x^{(j,k)*}|$ the pair $(x^{(j,k)*}, p^{(j)T}x^{(j,k)*})$ solves the linear programming problem in the statement of lemma 1$\}$.

It is easy to see that $j, k\epsilon\{1, \ldots, m\}$ and $x\epsilon\mathbb{R}^n$: $(2\alpha-1)p^{(j)T}x+ (1-\alpha)(p^{(k)T} + p^{(j)T})x = \alpha p^{(j)T}x+ (1-\alpha)p^{(k)T}x = (\alpha p^{(j)T}+ (1-\alpha)p^{(k)T})x$.▌

6. Linear Programming Formulation for OPT-α: The following lemma connects OPT- α with the kind of problems discussed in lemma 1.

Lemma 2: x^* solves OPT- α if and only if for all $(j^*, k^*) \epsilon [K]^+$ satisfying $j^* \epsilon \underset{h\epsilon\{1,\ldots,K\}}{\operatorname{argmin}} p^{(h)T}x^*$ and $k^* \epsilon \underset{h\epsilon\{1,\ldots,K\}}{\operatorname{argmax}} p^{(h)T}x^*$:

(i) $x^* \epsilon \underset{x\epsilon X(j^*,k^*)}{\operatorname{argmax}} \{\alpha p^{(j^*)T}x + (1-\alpha)p^{(k^*)T}x\}$.

(ii) For all $(j, k) \epsilon [K]$ and $x^{(j,k)*} \epsilon \underset{x\epsilon X(j,k)}{\operatorname{argmax}} \{\alpha p^{(j)T}x + (1-\alpha)p^{(k)T}x\}$ it is the case that $\alpha p^{(j^*)T}x^* + (1-\alpha)p^{(k^*)T}x^* \geq \alpha p^{(j)T}x^{(j,k)*} + (1-\alpha)p^{(k)T}x^{(j,k)*}$.

Proof: (i) Let $x^* \epsilon \underset{x\epsilon X(A,b)}{\operatorname{argmax}} \{\alpha \min_{h\epsilon\{1,\ldots,K\}} p^{(h)T}x + (1-\alpha) \max_{h\epsilon\{1,\ldots,K\}} p^{(h)T}x\}$.

Let $j^* \epsilon \underset{h\epsilon\{1,\ldots,K\}}{\operatorname{argmin}} p^{(h)T}x^*$ and $k^* \epsilon \underset{h\epsilon\{1,\ldots,K\}}{\operatorname{argmax}} p^{(h)T}x^*$.

Thus, $x^* \epsilon X(A,b)$ and $p^{(j^*)T}x^* \leq p^{(h)T}x^* \leq p^{(k^*)T}x^*$. Thus, $x^* \epsilon X(j^*, k^*)$.

Further, $\alpha p^{(j^*)T}x^* + (1-\alpha)p^{(k^*)T}x^* \geq \alpha \min_{h\epsilon\{1,\ldots,K\}} p^{(h)T}x + (1-\alpha) \max_{h\epsilon\{1,\ldots,K\}} p^{(h)T}x$ for all $x \epsilon X(A, b)$ implies $\alpha p^{(j^*)T}x^* + (1-\alpha)p^{(k^*)T}x^* \geq \alpha \min_{h\epsilon\{1,\ldots,K\}} p^{(h)T}x + (1-\alpha) \max_{h\epsilon\{1,\ldots,K\}} p^{(h)T}x$ for all $x \epsilon X(j^*, k^*)$.

$x \epsilon X(j^*, k^*)$ implies $p^{(j^*)T}x \leq p^{(h)T}x \leq p^{(k^*)T}x$ for all $h \epsilon \{1, \ldots, K\}$.

Thus, $x \epsilon X(j^*, k^*)$ implies $p^{(j^*)T}x = \min_{h\epsilon\{1,\ldots,K\}} p^{(h)T}x$ and $p^{(k^*)T}x = \max_{h\epsilon\{1,\ldots,K\}} p^{(h)T}x$.

Thus, $\alpha p^{(j^*)T}x^* + (1-\alpha)p^{(k^*)T}x^* \geq \alpha p^{(j^*)T}x + (1-\alpha)p^{(k^*)T}x$ for all $x \epsilon X(j^*, k^*)$ implies (i) is satisfied.

$x^* \epsilon \underset{x\epsilon X(A,b)}{\operatorname{argmax}} \{\alpha \min_{h\epsilon\{1,\ldots,K\}} p^{(h)T}x + (1-\alpha) \max_{h\epsilon\{1,\ldots,K\}} p^{(h)T}x\}$ implies
$\alpha \min_{h\epsilon\{1,\ldots,K\}} p^{(h)T}x^* + (1-\alpha) \max_{h\epsilon\{1,\ldots,K\}} p^{(h)T}x^* \geq \alpha \min_{h\epsilon\{1,\ldots,K\}} p^{(h)T}x + (1-\alpha) \max_{h\epsilon\{1,\ldots,K\}} p^{(h)T}x$ for all $x \epsilon X(A, b) = \bigcup_{(j,k)\epsilon[K]^+} X(j, k)$.

Thus, for all $(j, k) \epsilon [K]^+$: $\alpha \min_{h\epsilon\{1,\ldots,K\}} p^{(h)T}x^* + (1-\alpha) \max_{h\epsilon\{1,\ldots,K\}} p^{(h)T}x^* \geq \alpha p^{(j)T}x + (1-\alpha)p^{(k)T}x$.

Since, $j^* \epsilon \underset{h\epsilon\{1,\ldots,K\}}{\operatorname{argmin}} p^{(h)T}x^*$ and $k^* \epsilon \underset{h\epsilon\{1,\ldots,K\}}{\operatorname{argmax}} p^{(h)T}x^*$: for all $(j, k) \epsilon [K]^+$ and $x^{(j,k)*} \epsilon \underset{x\epsilon X(j,k)}{\operatorname{argmax}} [\alpha p^{(j)T}x + (1-\alpha)p^{(k)T}x]$ it is the case that $\alpha p^{(j^*)T}x^* + (1-\alpha)p^{(k^*)T}x^* \geq \alpha p^{(j)T}x^{(j,k)*} + (1-\alpha)p^{(k)T}x^{(j,k)*}$.

Thus (ii) is satisfied.

Now suppose x^* satisfies (i) and (ii).

Towards a contradiction suppose there exists xϵX(A, b) such that $\alpha \min_{h\epsilon\{1, ..., K\}} p^{(h)T}x +$ $(1- \alpha) \max_{h\epsilon\{1, ..., K\}} p^{(h)T}x > \alpha \min_{h\epsilon\{1, ..., K\}} p^{(h)T}x^* + (1- \alpha) \max_{h\epsilon\{1, ..., K\}} p^{(h)T}x^*$.

Since $j^*\epsilon \operatorname{argmin}_{h\epsilon\{1, ..., K\}} p^{(h)T}x^*$ and $k^*\epsilon \operatorname{argmax}_{h\epsilon\{1, ..., K\}} p^{(h)T}x^*$ it must be the case that $\min_{h\epsilon\{1, ..., K\}} p^{(h)T}x^* = p^{(j^*)T}x^*$ and $\max_{h\epsilon\{1, ..., K\}} p^{(h)T}x^* = p^{(k^*)T}x^*$.

Thus, $\alpha \min_{h\epsilon\{1, ..., K\}} p^{(h)T}x^* + (1- \alpha) \max_{h\epsilon\{1, ..., K\}} p^{(h)T}x^* = \alpha p^{(j^*)T}x^* + (1-\alpha)p^{(k^*)T}x^*$.

Let $j\epsilon \operatorname{argmin}_{h\epsilon\{1, ..., K\}} p^{(h)T}x$, $k \epsilon \operatorname{argmax}_{h\epsilon\{1, ..., K\}} p^{(h)T}x$, $x^{(j,k)*}\epsilon \operatorname{argmax}_{x\epsilon X(j,k)} [\alpha \min\{p^{(j)T}x, p^{(k)T}x\} + (1 - \alpha)\max\{p^{(j)T}x, p^{(k)T}x\}]$.

Then, $\alpha p^{(j)T}x^{(j,k)*} + (1-\alpha)p^{(k)T}x^{(j,k)*} \geq \alpha p^{(j)T}x + (1-\alpha)p^{(k)T}x > \alpha \min_{h\epsilon\{1, ..., K\}} p^{(h)T}x^*$ $+ (1- \alpha) \max_{h\epsilon\{1, ..., K\}} p^{(h)T}x^* = \alpha p^{(j^*)T}x^* + (1-\alpha)p^{(k^*)T}x^*$.

Thus, $\alpha p^{(j)T}x^{(j,k)*} + (1-\alpha)p^{(k)T}x^{(j,k)*} > \alpha p^{(j^*)T}x^* + (1-\alpha)p^{(k^*)T}x^*$ contrary to the requirement in (ii).

Thus, x^* solves OPT- α. Q.E.D.

The following is our main proposition of this section.

Proposition 1: Suppose $\alpha\epsilon(\frac{1}{2}, 1]$.

x^* solves OPT- α <u>if and only if</u> there exists a real number U^* and for every pair (j, k) $\epsilon[K]^+$ a real number $u^{(j,k)*}$ and an n-dimensional vector $x^{(j,k)*}$ such that:

(i) U^* and the array $<(x^{(j,k)*}, u^{(j,k)*})|$ (j, k)$\epsilon[K]^+>$ solve the linear programming problem: Maximize $2\sum_{(j,k)\epsilon[K]^+} [(2\alpha - 1)u^{(j,k)} + (1-\alpha)(p^{(j)T} + p^{(k)T})x^{(j,k)}]$ - U, subject to $u^{(j,k)} \leq p^{(j)T}x^{(j,k)}$, $x^{(j,k)}\epsilon$ X(j, k), $(2\alpha -1)u^{(j,k)} + (1-\alpha)(p^{(j)T} + p^{(k)T})x^{(j,k)} \leq U$, $u^{(j,k)}\epsilon\mathbb{R}$ for all (j, k)$\epsilon[K]^+$, U$\epsilon\mathbb{R}$.

(ii) $x^*\epsilon\{x^{(j,k)*}|(j,k)\epsilon[K]^+, x^{(j,k)*}\epsilon X^*(j,k)$ and $(\alpha p^{(j)T} + (1-\alpha)p^{(k)T})x^{(j,k)*} = U^*\}$.

Proof: Suppose x^* solves OPT- α.

For (j, k)$\epsilon[K]^+$, let $x^{(j,k)*}\epsilon \operatorname{argmax}_{x\epsilon X(j,k)} \{\alpha p^{(j)T}x^{(j,k)} + (1-\alpha)p^{(k)T}x^{(j,k)}\}$.

By (i) of lemma 1, (j, k)$\epsilon[K]^+$ for $(x^{(j,k)*}, u^{(j,k)*})$ satisfying $u^{(j,k)*} = p^{(j)T}x^{(j,k)*}$ solve: Maximize $[(2\alpha-1)u^{(j,k)} + (1-\alpha)(p^{(k)T} + p^{(j)T})x^{(j,k)}]$ subject to $u^{(j,k)} \leq p^{(j)T}x^{(j,k)}$, $x^{(j,k)}\epsilon$X(j, k). Let $U^* = \max_{(j,k)\epsilon[K]^+} [(2\alpha - 1)u^{(j,k)*} + (1-\alpha)(p^{(k)T} + p^{(j)T})x^{(j,k)*}]$.

Thus, U^* and the array $<(x^{(j,k)*}, u^{(j,k)*})|$ (j, k)$\epsilon[K]^+>$ satisfy all the constraints of the linear programming problem in (i) of this proposition.

Let $(j^0, k^0) \in \underset{(j,k)\in[K]^+}{\operatorname{argmax}} [(2\alpha - 1)u^{(j,k)*} + (1-\alpha)(p^{(k)T} + p^{(j)T})x^{(j,k)*}]$. Thus,
$2\sum_{(j,k)\in[K]^+} [(2\alpha - 1)u^{(j,k)*} + (1-\alpha)(p^{(j)T} + p^{(k)T})x^{(j,k)*}] - U^* =$
$2\sum_{(j,k)\in[K]^+\setminus\{(j^0,k^0)\}} [(2\alpha - 1)u^{(j,k)*} + (1-\alpha)(p^{(j)T} + p^{(k)T})x^{(j,k)*}] + [(2\alpha - 1)u^{(j^0,k^0)*} + (1-\alpha)(p^{(j^0)T} + p^{(k^0)T})x^{(j^0,k^0)*}$.

Towards a contradiction suppose there exists a real number U and for every pair (j, k) $\in [K]^+$ a real number $u^{(j,k)}$ and an n-dimensional vector $x^{(j,k)}$ such that U and the array $\langle (x^{(j,k)}, u^{(j,k)}) | (j, k) \in [K]^+ \rangle$ satisfy all the constraints of the linear programming problem in (i) of this proposition and $2\sum_{(j,k)\in[K]^+} [(2\alpha - 1)u^{(j,k)} + (1-\alpha)(p^{(j)T} + p^{(k)T})x^{(j,k)}] - U > 2\sum_{(j,k)\in[K]^+} [(2\alpha - 1)u^{(j,k)*} + (1-\alpha)(p^{(j)T} + p^{(k)T})x^{(j,k)*}] - U^*$.

It is easy to see that without loss of generality we may assume $U = \max_{(j,k)\in[K]^+} [(2\alpha - 1)u^{(j,k)} + (1-\alpha)(p^{(k)T} + p^{(j)T})x^{(j,k)}]$.

Clearly, $(x^{(j,k)}, u^{(j,k)})$ satisfy $u^{(j,k)} \leq p^{(j)T}x^{(j,k)}$, $x^{(j,k)} \in X(j, k)$

Let $(j^1, k^1) \in \underset{(j,k)\in[K]^+}{\operatorname{argmax}} [(2\alpha - 1)u^{(j,k)} + (1-\alpha)(p^{(k)T} + p^{(j)T})x^{(j,k)}]$.

Thus, $2\sum_{(j,k)\in[K]^+} [(2\alpha - 1)u^{(j,k)} + (1-\alpha)(p^{(j)T} + p^{(k)T})x^{(j,k)}] - U =$
$2\sum_{(j,k)\in[K]^+\setminus\{(j^1,k^1)\}} [(2\alpha - 1)u^{(j,k)} + (1-\alpha)(p^{(j)T} + p^{(k)T})x^{(j,k)}] + [(2\alpha - 1)u^{(j^1,k^1)} + (1-\alpha)(p^{(j^1)T} + p^{(k^1)T})x^{(j^1,k^1)}]$.

Thus, $2\sum_{(j,k)\in[K]^+\setminus\{(j^1,k^1)\}} [(2\alpha - 1)u^{(j,k)} + (1-\alpha)(p^{(j)T} + p^{(k)T})x^{(j,k)}] + [(2\alpha - 1)u^{(j^1,k^1)} + (1-\alpha)(p^{(j^1)T} + p^{(k^1)T})x^{(j^1,k^1)}] > 2\sum_{(j,k)\in[K]^+\setminus\{(j^0,k^0)\}} [(2\alpha - 1)u^{(j,k)*} + (1-\alpha)(p^{(j)T} + p^{(k)T})x^{(j,k)*}] + [(2\alpha - 1)u^{(j^0,k^0)*} + (1-\alpha)(p^{(j^0)T} + p^{(k^0)T})x^{(j^0,k^0)*}$.

Thus, there exists $(j,k) \in [K]^+$ such that $(2\alpha - 1)u^{(j,k)} + (1-\alpha)(p^{(j)T} + p^{(k)T})x^{(j,k)} > (2\alpha - 1)u^{(j,k)*} + (1-\alpha)(p^{(j)T} + p^{(k)T})x^{(j,k)*}$ contrary to the requirement that $(x^{(j,k)*}, u^{(j,k)*})$ satisfying $u^{(j,k)*} = p^{(j)T}x^{(j,k)*}$ solve: Maximize $[(2\alpha\text{-}1)u^{(j,k)} + (1\text{-}\alpha)(p^{(k)T} + p^{(j)T})x^{(j,k)}]$ subject to $u^{(j,k)} \leq p^{(j)T}x^{(j,k)}$, $x^{(j,k)} \in X(j, k)$.

Thus, (i) in the statement of this proposition holds.

By lemma 2, for all $(j^*, k^*) \in [K]^+$ satisfying $j^* \in \underset{h\in\{1, \ldots, K\}}{\operatorname{argmin}} p^{(h)T}x^*$ and $k^* \in \underset{h\in\{1, \ldots, K\}}{\operatorname{argmax}} p^{(h)T}x^*$ it is the case that $x^* \in \underset{x\in X(j^*,k^*)}{\operatorname{argmax}} \{\alpha p^{(j^*)T}x + (1-\alpha)p^{(k^*)T}x\}]$ and for all $(j, k) \in [K]^+$, and and $\alpha p^{(j^*)T}x^* + (1-\alpha)p^{(k^*)T}x^* \geq \alpha p^{(j)T}x^{(j,k)*} + (1-\alpha)p^{(k)T}x^{(j,k)*}$ for all $x^{(j,k)*} \in \underset{x\in X(j,k)}{\operatorname{argmax}} \{\alpha p^{(j)T}x + (1-\alpha)p^{(k)T}x\}$.

Thus, by lemma 1, for all $(j^*, k^*) \in [K]^+$ satisfying $j^* \in \underset{h\in\{1, \ldots, K\}}{\operatorname{argmin}} p^{(h)T}x^*$ and $k^* \in \underset{h\in\{1, \ldots, K\}}{\operatorname{argmax}} p^{(h)T}x^*$ it is the case that $(x^*, p^{(j^*)T}x^*) \in \underset{x\in X(j^*,k^*)}{\operatorname{argmax}} \{(2\alpha - 1)u^{(j^*, k^*)} + (1 -$

$\alpha)(p^{(j^*)T} + p^{(k^*)T})x^{(j^*,k^*)}\}]$. Further, by (ii) of lemma 2, for all (j, k)$\epsilon[K]^+$ it is the case that $\alpha p^{(j^*)T}x^* + (1-\alpha)p^{(k^*)T}x^* \geq \alpha p^{(j)T}x^{(j,k)*} + (1-\alpha)p^{(k)T}x^{(j,k)*}$ for all $x^{(j,k)*} \epsilon \underset{x\epsilon X(j^*,k^*)}{\operatorname{argmax}}\{(2\alpha-1)u^{(j,k)} + (1-\alpha)(p^{(j)T} + p^{(k)T})x^{(j,k)}\}$.

Since $U^* = \max_{(j,k)\epsilon[K]^+}[(2\alpha-1)u^{(j,k)*} + (1-\alpha)(p^{(k)T} + p^{(j)T})x^{(j,k)*}]$ and for all (j, k)$\epsilon[K]^+$, $u^{(j,k)*} = p^{(j)T}x^{(j,k)*}$, it follows that $x^*\epsilon\{x^{(j^*,k^*)}|(j^*,k^*)\epsilon[K]^+, x^{(j^*,k^*)}\epsilon X^*(j^*,k^*)$ and $(\alpha p^{(j^*)T} + (1-\alpha)p^{(k^*)T})x^{(j^*,k^*)} = U^*\}$. Thus (ii) is satisfied.

Now suppose there exists a real number U^* and for every pair (j, k) $\epsilon[K]^+$ a real number $u^{(j,k)*}$ and an n-dimensional vector $x^{(j,k)*}$ such that along with x^* they satisfy (i) and (ii) in the statement of this proposition.

Thus:

(i) U^* and the array $<(x^{(j,k)*}, u^{(j,k)*})|$ (j, k)$\epsilon[K]^+>$ solve the linear programming problem: Maximize $2\sum_{(j,k)\epsilon[K]^+}[(2\alpha-1)u^{(j,k)} + (1-\alpha)(p^{(j)T} + p^{(k)T})x^{(j,k)}]$ - U, subject to $u^{(j,k)} \leq p^{(j)T}x^{(j,k)}$, $x^{(j,k)}\epsilon X(j,k)$, $(2\alpha-1)u^{(j,k)} + (1-\alpha)(p^{(j)T} + p^{(k)T})x^{(j,k)} \leq U$, $u^{(j,k)}\epsilon\mathbb{R}$ for all (j, k)$\epsilon[K]^+$, $U\epsilon\mathbb{R}$,

<u>&</u>

(ii) $x^*\epsilon\{x^{(j,k)}|(j,k)\epsilon[K]^+, x^{(j,k)}\epsilon X^*(j,k)$ and $(\alpha p^{(j)T} + (1-\alpha)p^{(k)T})x^{(j,k)} = U^*\}$.

Since $2\alpha - 1 > 0$, it must be the case that for all (j,k)$\epsilon[K]^+$ and $x^{(j,k)*}\epsilon X^*(j,k)$, $u^{(j,k)*} = p^{(j)T}x^{(j,k)*}$ so that $(2\alpha-1)u^{(j,k)*} + (1-\alpha)(p^{(j)T} + p^{(k)T})x^{(j,k)*} = (\alpha p^{(j)T} + (1-\alpha)p^{(k)T})x^{(j,k)*}$.

Let $(j^*, k^*)\epsilon[K]^+$ such that $x^*\epsilon X^*(j^*,k^*)$ and hence $u^{(j^*,k^*)} = p^{(j^*)T}x^*$.

Towards a contradiction suppose there exists yϵX(A, b) such that $\alpha \min_{h\epsilon\{1,\ldots,K\}} p^{(h)T}y + (1-\alpha)\max_{h\epsilon\{1,\ldots,K\}} p^{(k)T}y > \alpha \min_{h\epsilon\{1,\ldots,K\}} p^{(h)T}x^* + (1-\alpha)\max_{h\epsilon\{1,\ldots,K\}} p^{(k)T}x^*$.

Let $\hat{j} \epsilon \underset{h\epsilon\{1,\ldots,K\}}{\operatorname{argmin}} p^{(h)T}y$ and $\hat{k} \epsilon \underset{h\epsilon\{1,\ldots,K\}}{\operatorname{argmax}} p^{(h)T}y$.

Thus, $x^{(\hat{j},\hat{k})*}\epsilon X^*(\hat{j},\hat{k})$ implies $(\alpha p^{(\hat{j})T} + (1-\alpha)p^{(\hat{k})T})x^{(\hat{j},\hat{k})*} \geq (\alpha p^{(\hat{j})T} + (1-\alpha)p^{(\hat{k})T})y > (\alpha p^{(j^*)T} + (1-\alpha)p^{(j^*)T})x^*$, with $x^*\epsilon\{x^{(j^*,k^*)*}\epsilon X^*(j,k)\}$.

Since $U^* \geq (2\alpha-1)u^{(j,k)*} + (1-\alpha)(p^{(j)T} + p^{(k)T})x^{(j,k)*} = (\alpha p^{(j)T} + (1-\alpha)p^{(k)T})x^{(j,k)*}$ for all (j, k)$\epsilon[K]^+$ and $x^*\epsilon\{x^{(j,k)}|(j,k)\epsilon[K]^+, x^{(j,k)}\epsilon X^*(j,k)$ and $(\alpha p^{(j)T} + (1-\alpha)p^{(k)T})x^{(j,k)} = U^*\}$, it must be the case that $(\alpha p^{(j^*)T} + (1-\alpha)p^{(j^*)T})x^* = U^* \geq (\alpha p^{(\hat{j})T} + (1-\alpha)p^{(\hat{k})T})x^{(\hat{j},\hat{k})*}$, contradicting $(\alpha p^{(\hat{j})T} + (1-\alpha)p^{(\hat{k})T})x^{(\hat{j},\hat{k})*} > (\alpha p^{(j^*)T} + (1-\alpha)p^{(j^*)T})x^*$.

Thus, x^* solves OPT-α. Q. E. D.

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