

Effects of plastic anisotropy on localization in orthotropic materials: new explicit expressions for the orientation of localization bands in flat specimens subjected to uniaxial tension

O. Cazacu^{1,2,*}, J. A. Rodríguez-Martínez²

¹Department of Mechanical and Aerospace Engineering, University of Florida, REEF, 1350 N. Poquito Rd., Shalimar, FL 32579, USA

²Department of Continuum Mechanics and Structural Analysis, University Carlos III of Madrid, Avda. de la Universidad, 30, 28911 Leganés, Madrid, Spain

Abstract. This paper presents a theoretical investigation on the inception of plastic localization bands in specimens taken from orthotropic metallic sheets, and subjected to uniaxial tension. For the first time, it is shown that the orientations of the localization bands can be obtained directly from experimental measurements of the uniaxial tensile flow stresses and Lankford coefficients (r -values) of the metallic sheet. In contrast to isotropic materials, it is shown that localization bands equally inclined with respect to the loading axis develop only for the samples of orientations θ^* corresponding to the extrema of the uniaxial tensile flow stresses in the plane of the sheet. Moreover, the expression for the bands angle depends solely on the Lankford coefficient $r(\theta^*)$. For specimen orientations other than θ^* , the two localization bands have different inclinations with respect to the loading axis. If for a given specimen orientation experimental values are not available, we show that the orientations of the localization bands can be obtained using the theoretical r -values and uniaxial flow stresses calculated using any orthotropic plastic potential. As an example, explicit expressions for the band angles obtained using Hill (1948) and Cazacu (2018) orthotropic plastic potentials are provided, and further applied to two textured sheets: a 2090-T3 Al alloy, and a 99% Al alloy. The results obtained are compared, and the sensitivity of the orientation of the localization bands to the constitutive model used for description of plastic anisotropy is brought to light.

1. Introduction

Localized deformation such as Luders bands or necking in flat specimens subjected to uniaxial tension has been the object of numerous investigations (e.g. Nadai (1931), Tvergaard (1993), Ikeda et al. (2001), Rusinek et al. (2005), Okazawa (2010), N'souglo and Rodríguez-Martínez (2018)). Beginning with the pioneering papers of Hill (1948, 1952), Hutchinson and Miles (1974), Rudnicki and Rice (1975), Hill and Hutchinson (1975), Rice (1976), Hutchinson and Neale (1978) and Christoffersen and Hutchinson (1979), the localization phenomenon is modeled as a bifurcation problem for the velocity gradient (or incremental displacement field).

This incremental non-uniqueness, which occurs even for small strains and deformations, has been amply discussed and investigated for geological materials, which can be considered as isotropic elastic-plastic solids with pressure-dependent yielding. The literature is extensive and, as an example, we mention the seminal studies of Rudnicki and Rice (1975) for the case when yielding

*Corresponding author. Tel. (850) 833-9355 ext 241; Fax: (850) 833-9366.
Email address: cazacu@reef.ufl.edu

is governed by Drucker-Prager yield criterion and non-associated flow rule, and Vermeer and de Borst (1984) for the case when yielding is described by the pressure-dependent Mohr-Coulomb's yield criterion. For isotropic elastic-plastic metallic materials for which yielding is pressure-independent, analytical expressions for the corresponding bifurcations directions were derived for either plane stress or plane strain conditions (see for instance Nielsen and Schreyer (1993)). In particular, assuming associated flow rule, a metallic flat specimen subjected to uniaxial tension will develop localization bands (from now on indistinctly referred to as necking bands) that are equally inclined with respect to the loading axis, and the band angle, β_{iso} , should satisfy:

$$\tan^2(\beta_{iso}) = -\eta_1 / \eta_2, \quad (1)$$

where $\eta_{1,2}$ are the principal values of the plastic strain increment, with $\eta_1 > 0 > \eta_2$. Obviously, since for an isotropic material the principal directions for the stress and plastic strain increment coincide, and under uniaxial tension along an arbitrary $\mathbf{x}$ -direction, $\eta_1 = d\epsilon_{xx}$ and $\eta_2 = d\epsilon_{yy} = -d\epsilon_{xx} / 2$, it follows that irrespective of the expression of the plastic potential describing the plastic behavior of the isotropic material, the necking bands are at

$$\beta_{iso} = \pm 54.76^\circ$$

to the loading axis.

However, metallic sheets are usually anisotropic. Thermo-mechanical processes such as rolling lead to preferential orientations of the constituent grains, i.e. texture, which in turn induces to anisotropy of the mechanical properties in the plastic regime. Modeling of the effects of anisotropy on the onset of plastic deformation has received a lot of attention and very versatile orthotropic yield criteria that are applicable for fully three-dimensional (3-D) loadings have been developed. Examples of such orthotropic 3-D yield criteria include Hill (1948), Cazacu and Barlat (2001, 2003, 2004), Bron and Beson (2004), Barlat et al. (2005), Nixon et al. (2010), etc. The effect of texture on the plastic anisotropy of rolled metallic sheets, and particularly the anisotropy in uniaxial yield stresses and Lankford coefficients (plastic strain ratios or r -values) has been investigated extensively. The consensus is that the aforementioned orthotropic yield criteria describe well the plastic behavior of anisotropic metallic sheets of various crystal structures (for more examples and in-depth discussion, the reader is referred to the recent book of Cazacu et al. (2018)). However, the analysis of localization of deformation in tensile specimens taken from orthotropic rolled sheets has received much less attention from both experimental and theoretical standpoints.

An experimental investigation on the influence of anisotropy on plastic localization under uniaxial tensile loading was reported by Korber and Hoff (1928). These authors tested samples taken from rolled sheets of various metallic materials (Al, Cu, Brass, Iron, Ni). For any given sheet material,

experiments were carried out using specimens taken at different orientations θ with respect to the rolling direction (RD). The authors reported, for each test, the yield stress and the orientation of the localization (necking) bands that developed in the specimens prior to failure. Irrespective of the rolled sheet material tested, it was observed that the directions of necking bands that developed in the specimens were dependent on the sample orientation θ with respect to the rolling direction of the sheet. Furthermore, in contrast to the case of isotropic materials, the orientation of the localization bands did not make an angle of $\sim 55^\circ$ with the loading axis.

A qualitative discussion of the experimental results of Korber and Hoff (1928) was done by Hill (1948) in the same paper where he proposed the extension of the von Mises yield criterion to orthotropy. In that work, Hill also provided explicit expressions for the orientation of the necking bands in terms of the anisotropy coefficients F, G, H, N involved in the proposed orthotropic yield criterion. Let us recall that in the coordinate system Oxyz associated with the rolling direction (RD), transverse direction (TD) and normal direction (ND) to the sheet plane (i.e. the axes of orthotropy of the material), Hill (1948) yield criterion is expressed as:

$$F(\sigma_{yy} - \sigma_{zz})^2 + G(\sigma_{zz} - \sigma_{xx})^2 + H(\sigma_{xx} - \sigma_{yy})^2 + 2L\sigma_{yz}^2 + 2M\sigma_{zx}^2 + 2N\sigma_{xy}^2 = \bar{\sigma}^2, \quad (2)$$

where F, G, H, L, M, N are anisotropy coefficients, and $\bar{\sigma}$ denotes the equivalent stress which is defined such that it coincides with the uniaxial yield stress in the $\mathbf{x}$ -direction. Using Eq. (2) for the yielding description, and associated flow rule for calculation of the plastic strain increments, Hill (1948) showed that necking bands that are equally inclined with respect to the direction of loading develop for the RD specimen (i.e. specimen for which the loading axis is at $\theta = 0^\circ$, see Fig.1), and the TD specimen ($\theta = 90^\circ$, see Fig.1). Specifically, for the RD specimen the localization bands are inclined at

$$\beta^{RD} = \pm \text{atan}\left(\sqrt{(G + H) / H}\right), \quad (3)$$

while for the TD specimen the bands are at:

$$\beta^{TD} = \pm \text{atan}\left(\sqrt{(F + H) / H}\right). \quad (4)$$

Moreover, if $(N - G - 2H)$ and $(N - F - 2H)$ have the same sign, there exists an intermediate orientation θ^* for which localization bands of equal inclination with respect to the direction of loading will develop, and θ^* can be expressed in terms of the anisotropy coefficients as:

$$\tan^2(\theta^*) = (N - G - 2H) / (N - F - 2H) \quad (5)$$

Moreover, using the above equations Hill discussed the numerical values of the necking band angles to be expected for the RD and TD specimens. Using a special case of Hill's (1948)

orthotropic criterion, which involves only three independent coefficients, Steinmann et al. (1994) analyzed localization of deformation in specimens loaded in quasi-static uniaxial tension and plane strain compression along the directions of orthotropy. Very recently, N'souglo et al. (2019) investigated, using finite element simulations, the combined effects of anisotropy and tension-compression asymmetry on the development of localization bands in tensile specimens subjected to dynamic loading. Nixon et al (2010) orthotropic yield criterion was used, with parameters values corresponding to a hexagonal closed-packed Ti plate. It was shown that, irrespective of the imposed loading velocity, for the respective parameters values, there are only three orientations, namely $\theta = 0^\circ$ (RD specimen), $\theta = 90^\circ$ (TD specimen), and $\theta = \theta^* \sim 45^\circ$ for which the localization bands that develop in the specimen are equally inclined to the loading direction. For specimen orientations $\theta < \theta^*$, there are two bands of different inclinations, the band that grows faster making an acute clockwise angle with the loading axis. On the other hand, for specimens with orientations $\theta > \theta^*$, the band that grows faster makes an acute counter-clockwise angle with the loading axis. The same authors performed numerical calculations of localization bands for quasi-static uniaxial tensile loadings using the same yield criterion of Nixon et al. (2010) and Hill's condition for localization (i.e. localization bands in flat metallic specimens are along directions of zero extension). It was shown that the influence of the anisotropy on the localization of deformation under uniaxial tensile loadings is the same irrespective of the loading rate (i.e. reported same trends for quasi-static and dynamic uniaxial loadings).

In this paper, we analyze the effect of anisotropy on the inception of necking bands in flat metallic specimens subjected to quasi-static uniaxial tension. We do not make any assumption on the specific mathematical form of the criterion that describes the plastic behavior of the material. The only requirement to be satisfied by the criterion is that of orthotropic symmetry and insensitivity to hydrostatic pressure. In section 2 it is shown that the band angles to be expected can be inferred solely from the experimental observations of the anisotropy in uniaxial flow stresses $\sigma(\theta)$ and Lankford coefficients $r(\theta)$ of the given metallic sheet. This is an original outcome of this investigation that, to the authors' knowledge, has not been reported before in the literature. Moreover, for the orientations θ with respect to the rolling direction for which the bands are equally inclined, an explicit formula is obtained for the bands angles. For all the other specimen orientations, it is shown that bands of different angles develop. In section 3, we present illustrative examples for orthotropic aluminum sheets. It is shown how the experimental information on flow stresses and Lankford coefficients that is already available in the literature can be used in conjunction with the analytical formulas provided in this paper to explain the band angles that may develop in aluminum flat tensile specimens. Moreover, we discuss the influence that the choice of a given plastic potential used for the description of plastic anisotropy has on the predictions of localization bands in flat tensile specimens. A summary of the main findings of the paper is given in section 4.

2. Theoretical results

We consider plastically orthotropic metallic materials. As already mentioned, we do not make any assumption on the mathematical form of the criterion that describes the plastic behavior of the material. The only restriction is that it is insensitive to hydrostatic pressure and that it is invariant to any orthogonal transformation belonging to the symmetry group of the material. Let us recall that the generators of the symmetry group of an orthotropic material are $R_{\mathbf{e}_x}^\pi$ and $R_{\mathbf{e}_y}^\pi$, i.e. the rotations of 180° with respect to the unit vectors of the directions of orthotropy $\mathbf{x}$ and $\mathbf{y}$, respectively, denoted as $\mathbf{e}_x$ and $\mathbf{e}_y$ (see Coleman and Noll, 1963; Teodosiu, 1982). As already mentioned, for a rolled sheet, the orthotropy axes (also referred to as material axes) are the rolling direction, the transverse direction, and the direction along the thickness of the sheet. The yield condition is thus of the general form:

$$\bar{\sigma} = Y, \quad (6)$$

where $\bar{\sigma}$ is the effective stress associated to the yield criterion, and Y is the yield stress along one of the orthotropic axes, generally the x -axis. The evolution of plastic strain is given by an associated flow rule. To simplify the writing, we denote the plastic strain increment as $d\boldsymbol{\varepsilon}$, so

$$d\boldsymbol{\varepsilon} = d\lambda \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}}, \quad (7)$$

where $d\lambda \geq 0$ denotes the plastic multiplier.

Consider a flat tensile specimen cut along a direction $\mathbf{x}'$ at an angle θ to the x -axis in the plane of the sheet, i.e. $\mathbf{x}' = (\cos\theta)\mathbf{e}_x + (\sin\theta)\mathbf{e}_y$, and let us denote by $Ox'y'z'$ the loading frame (see Fig.1).

With respect to the material axes $Oxyz$, the non-zero stress components are

$$\begin{aligned} \sigma_{xx} &= \sigma'_{xx} \cos^2 \theta \\ \sigma_{yy} &= \sigma'_{xx} \sin^2 \theta \\ \sigma_{xy} &= \sigma_{yx} = \sigma'_{xx} \sin \theta \cos \theta \end{aligned} \quad (8)$$

Let us denote $\sigma(\theta)$ the flow stress in uniaxial tension along the direction $\mathbf{x}'$ (see Fig.1). The anisotropy in flow stresses according to any given criterion, i.e. the dependence of $\sigma(\theta)$ on the loading axis orientation θ , is obtained by substituting Eq. (8) in Eq. (6).

The ratio between the in-plane transverse and through-thickness plastic strain increments that develop in the specimen, the so-called Lankford coefficient, is given by:

$$r(\theta) = \frac{d\varepsilon'_{yy}}{d\varepsilon'_{zz}} = - \frac{\sin^2 \theta d\varepsilon_{xx} - \sin(2\theta) d\varepsilon_{xy} + \cos^2 \theta d\varepsilon_{yy}}{d\varepsilon_{xx} + d\varepsilon_{yy}} \quad (9)$$

Proposition 1:

Consider a flat specimen cut at an angle θ to the x -axis in the plane of the sheet. When subjected to uniaxial tension in the plastic regime, two necking bands develop. Their orientations with respect to the loading axis are the solutions of the quadratic algebraic equation:

$$A \left(u^2 - \frac{1+r(\theta)}{r(\theta)} \right) + 2Bu = 0, \quad (10)$$

with $u = \tan \beta$

$$A = (\sin^2 \theta) \frac{\partial \bar{\sigma}}{\partial \sigma_{xx}} - \sin(2\theta) \frac{\partial \bar{\sigma}}{\partial \sigma_{xy}} + (\cos^2 \theta) \frac{\partial \bar{\sigma}}{\partial \sigma_{yy}} \quad (11)$$

$$B = \sin \theta \cos \theta \left(\frac{\partial \bar{\sigma}}{\partial \sigma_{yy}} - \frac{\partial \bar{\sigma}}{\partial \sigma_{xx}} \right) + \cos(2\theta) \left(\frac{\partial \bar{\sigma}}{\partial \sigma_{xy}} \right) \quad (12)$$

Proof: Indeed, based on the classical bifurcation criterion (e.g. see Rudnicki and Rice (1975)) a necessary condition for the existence of bifurcation is that the characteristic tangent stiffness tensor becomes singular. For uniaxial tension, the bands directions correspond to the directions of zero extension. Let us denote by ξ the angle between the direction of zero extension, Ox'' , and Ox . Therefore,

$$d\varepsilon''_{xx} = d\varepsilon_{xx} \cos^2 \xi + d\varepsilon_{yy} \sin^2 \xi + 2d\varepsilon_{xy} \sin \xi \cos \xi = 0 \quad (13)$$

Because the Lankford coefficient in the x -axis (i.e. $r(0^\circ)$) is always non-zero, $\xi \neq 90^\circ$, and Eq. (13) can be rewritten as:

$$d\varepsilon_{yy} \tan^2 \xi + 2d\varepsilon_{xy} \tan \xi + d\varepsilon_{xx} = 0$$

Making use of the trigonometric identity,

$$\tan(\theta + \beta) = \frac{\tan \theta + \tan \beta}{1 - \tan \theta \tan \beta},$$

the flow rule (see Eq.(7)), and the definition of the Lankford coefficient $r(\theta)$ (see Eq. (9)), we arrive at the quadratic algebraic equation for $u = \tan \beta$ given by Eq. (10).

Remark 1:

It is important to note that in Eq. (10) the coefficient of u^2 is always negative, i.e. $A < 0$. Indeed, using the transformation rules from the material axes to the loading axes, it follows that the plastic strain increment in the y' -direction is:

$$d\varepsilon'_{yy} = \sin^2 \theta d\varepsilon_{xx} - \sin(2\theta) d\varepsilon_{xy} + \cos^2 \theta d\varepsilon_{yy}.$$

Because the coefficient A is proportional to the plastic strain increment in the transverse y' -direction (see Eq. (11)), it is necessarily negative.

Remark 2:

It will be demonstrated later on that the coefficient B in Eq. (10) is zero only for the orientations θ^* which correspond to the minimum or maximum of the flow stress in uniaxial tension (i.e. minimum or maximum of the function $\sigma(\theta)$). Moreover, irrespective of the yield criterion used for description of plastic anisotropy, the following results hold true:

Proposition 2:

For orthotropic materials subject to uniaxial tension, for the orientations θ^* which correspond to a minimum or maximum of the function $\sigma(\theta)$, the necking bands are equally inclined to the loading axis at an angle β equal to:

$$\beta = \pm \text{atan} \left(\sqrt{(1+r(\theta^*)) / r(\theta^*)} \right) \quad (14)$$

Proof: The orientations θ^* which correspond to the extrema in the variation of the flow stress in uniaxial tension $\sigma(\theta)$ with the orientation θ satisfy

$$\frac{\partial \bar{\sigma}}{\partial \theta} = 0.$$

Using the chain rule,

$$\frac{\partial \bar{\sigma}}{\partial \theta} = \left(\frac{\partial \bar{\sigma}}{\partial \sigma_{xx}} \right) \left(\frac{\partial \sigma_{xx}}{\partial \theta} \right) + \left(\frac{\partial \bar{\sigma}}{\partial \sigma_{yy}} \right) \left(\frac{\partial \sigma_{yy}}{\partial \theta} \right) + 2 \left(\frac{\partial \bar{\sigma}}{\partial \sigma_{xy}} \right) \left(\frac{\partial \sigma_{xy}}{\partial \theta} \right) \quad (15)$$

and Eq. (8), we obtain that the extrema correspond to the orientations θ for which:

$$2 \sin \theta \cos \theta \left(\frac{\partial \bar{\sigma}}{\partial \sigma_{yy}} - \frac{\partial \bar{\sigma}}{\partial \sigma_{xx}} \right) + 2 \cos(2\theta) \left(\frac{\partial \bar{\sigma}}{\partial \sigma_{xy}} \right) = 0. \quad (16)$$

i.e. $B=0$ (see Eq.(12)). Therefore, for these specimen orientations Eq. (10) has two roots of the same absolute value, $u_{1,2} = \pm \sqrt{(1+r(\theta^*))}/r(\theta^*)$, so the bands are equally inclined with respect to the loading axis, the respective band angles being given by Eq.(14).

Remark 3:

It is also worth noting the novelty and implications of these general results. The novel finding is that the bands orientations can be obtained directly from the experimental measurements without making use of any specific model. Namely, based on the experimentally observed in-plane anisotropy of the uniaxial tensile flow stresses of a sheet material, one can determine which are the loading directions for which two necking bands that are equally inclined with respect to the loading axis will develop. Furthermore, the respective band angles can be easily calculated using the experimental r -values in conjunction with Eq.(14). Moreover, if $r(\theta^*) < 1$, the absolute value of the bands angles of the specimen of orientation θ^* are greater than 54.76° , and vice-versa if $r(\theta^*) > 1$. Obviously, for an isotropic material $r(\theta) = 1$ irrespective of the loading direction θ , and the analytical formula given by Eq. (14) reduces to Eq. (1). Therefore, the corresponding band angles are equal to: $\beta_{iso} = \pm 54.76^\circ$.

Remark 4:

It is also worth verifying that if the plastic anisotropy of the material is described by Hill's (1948) criterion (see Eq. (2)), Proposition 2 leads to the expressions of the bands angles obtained by Hill (see Eq.(3)- (5)).

Indeed, according to the orthotropic yield criterion of Hill (1948) (see Eq.(2)), in the (RD,TD) plane the variation of the normalized uniaxial flow stress $\sigma(\theta)/\bar{\sigma} = \sigma(\theta)/\sigma(0)$ is given by:

$$\sigma(\theta)/\sigma(0) = \frac{1}{\sqrt{F \sin^4 \theta + G \cos^4 \theta + H (\cos^2 \theta - \sin^2 \theta)^2 + 2N \sin^2 \theta \cos^2 \theta}}.$$

Using the above formula, it follows that irrespective of the values of the coefficients F , G , H , and N , extrema of $\sigma(\theta)$ vs. (θ) correspond to: $\theta = 0^\circ$ and $\theta = 90^\circ$. If $(N - G - 2H)$ and $(N - F - 2H)$ have the same sign, there exists an additional minimum or maximum which corresponds to the orientation:

$$\theta^* = \text{atan}\left(\sqrt{(N - G - 2H)/(N - F - 2H)}\right)$$

i.e. we recover Eq. (5). Moreover, according to Hill (1948) yield criterion and associated flow rule, the expression for the variation of the Lankford coefficients with the loading orientation is:

$$r(\theta) = \frac{(2N - G - F)\sin^2 \theta \cos^2 \theta + H \cos^2(2\theta)}{F \sin^2 \theta + G \cos^2 \theta}. \quad (17)$$

It follows that:

- For the RD specimen ($\theta = 0^\circ$), $r(0^\circ) = H / G$, so further substitution in Eq. (14) leads to the expression given by Eq. (3).
- For the specimen of orientation θ^* , given by Eq.(5), the bands angles are

$$\beta^* = \pm \text{atan} \sqrt{\frac{4(F + H)\tan^4(\theta^*) + (3F + 3G + 2N)\tan^2(\theta^*) + 4(G - H)}{(2N - G - F)\tan^2(\theta^*) + 4H(\tan^4(\theta^*) - 1)}} \quad (18)$$

- For the TD specimen ($\theta = 90^\circ$), $r(90^\circ) = H / F$, and the respective band angles are:
 $\beta^{TD} = \pm \text{atan}(\sqrt{(F + H) / H})$, i.e. we recover the expression given by Eq. (4).

In summary, for the case when the plastic behavior is described by the orthotropic yield criterion of Hill (1948), we recover the expressions of the bands angles for the RD and TD specimens in terms of the anisotropy coefficients F, G, H that were derived by this author.

Remark 5:

It is also important to note that irrespective of the model used to describe the plastic anisotropy of the material, the only orientations for which the bands are equally inclined are also the orientations for which the principal directions of stress and plastic strain increment coincide.

Indeed, for uniaxial tension along a direction θ with respect to the $\mathbf{x}$ -direction:

$$\begin{aligned} d\varepsilon'_{xx} &= d\varepsilon_{xx} \cos^2 \theta + d\varepsilon_{yy} \sin^2 \theta + \sin(2\theta) d\varepsilon_{xy} \\ d\varepsilon'_{yy} &= d\varepsilon_{xx} \sin^2 \theta + d\varepsilon_{yy} \cos^2 \theta - \sin(2\theta) d\varepsilon_{xy} \\ d\varepsilon'_{xy} &= \frac{1}{2}(d\varepsilon_{yy} - d\varepsilon_{xx})\sin(2\theta) + d\varepsilon_{xy} \cos(2\theta) \end{aligned} \quad (19)$$

Therefore, the loading directions θ for which the principal axes of stress and plastic strain increments coincide correspond to:

$$(d\varepsilon_{yy} - d\varepsilon_{xx})\sin(2\theta) + 2(d\varepsilon_{xy})\cos(2\theta) = 0 \quad (20)$$

Using the flow rule (see Eq. (7)), it follows that Eq. (16) coincides with Eq. (20).

Proposition 3:

For all other specimen orientations than those corresponding to the extrema of $\sigma(\theta)$, the two bands have different inclinations, say $\beta_1 > 0$ and $\beta_2 < 0$ with $\beta_1 \neq |\beta_2|$.

- If the $\sigma(\theta)$ vs. θ variation admits one absolute minimum $0^\circ < \theta_1^* < 90^\circ$: for specimen orientations $\theta < \theta_1^*$ the inclination of the band which makes an acute counter-clockwise angle with the loading direction is greater than the inclination of the band which makes an acute clockwise angle with the loading direction, i.e. $\beta_1 > |\beta_2|$ (see Fig. 1). For specimen orientations $\theta > \theta_1^*$, the reverse occurs, i.e. $\beta_1 < |\beta_2|$.
- If the $\sigma(\theta)$ vs. θ variation admits one absolute maximum $0^\circ < \theta_2^* < 90^\circ$: for $\theta < \theta_2^*$ we have that $\beta_1 < |\beta_2|$ and vice-versa for $\theta > \theta_2^*$.

Proof: The product of the roots of Eq. (10) is equal to $-\left(\frac{1+r(\theta)}{r(\theta)}\right)$. Therefore, it is always negative. It follows that the band angles should be opposite in sign. Since $A < 0$, the sign of the sum of the roots is given by the sign of B . The sign of B is the opposite of the sign of $\partial\sigma(\theta)/\partial\theta$. Therefore, if $0^\circ < \theta_1^* < 90^\circ$ is an absolute minimum for $\sigma(\theta)$ vs. θ , then the coefficient B is positive for $\theta < \theta_1^*$ and negative for $\theta > \theta_1^*$. It means that $\beta_1 > |\beta_2|$ for $\theta < \theta_1^*$, and vice-versa for $\theta > \theta_1^*$. Illustrative examples for aluminum alloys will be presented in the next section.

3. Applications

As demonstrated in the previous section, for any given material the inclinations of the localization bands and the overall trends (i.e. which band forms greater angle with respect to the loading axis) can be predicted without using any given model. In this section, we present applications for two initially textured aluminum sheets. For each material, we first present a qualitative analysis based solely on experimental flow stresses and r -values. Next, for any specimen orientation we obtain the numerical values of the necking bands angles using the theoretical $\sigma(\theta)$ and $r(\theta)$ according to two orthotropic criteria, namely Hill (1948) and Cazacu (2018).

3.1. Analysis of necking band orientations based solely on experimental r-values and flow stress data

The first example pertains to a 2090-T3 aluminum alloy sheet for which the uniaxial flow stresses (normalized by the uniaxial flow stress in the rolling direction) and r-values for seven in-plane loading orientations were reported by Barlat et al. (1997) (see Table 1). The strong anisotropy in uniaxial plastic properties of this material is due to its pronounced crystallographic texture (for the (111) pole figure of this sheet the reader is referred to Barlat et al. (2005)).

Specimen orientation to the rolling direction, θ (degrees)	Experimental $r(\theta)$	Experimental normalized flow stress $\sigma(\theta)/\sigma(0)$
0	0.212	1
15	0.327	0.96
30	0.692	0.91
45	1.577	0.81
60	1.038	0.81
75	0.538	0.88
90	0.692	0.91

Table 1: Experimental Lankford coefficients $r(\theta)$ and uniaxial flow stresses (normalized by the RD uniaxial flow stress) for 2090-T3 Al sheet for seven in-plane loading orientations θ (after Barlat et al., 1997).

Based on available data, $\theta_1^* = 45^\circ$ and $\theta_2^* = 60^\circ$ correspond to minimum values for the flow stresses. Provided that this is the case, according to Proposition 2, we have localization bands of equal inclinations for RD. 45° , 60° , and TD. The respective band orientations, calculated using the experimental r-values, are given in Table 2. Since for this material, both $r(0^\circ)$ and $r(90^\circ)$ are less than 1, for the RD and TD specimens the bands are inclined at an angle greater than $\beta_{iso} = 54.736^\circ$ which corresponds to $r = 1$. In addition, since $r(0^\circ) < r(90^\circ)$, the inclination of the bands is greater for the RD specimen than for the TD specimen (see Eq. (14)). On the other hand, for the 45° and 60° specimens, for which the r-values are greater than one, the inclination of the bands is less than 54.736° .

Specimen orientation to the rolling direction, θ (degrees)	Predicted bands orientation, β (degrees)
0	± 67.30
45	± 51.96
60	± 54.49
90	± 57.40

Table 2: Numerical values of the bands angles obtained from experimental r-values and Eq. (15) for the 2090-T3 Al alloy specimens orientations for which localization bands are equally inclined to the loading axis.

Since $\sigma(\theta)$ function admits minima at $\theta_1^*=45^\circ$ and $\theta_2^*=60^\circ$, according to Proposition 3 for the specimens of orientation $\theta < \theta_1^*$, the angle of the band corresponding to $\beta_1 > 0$ is numerically larger than $|\beta_2|$, and vice-versa for the specimens of orientation $\theta > \theta_2^*$.

One may argue that for AA 2090-T3 the function $\sigma(\theta)$ most likely has an absolute minimum θ^* somewhere between 45° and 60° . Since $r(\theta^*) > 1$, the inclination of the bands corresponding to the θ^* specimen is less than the isotropic value. And the same conclusions hold concerning the bands angles for specimens of orientation $\theta < 45^\circ$ and $\theta > 60^\circ$, respectively.

A second example concerns a cold-rolled 99% Al sheet (containing 0.4% Fe, 0.09% Mn, and 0.38% Si) cold rolled to 94% for which experimental in-plane uniaxial flow stresses were reported by Korber and Hoff (1928). For each specimen tested, these authors also measured the inclination of the predominant band, i.e. the band that grows faster and leads to specimen failure (see Table 3). Transverse strains data, necessary for the calculation of the experimental r-values, were not reported.

Specimen orientation to the rolling direction, θ (degrees)	Experimental normalized flow stresses $\sigma(\theta)/\sigma(0)$	Experimental band orientation, β (degrees)
0	1	± 66
22.5	0.99	-57
45	0.97	± 59
67.5	1.01	+57
90	1.03	± 60

Table 3: Experimental uniaxial flow stresses and the orientations of the localization bands reported by Korber and Hoff (1928) for specimens cut from a 99% Al sheet, rolled to 94%.

Note that the data indicate that for $0^\circ < \theta < 45^\circ$ the predominant band is the one that forms an acute clockwise angle to the loading axis, and vice-versa for $45^\circ < \theta < 90^\circ$. From the available data, it appears that $\theta_1^*=45^\circ$ orientation is a minimum for the uniaxial flow stresses. According to Proposition 3 (see section 2), for the specimens with $\theta < 45^\circ$ the band that forms greater angle with the loading axis corresponds to a positive value of β (i.e. $\beta_1 > |\beta_2|$), and vice-versa for the specimens with $\theta > 45^\circ$.

It is also interesting to note that in the F.E. study of N'souglo et al (2019) on a Ti rolled sheet that displays a minimum in uniaxial flow stresses at 45° , it was shown that for specimens with $\theta < 45^\circ$ the band that develops faster is the one that forms an acute clockwise angle to the RD, i.e. same trends as those observed by Korber and Hoff (1928) for the Al sheet.

3.2. Comparison between the bands orientations predicted by Hill (1948) and Cazacu (2018) criteria for the two aluminum alloys

For any specimen orientation, using Proposition 1 (see section 2), one can easily calculate the necking band inclinations by solving the quadratic algebraic Eq. (10). If for a given specimen orientation, the flow stress and r-value are not known, the necking bands can be obtained based on theoretical r-values and uniaxial flow stresses obtained using any given orthotropic yield criterion (plastic potential) or crystal plasticity models. In the following, for the two aluminum alloys analyzed in section 3.1., we will use Hill (1948) and Cazacu (2018) orthotropic yield criteria, respectively to obtain the numerical values of the necking band angles as a function of the specimen orientation θ . While the expression of the Hill (1948) criterion was already given in section 1, below we present the expression and the main features of Cazacu (2018) criterion. Cazacu (2018) orthotropic criterion is expressed as:

$$F(J_2^0, J_3^0) = (J_2^0)^4 - \alpha (J_2^0)(J_3^0)^2 \quad (21)$$

In Eq.(21), α is a parameter, and J_2^0, J_3^0 are the orthotropic invariants of the stress deviator, σ' , their expressions relative to the coordinate system associated with the orthotropy axes of the material being given by:

$$J_2^0 = \frac{a_1}{6}(\sigma'_{xx} - \sigma'_{yy})^2 + \frac{a_2}{6}(\sigma'_{yy} - \sigma'_{zz})^2 + \frac{a_3}{6}(\sigma'_{xx} - \sigma'_{zz})^2 + a_4 \sigma'^2_{xy} + a_5 \sigma'^2_{xz} + a_6 \sigma'^2_{yz}, \quad (22)$$

and

$$\begin{aligned} J_3^0 = & \frac{1}{27}(b_1 + b_2)\sigma'^3_{xx} + \frac{1}{27}(b_3 + b_4)\sigma'^3_{yy} + \frac{1}{27}[2(b_1 + b_4) - b_2 - b_3]\sigma'^3_{zz} \\ & - \frac{1}{9}(b_1\sigma'_{yy} + b_2\sigma'_{zz})\sigma'^2_{xx} - \frac{1}{9}(b_3\sigma'_{zz} + b_4\sigma'_{xx})\sigma'^2_{yy} \\ & - \frac{1}{9}[(b_1 - b_2 + b_4)\sigma'_{xx} + (b_1 - b_3 + b_4)\sigma'_{yy}]\sigma'^2_{zz} \\ & + \frac{2}{9}(b_1 + b_4)\sigma'_{xx}\sigma'_{yy}\sigma'_{zz} - \frac{\sigma'^2_{xz}}{3}[2b_9\sigma'_{yy} - b_8\sigma'_{zz} - (2b_9 - b_8)\sigma'_{xx}] \\ & - \frac{\sigma'^2_{xy}}{3}[2b_{10}\sigma'_{zz} - b_5\sigma'_{yy} - (2b_{10} - b_5)\sigma'_{xx}] - \frac{\sigma'^2_{yz}}{3}[2b_7\sigma'_{xx} - b_6\sigma'_{yy} - (2b_7 - b_6)\sigma'_{zz}] \\ & + 2b_{11}\sigma'_{xy}\sigma'_{xz}\sigma'_{yz} \end{aligned} \quad (23)$$

with a_i ($i=1...6$) and b_k ($k=1...11$) being constants. Note that J_2^0 is a homogenous polynomial of degree two in its arguments, insensitive to hydrostatic pressure, which satisfies the symmetry restrictions associated with orthotropy. When all the coefficients $a_i=1$, J_2^0 reduces to the classic

isotropic invariant J_2 . On the other hand, J_3^o is a homogenous third-order polynomial in stresses that reduces to the isotropic invariant J_3 if all $b_k=1$, it is insensitive to hydrostatic pressure, and satisfies orthotropic symmetries. For more details concerning the derivation of these orthotropic invariants, see Cazacu and Barlat (2001) and the monograph of Cazacu et al. (2018). The effective stress, $\bar{\sigma}$ associated to the Cazacu (2018) criterion is given by:

$$\bar{\sigma} = m \left[\left(J_2^o \right)^4 - \alpha \left(J_2^o \right) \left(J_3^o \right)^2 \right]^{1/8}, \quad (24)$$

with m being a constant defined such that for uniaxial tension in the $\mathbf{x}$ -direction the effective stress reduces to the yield stress, i.e.

$$m = \frac{3\sqrt{2}}{\left\{ \left[27(a_1 + a_3)^3 - 8\alpha(b_1 + b_2)^2 \right] (3a_1 + 3a_3) \right\}^{1/8}} \quad (25)$$

For general 3-D stress conditions Cazacu (2018) orthotropic criterion involves 17 anisotropy coefficients (for proof of this statement see Cazacu (2018)).

In the plane of the sheet, this orthotropic criterion predicts the following dependence of the normalized uniaxial flow stress $\sigma(\theta)/\sigma(0)$ on the angle θ :

$$m(\sigma(\theta)/\sigma(0)) = \left\{ \left[\left(a_1/6 + a_3/6 \right) \cos^4 \theta + \left(a_4 - a_1/3 \right) \cos^2 \theta \sin^2 \theta + \left(a_1/6 + a_2/6 \right) \sin^4 \theta \right]^4 \right. \\ \left. - \alpha \left[\begin{array}{l} \cos^6 \theta (b_1 + b_2)/27 + \sin^6 \theta (b_3 + b_4)/27 \\ -\sin^2 \theta \cos^4 \theta (b_1 + 3b_5 - 6b_{10})/9 \\ -\sin^4 \theta \cos^2 \theta (b_4 - 3b_5)/9 \end{array} \right]^2 \times \left[\begin{array}{l} \left(a_1/6 + a_3/6 \right) \cos^4 \theta \\ + \left(a_4 - a_1/3 \right) \cos^2 \theta \sin^2 \theta \\ + \left(a_1/6 + a_2/6 \right) \sin^4 \theta \end{array} \right] \right\}^{-1/8} \quad (26)$$

Assuming associated flow rule, the Lankford coefficients $r(\theta)$ are calculated using Eq. (9) with the specific expressions for $\frac{\partial \bar{\sigma}}{\partial \sigma_{ij}}$ $i, j=1...3$ given by

$$\frac{\partial \bar{\sigma}}{\partial \sigma_{ij}} = \left(\frac{m}{8} \right) \left\{ \left[4 \left(J_2^o \right)^3 - \alpha \left(J_3^o \right)^2 \right] \frac{\partial J_2^o}{\partial \sigma_{ij}} - 2\alpha \left(J_3^o \right) \left(J_2^o \right) \frac{\partial J_3^o}{\partial \sigma_{ij}} \right\} \times \\ \left[\left(J_2^o \right)^4 - \alpha \left(J_2^o \right) \left(J_3^o \right)^2 \right]^{-7/8} \quad (27)$$

It is also worth noting that if in the Cazacu (2018) orthotropic criterion the coefficient α is set equal to zero, Hill's orthotropic criterion is recovered (compare Eq. (24) with $\alpha=0$, and Eq. (2)).

Application to AA 2090-T3

For this material, the numerical values of the anisotropy parameters involved in Cazacu (2018) criterion are: $a_1=1.76$, $a_2=1.715$; $a_3=1.16$, $a_4=0.91$, $b_1=-6.4$, $b_2=-0.006$, $b_3=2.61$, $b_4=4.88$, $b_5=6.6$, $b_{10}=0.913$, and $\alpha=1.2$. For details concerning the identification procedure the reader is referred to Cazacu (2018). The numerical values of the parameters involved in Hill (1948) yield criterion (see Eq. (2)), which were obtained using the experimental r -values in the 0° , 45° and 90° directions are: $F = 0.252$, $G = 0.825$, $H = 0.175$, $N = 2.238$. Fig. 2 shows a comparison between the experimental r -values and uniaxial flow stresses, and the predictions provided by the two orthotropic criteria. It becomes apparent that Cazacu (2018) criterion provides a better description of the mechanical response of the AA 2090-T3 under uniaxial tension. This is to be expected given the fact that the latter criterion is non-quadratic and as such involves more anisotropy coefficients.

“As already mentioned, for AA 2090-T3 the band angles were not measured. However, for each specimen orientation the band angles can be calculated using Eq. (10) in conjunction with any given yield criterion. As an example, in Figures 3-4 are shown the predicted values of the bands angles according to Hill (1948) and Cazacu (2018) criteria, respectively. The coefficients A and B of Eq. (10) as well as $r(\theta)$ were calculated according to the respective yield criterion, namely for Hill (1948), Eq.(2) was used for calculating $\bar{\sigma}$, and its derivatives while for Cazacu (2018) Eq. (24) was used.

Note that for AA 2090-T3, $N > F+2H$ and $N > G+2H$. Therefore, according to Hill (1948) criterion the extrema for the in-plane uniaxial flow stresses correspond to RD, TD, and the intermediate orientation $\theta^*=39.42^\circ$ (see Eq. (5)). Moreover, according to Proposition 3 (see section 2), for specimens with orientation $0<\theta<39.42^\circ$, we have that $\beta_1 > |\beta_2|$ and the maximum of $(\beta_1 - |\beta_2|) \cong 17^\circ$. On the other hand, for specimens with $\theta>39.42^\circ$, the angle for the necking band corresponding to β_2 is larger in absolute value, and the maximum of $(|\beta_2| - \beta_1) \cong 37^\circ$ (see Fig. 3). In addition, using Eq. (3), Eq. (4) and Eq. (18), we obtain that for the 0° specimen, the specimen with $\theta^*=39.42^\circ$ and the 90° specimen, the bands are inclined $\pm 67.3^\circ$, $\pm 53.4^\circ$ and $\pm 57.4^\circ$ with respect to the loading axis. Note that according to Hill (1948), the bands angles for the RD and TD specimens coincide with the values reported in Table 2, which were calculated using the experimental $r(0^\circ)$ and $r(90^\circ)$ values. This is to be expected since calibration of Hill (1948) was done by imposing that the theoretical and experimental values $r(0^\circ)$ and $r(90^\circ)$ coincide (see also Fig 2(b)).

Therefore, according to Hill (1948) it is also predicted that for these orientations the bands are inclined at an angle greater than the isotropic value and moreover, the inclination of the bands is greater for the RD specimen than for the TD specimen. On the other hand, for the specimen

corresponding to $\theta^* = 39.42^\circ$ for which the predicted r-value is greater than one, the inclination of the bands is less than 54.736° . It is also worth noting that if Hill (1948) yield criterion is used to describe the plastic anisotropy of this material, a slight misalignment between the loading axis and RD (i.e. loading axis corresponding to $\theta = 6.48^\circ$) will result in a band oriented at $\sim 70^\circ$ to the loading axis (see Fig. 3).

On the other hand, if the plastic behavior of the same material is described using Cazacu (2018) criterion, the bands angles and the specimen orientations for which the bands angles are equal and opposite in sign will be different. Indeed, for the given values of the anisotropy coefficients a_i and b_k , Cazacu (2018) orthotropic criterion predicts that the minima for the in-plane uniaxial flow stresses for 2090-T3 aluminum alloy correspond to RD, TD, and an intermediate orientation $\theta^* = 60^\circ$ (see Fig. 2(a)). Moreover, according to Proposition 3 (see section 2), for specimens with orientation $0 < \theta < 60^\circ$, we have that $\beta_1 > |\beta_2|$, and the maximum of $(\beta_1 - |\beta_2|) \cong 13^\circ$. On the other hand, for specimens with $\theta > 60^\circ$, the angle for the necking band corresponding to β_2 is larger in absolute value and the maximum $(|\beta_2| - \beta_1) \cong 18.26^\circ$ (see Fig. 4). For the RD, $\theta^* = 60^\circ$ and 90° specimens, it is predicted that the bands are equally inclined at angles of $\pm 67.3^\circ$, $\pm 54.147^\circ$ and $\pm 57.4^\circ$, respectively. Note the good agreement between the results for the 0° , 60° and 90° specimens and the band angles values reported in Table 2, which were calculated using the experimental r-values. This is because the calibration of the parameters involved in Cazacu's model was done by minimizing the error between calculated and predicted r-values and yield stresses along several orientations, including these specific values of θ . Note also that according to Cazacu's model, a slight misalignment between the loading axis and RD (corresponding to $\theta = 6.84^\circ$) will result in a necking band oriented at $\sim 70^\circ$ to the loading axis (see Fig. 4).

The comparison between the results presented in Figs. 3 and 4 makes apparent the important role played by the yield criterion in the prediction of the orientation of the necking bands that define the localization process in anisotropic flat specimens subjected to uniaxial tension. Recall that the intermediate orientation θ^* for which localization bands of equal inclination develop is $\theta^* = 60^\circ$ according to Cazacu's (2018) criterion, and $\theta^* = 39.42^\circ$ according to Hill's (1948) criterion. Note also that, the maximum of $(|\beta_2| - \beta_1)$ is significantly smaller, 18.26° versus 37° for the model of Hill. This shows that the predicted bands angles depend significantly on the constitutive model used for the description of material plastic anisotropy.

Application to the 99% Al alloy

As mentioned before, for this material transverse strains data, necessary for the calculation of the experimental r-values were not reported. Therefore, the identification of a non-quadratic yield criterion like Cazacu's model (2018), which involves more than four anisotropy parameters for in-plane loadings is not possible. However, identification of the parameters F , G , H , N involved in the quadratic Hill (1948) yield criterion can be done using the available experimental uniaxial flow

stresses (see Table 3). The following numerical values of the anisotropy coefficients: $F=0.469$, $G=0.531$, $H=0.469$, $N=1.637$ are obtained. The comparison between the experimental and simulated uniaxial flow stress variation is shown in Fig 5(a), and the projection of the yield surface in the (RD, TD) plane in Fig. 5 (b). Using Eq. (18), the predicted necking angles β_1 and $|\beta_2|$ as a function of the angle θ between the specimen axis and the RD axis are shown in Fig. 6. Note that since Hill's model was calibrated using the experimental uniaxial flow stresses, it captures qualitatively the experimental trends, namely that for specimens with orientation $\theta < 45^\circ$, $\beta_1 > |\beta_2|$ and vice-versa for the specimens with $\theta > 45^\circ$. The differences between the experimental and simulated values of β_1 and $|\beta_2|$ show that in order for Hill (1948) criterion to capture with increased accuracy the orientation of the necking bands that develop in the specimen, identification using r-values is required.

This reinforces the idea that the use of orthotropic plastic potentials that capture simultaneously the anisotropy in uniaxial flow stresses and r-values is essential for predicting the plastic localization in metallic sheets subjected to uniaxial tension.

4. Conclusions

In this paper we have investigated the effect of the loading orientation on the inception of localization bands in orthotropic metallic sheets subjected to uniaxial tension. For this purpose, specimens with the loading axis oriented at different angles $0^\circ \leq \theta \leq 90^\circ$ to the rolling direction of the sheet were investigated. Based on classical results of bifurcation analysis, which show that under uniaxial tension the localization bands are along the directions of zero extension, we have demonstrated that irrespective of the mathematical form of the plastic potential used for the description of plastic anisotropy, the bands angles are the solutions of a second-order algebraic equation. Specifically, it was demonstrated that localization bands equally inclined develop only for the specimens of orientations θ^* corresponding to the extrema of the uniaxial tensile flow stresses in the plane of the sheet. These orientations θ^* are also the only ones for which the principal directions of stress and plastic strain increments coincide, and explicit expressions for the bands angles in terms on the Lankford coefficient $r(\theta^*)$ were provided. This is a key result of our investigation that shows that the bands angles can be obtained directly from experimental measurements of r-values and flow stresses. To the authors' knowledge, this has not been reported before.

For specimen orientations other than θ^* , the two localization bands have different inclinations with respect to the loading axis. Specifically if θ^* is a minimum for the uniaxial tensile flow stresses, the analytical expressions developed for the bands angles indicate that for specimens with $\theta < \theta^*$ we have that $\beta_1 > |\beta_2|$, i.e. the band that forms greater angle with the loading axis corresponds to

a positive value of β , and vice-versa for $\theta > \theta^*$. On the other hand if θ^* is a maximum, the reverse occurs, namely for $\theta < \theta^*$ we have $\beta_1 < |\beta_2|$, while for $\theta > \theta^*$, we have $\beta_1 > |\beta_2|$. Most importantly, these trends can be inferred solely on the basis of the experimental variation of the uniaxial flow stresses and r-values of the metallic sheet.

Also, for the specimen orientations θ different from θ^* , the numerical values of the localization bands angles can be obtained using Eq. (10) in conjunction with any orthotropic plastic potential for the description of the plastic properties of the material. As an example, we provided the explicit expressions for the band angles obtained using Hill (1948) and Cazacu (2018) orthotropic plastic potentials, and further presented applications to two textured aluminum sheets: a 2090-T3 Al alloy, and a 99% Al alloy. The results obtained show the great sensitivity of the orientations of the localization bands to the plastic potential used to describe the mechanical response of the material.

The importance of the anisotropy of the material on the bands angles that was put into evidence leads to another key outcome of this investigation, namely that it is possible to engineer the localization path by producing metallic sheets with specific textures. Indeed, since cracks will ultimately develop from the necking bands, it is possible to engineer the crack path by controlling the material's initial microstructure, either by including specific distributions of second phase particles as suggested by Srivastava et al. (2017), or tailoring specific textures.

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