Reference: E.I.Rashba, Proc. Inst. Civil Engin. of Acad. Sci. of Ukrain. SSR 18, 23-27 (1953)

Determining stresses in massive structures originating from their own weight: The effect of the construction sequence

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The article considers the stress profile that develops in a massive structure due to its own weight, with the account taken of the sequence of the construction. It is demonstrated that the conventional approach that disregards the construction sequence can in general lead to significant errors.

The stressed state of an isotropic massive body that develops under the action of its own weight is determined by the equations of elastic equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \varrho g = 0; \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{1}$$

(the *x* axis is directed downward), which are usually combined with the strain compatibility equation

$$\nabla^2 (\sigma_x + \sigma_y) = 0 \tag{2}$$

subject to boundary conditions (given here for a planar problem).

While equations (1) are not subject to any special assumptions and are very general, the derivation of the compatibility equation (2) implies that the elastic body has been formed in the absence of external forces ("zero gravity") and the loads were applied afterwards. However, in the construction of massive structures in which the effects of their own weight can become significant, the weight of each layer creates deformations only in the layers laid previously; this happens before the subsequent layers are laid above this layer. Therefore, the compatibility equation (2) expressing the condition of the preservation of the elastic body continuity after application of the load loses its validity. This is so because the formation of the elastic body actually occurs in the loading process and its configuration is constantly changing.

From what has been said above it follows that the stressed state depends on the order of the construction of the structure. The stresses in each layer are obtained by adding the contributions of all layers laid above a given layer; the contribution due to each newly added layer should be determined from the configuration that the structure had before adding this layer. As seen from the example described below, the stressed state found in this way can differ significantly from that determined by the system of equations (1) and (2).

As an example, we consider the stressed state of an infinite wedge (Fig. 1) arising due to its own weight. The stresses in such a wedge that originate from its own weight [1] and satisfy the system of equations (1) and (2) are given by

$$\sigma_x = \varrho g\left[y\,\frac{\cos\alpha\cos\beta\sin(\beta-\alpha)}{\sin^2(\beta+\alpha)} - x\left(1-\frac{1}{2}\,\frac{\sin 2\alpha\sin 2\beta}{\sin^2(\beta+\alpha)}\right)\right];$$

$$\sigma_y = -\frac{\varrho g\sin\alpha\sin\beta}{\sin^2(\beta+\alpha)}\left[y\sin(\beta-\alpha)+2x\sin\alpha\sin\beta\right]; \tag{3}$$

$$\tau_{xy} = -\frac{\varrho g\sin\alpha\sin\beta}{\sin^2(\beta+\alpha)}\left[2y\cos\alpha\cos\beta - x\sin(\beta-\alpha)\right].$$

These relations predict an unphysical behavior when the angles α and β grow and approach π/2 (the case of large angles is important, in particular, in calculating earth dams). The tangential stresses σ_y and τ_{xy} grow quickly and in the limit α=β=π/2 diverge instead of assuming zero limiting values, as expected for tangential stresses in a heavy uniform half-plane.

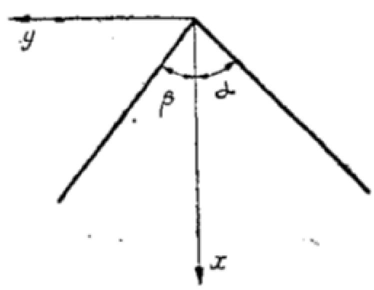

Fig. 1

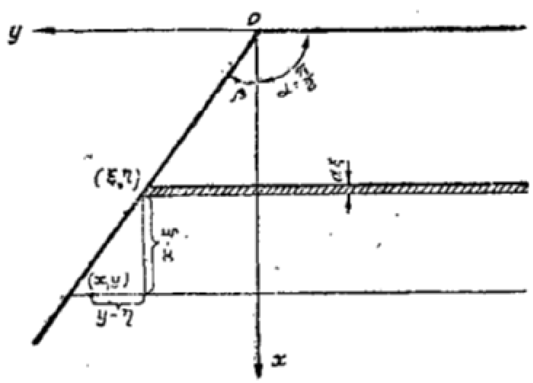

Fig. 2

In order to quantitatively compare the stresses determined by the conventional method from equations (1) and (2) with those obtained by accounting only for the influence of the layers laid over the given layer, we consider an infinite wedge with an angle α=π/2 (Fig. 2), built by stacking infinitesimally thin horizontal layers.

Equations (3) for $\alpha=\pi/2$ predict the stress profile

$$\sigma_x = -\varrho g x; \quad \sigma_y = \varrho g \operatorname{tg}\beta (y - 2x \operatorname{tg}\beta); \qquad (4)$$
$$\tau_{xy} = -\varrho g x \operatorname{tg}\beta.$$

Notably, the tangential stresses σ_y and τ_{xy} given by Eq.(4) feature a divergence when the wedge angle β grows and approaches $\pi/2$. However, as noted above, in this limit the tangential stresses are expected to vanish. This conundrum can be resolved by using the method that accounts for the construction sequence.

When performing the calculation according to the method proposed above, we must model the effect of each new layer with a uniform vertical load and integrate over all overlaying layers. In doing so one must account for the profile changing gradually in the process. If a uniformly distributed normal load *q* is applied along the horizontal edge of an infinite wedge with the angle $\alpha=\pi/2$, with the vertex located at the point (ξ,η), then the stresses arising in it are given by [1]:

$$\sigma_x(x-\xi, y-\eta) = -\frac{q}{\operatorname{ctg}\beta+\beta+\frac{\pi}{2}}\left[\operatorname{ctg}\beta+\beta-\operatorname{arc\,tg}\frac{y-\eta}{x-\xi}-\frac{(x-\xi)(y-\eta)}{(x-\xi)^2+(y-\eta)^2}\right];$$
$$\sigma_y(x-\xi, y-\eta) = -\frac{q}{\operatorname{ctg}\beta+\beta+\frac{\pi}{2}}\left[\beta-\operatorname{arc\,tg}\frac{y-\eta}{x-\xi}+\frac{(x-\xi)(y-\eta)}{(x-\xi)^2+(y-\eta)^2}\right]; \qquad (5)$$
$$\tau_{xy}(x-\xi, y-\eta) = -\frac{q}{\operatorname{ctg}\beta+\beta+\frac{\pi}{2}}\cdot\frac{(x-\xi)^2}{(x-\xi)^2+(y-\eta)^2}.$$

After integrating over the overlaying layers we obtain the stresses in a heavy wedge with the angle $\alpha=\pi/2$ and a vertex taken to be at the origin

$$\sigma_x^*(x, y) = \varrho g \int_0^x \sigma_x(x-\xi, y-\xi\operatorname{tg}\beta)\, d\xi \qquad (6)$$

The remaining quantities are given by similar expressions. As a result of integration, we obtain (here $r = (x^2+y^2)^{1/2}$)

$$\sigma_x^* = \frac{\varrho g}{\operatorname{ctg}\beta+\beta+\frac{\pi}{2}}\Big\{-\Big[\beta(1+2\sin^2\beta\cos^2\beta)+\cos^2\beta\operatorname{ctg}\beta+\frac{\pi}{2}\sin^2\beta\Big]x+$$

$$+\sin\beta\cos\beta\Big(2\beta\cos^2\beta+\frac{\pi}{2}\Big)y+2\cos^4\beta(x\operatorname{tg}\beta-y)\ln\frac{x\operatorname{tg}\beta-y}{r}+$$

$$+\cos\beta(x\cos\beta+y\sin\beta)\operatorname{arc\,tg}\frac{y}{x}-$$

$$-2\sin\beta\cos^3\beta(x\operatorname{tg}\beta-y)\operatorname{arc\,tg}\frac{x\cos\beta+y\sin\beta}{\cos\beta(x\operatorname{tg}\beta-y)}\Big\}; \qquad (7a)$$

$$\sigma_y^* = \frac{\varrho g}{\operatorname{ctg}\beta+\beta+\frac{\pi}{2}}\Big\{-\Big[\beta(1-2\sin^2\beta\cos^2\beta)+\sin\beta\cos\beta+\frac{\pi}{2}\sin^2\beta\Big]x-$$

$$-\sin\beta\cos\beta\Big(2\beta\cos^2\beta-\frac{\pi}{2}\Big)y+2\sin^2\beta\cos^2\beta(x\operatorname{tg}\beta-y)\ln\frac{x\operatorname{tg}\beta-y}{r}+$$

$$+\cos\beta(x\cos\beta+y\sin\beta)\operatorname{arc\,tg}\frac{y}{x}+$$

$$+2\sin\beta\cos^3\beta(x\operatorname{tg}\beta-y)\operatorname{arc\,tg}\frac{x\cos\beta+y\sin\beta}{\cos\beta(x\operatorname{tg}\beta-y)}\Big\}; \qquad (7b)$$

$$\tau_{xy}^* = \frac{\varrho g}{\operatorname{ctg}\beta+\beta+\frac{\pi}{2}}\Big\{-x\cos^2\beta+2\sin\beta\cos^3\beta(x\operatorname{tg}\beta-y)\ln\frac{x\operatorname{tg}\beta-y}{r}+$$

$$+\cos^2\beta(\cos^2\beta-\sin^2\beta)(x\operatorname{tg}\beta-y)\Big[\operatorname{arc\,tg}\frac{x\cos\beta+y\sin\beta}{\cos\beta(x\operatorname{tg}\beta-y)}+\beta\Big]\Big\}. \qquad (7c)$$

The qualitative behavior of these dependences lends strong support to Eqs. (7) over the relations (4). Indeed, the stress values predicted by Eqs. (7) are everywhere finite at a fixed depth *x*, transforming into the relations for a heavy half-plane for $\beta=\pi/2$; asymptotically, for *y* values that are large and negative,

$$y \to -\infty$$

the tangential stress

$$\tau_{xy}^*$$

defined by Eq. (7c) vanishes.

(In all fractions commas should be understood as dots) Table 1

β	$-\frac{\sigma_x^*}{\varrho g x}$ from Eq.(7a)	$-\frac{\sigma_x}{\varrho g x}$ from Eq.(4)	$-\frac{\sigma_y^*}{\varrho g x}$ from Eq.(7b)	$-\frac{\sigma_y}{\varrho g x}$ from Eq.(4)	$-\frac{\tau_{xy}^*}{\varrho g x}$ from Eq.(7c)	$-\frac{\tau_{xy}}{\varrho g x}$ from Eq.(4)
0°	1,000	1	0	0	0	0
5°	0,867	1	0,007	0,008	0,076	0,088
10°	0,741	1	0,023	0,031	0,131	0,176
15°	0,626	1	0,045	0,072	0,168	0,268
30°	0,340	1	0,113	0,333	0,196	0,577
45°	0,149	1	0,149	1,000	0,149	1,000
60°	0,045	1	0,136	3,000	0,078	1,732
75°	0,006	1	0,079	13,9	0,021	3,732
90°	0	1	0	∞	0	∞

Table 1 compares the stresses expressed in units of $\rho g x$ at the inclined edge of the wedge $y = x \tan\beta$ calculated from expressions (4) and (7) for various values of the angle β. The values shown indicate that the stresses calculated from Eqs. (4) exceed in magnitude the stresses calculated Eqs. (7), becoming extremely large for large β values. The discrepancy becomes dramatic as the angle β increases. Indeed, for β=0, the stresses calculated Eqs. (4) and (7) coincide with each other; however, for β=30° they differ by about 3 times, with β=60° – by about 22 times, and when β=75° – by about 175 times.

(In all fractions commas should be understood as dots) Table 2

$\frac{y}{x}$	$-\frac{\sigma_x^*}{\varrho g x}$ from Eq.(7a)	$-\frac{\sigma_x}{\varrho g x}$ from Eq.(4)	$-\frac{\sigma_y^*}{\varrho g x}$ from Eq.(7b)	$-\frac{\sigma_y}{\varrho g x}$ from Eq.(4)	$-\frac{\tau_{xy}^*}{\varrho g x}$ from Eq.(7c)	$-\frac{\tau_{xy}}{\varrho g x}$ from Eq.(4)
1,732	0,045	1	0,136	3	0,078	1,7321
1,155	0,412	1	0,356	4	0,171	1,7321
0,577	0,701	1	0,463	5	0,173	1,7321
0	0,889	1	0,536	6	0,120	1,7321
− 0,577	0,966	1	0,603	7	0,064	1,7321
− 1,155	0,988	1	0,655	8	0,034	1,7321
− 2,309	0,997	1	0,714	10	0,013	1,7321
− 4,042	0,999	1	0,752	13	0,005	1,7321
− 5,774	1,000	1	0,770	16	0,003	1,7321
− 8,660	1,000	1	0,785	21	0,001	1,7321
−14, 43	1,000	1	0,798	31	0,000	1,7321
−∞	1	1	0,819	∞	0	1,7321

Table 2 gives the stresses expressed in units of $\rho g x$ in a generic horizontal section of the wedge for β=60°. The spatial dependence of the quantities

$$\frac{\sigma_x^*}{\varrho g x}, \quad \frac{\sigma_y^*}{\varrho g x} \text{ and } \frac{\tau_{xy}^*}{\varrho g x}$$

constructed according to Table 2 is presented in Fig. 3. For comparison, the quantity

$$\frac{\sigma_x}{\varrho g x}$$

is shown in dashed line. The quantities

$$\frac{\sigma_y}{\varrho g x} \text{ and } \frac{\tau_{xy}}{\varrho g x}$$

are significantly larger in magnitude than

$$\frac{\sigma_y^*}{\varrho g x} \text{ and } \frac{\tau_{xy}^*}{\varrho g x}$$

The corresponding values are too large to be shown to scale on the plot.

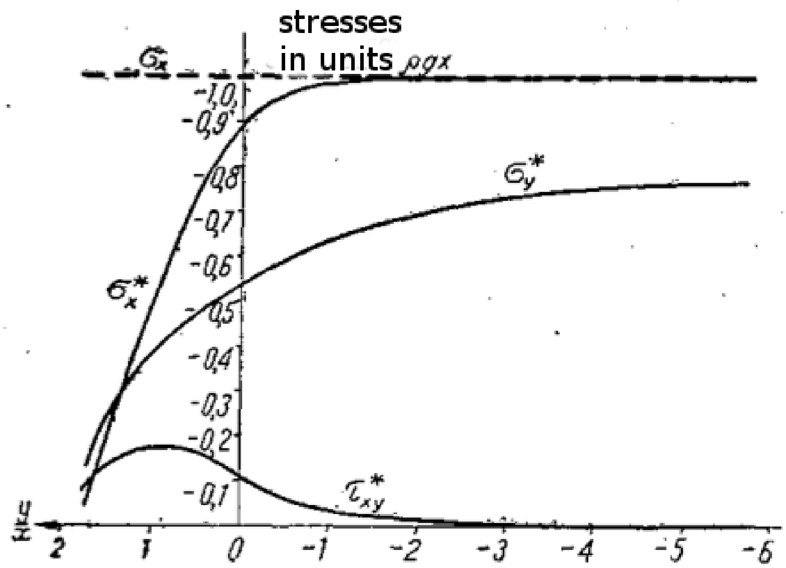

Fig. 3

As can be seen from the example presented above, the dependence of the stress state on the order of construction of the structure can be very significant and its effect cannot be neglected.

Bibliography:

1. B. G. Galerkin, Collection of LIIPS, no. 99, 1929.

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